非平衡統計力学の数理モデル

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1. 量子化する爆発機構

Model

Non-equilibrium thermo-dynamics

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$\left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \;\; u|_{t=0} = u_0(x) > 0$$

2. Poisson Part

$$-\Delta v = u, \;\; v|_{\partial\Omega} = 0$$

Green's function

$$G(x, x') = G(x', x)$$

$u = u(x,t) \geq 0$ density

$j = -\nabla u + u\nabla v$ flux

diffusion mass velocity

$u_t + \nabla \cdot j = 0$ conservation law

$v = (-\Delta)^{-1} u$ potential

attractive (chemotaxis, gravitation)
action at a distance (long range potential)
symmetry (action-reaction)

thermally closed system (canonical ensemble)

1. total mass conservation $\displaystyle \frac{d}{dt}\|u(t)\|_1 = 0$

2. free energy decreasing

$$A = U - TS$$

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u \otimes u$$

-(entropy) inner energy

$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 \leq 0$$

stationary state

$$u = \frac{\lambda e^v}{\int_\Omega e^v dx}$$

Boltzmann equation

scaling

$$u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t), \; \mu > 0$$

$$u_\mu(x) = \mu^2 u(\mu x), \;\; \mu > 0$$

$$\|u\|_1 = \|u_\mu\|_1 \equiv \lambda \;\Leftrightarrow\; n = 2$$ *critical dimension*

$$\mathcal{F}(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2}\langle \Gamma * u, u\rangle, \;\; \Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$\mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi}\right)\log\mu + \mathcal{F}(u)$$ *critical mass* $\lambda = 8\pi$

Boltzmann Poisson equation ~ stationary state

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}$$

Euler's equation of motion

$$v_t + (v \cdot \nabla) v = -\nabla p, \ \ \nabla \cdot v = 0, \ \ \nu \cdot v|_{\partial\Omega} = 0$$

2D $\omega = \nabla^{\perp} \cdot v$ ➡ $\omega_t + \nabla \cdot (v\omega) = 0, \ \ \nabla \cdot v = 0$

$v = \nabla^{\perp}\psi$ stream function $\nabla^{\perp} = \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}$

$$\omega_t + \nabla \cdot (\omega \nabla^{\perp}\psi) = 0, \ \ -\Delta\psi = \omega$$ vorticity equation

boundary condition ➡ $\psi|_{\partial\Omega} = 0$

Green function

$$-\Delta_x G(x, x') = \delta_{x'}(dx), \ \ G(x, x')|_{x \in \partial\Omega} = 0$$ ➡

$$\omega_t + \nabla \cdot (\omega \nabla^{\perp}\psi) = 0, \ \ \psi(\cdot, t) = \int_\Omega G(\cdot, x')\omega(x', t) dx'$$

$G(x, x') = G(x', x)$ action reaction law

$\varphi \in C^1(\overline{\Omega}), \ \ \varphi|_{\partial\Omega} = 0$ ➡ $\omega \otimes \omega = \omega(x, t)\omega(x', t)$

$$\frac{d}{dt}\int_\Omega \varphi\omega = \int_\Omega \omega \nabla^{\perp}\psi \cdot \nabla\varphi$$

$$= \int\int_{\Omega\times\Omega} \nabla_x^{\perp} G(x, x') \nabla\varphi(x) \ \omega(x, t)\omega(x', t) \ dxdx'$$

$$= \frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\varphi(x, x') \omega \otimes \omega \ dxdx'$$

$$\rho_\varphi(x, x') = \nabla\varphi(x) \cdot \nabla_x^{\perp} G(x, x') + \nabla\varphi(x') \cdot \nabla_{x'}^{\perp} G(x, x') \in L^\infty(\Omega \times \Omega)$$

$$\omega(dx, t) = \sum_{i=1}^{\ell} \alpha_i \delta_{x_i(t)}(dx)$$ point vortex system

local second moment p.v. ➡ **Kirchhoff equation** $\frac{dx_i}{dt} = \nabla_{x_i}^{\perp} H_\ell$

point vortex Hamiltonian

$$H_\ell(x_1, \ldots, x_\ell) = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i\alpha_j G(x_i, x_j)$$

Robin function $R(x) = \left[G(x, x') + \frac{1}{2\pi} \log|x - x'| \right]_{x'=x}$

Thermal equilibrium

Gibbs theory of statistical mechanics

micro-canonical statistics

total energy

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \ 1 \leq i \leq \ell$$

$$\mathbf{R}^{6\ell}/\{H = E\}$$

co-area formula

$$x = (q_1, \ldots, q_\ell, p_1, \ldots, p_\ell), \ dx = dE \cdot \frac{d\Sigma(E)}{|\nabla H|}$$

$$d\Sigma(E) \ \leftrightarrow \ \left\{ x \in \mathbf{R}^{6\ell} \mid H(x) = E \right\}$$

micro-canonical measure

weight factor

$$d\mu^{E,N} = \frac{1}{W(E)} \cdot \frac{d\Sigma(E)}{|\nabla H|}, \ W(E) = \int_{\{H=E\}} \frac{d\Sigma(E)}{|\nabla H|}$$

$$H_\ell(x_1, \ldots, x_\ell) = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

principle of equal probability

$$\alpha_i = \hat{\alpha}, \ \hat{\alpha}\ell = 1, \ \hat{H}_\ell = H, \ \hat{\alpha}^2 \ell \hat{\beta} = \beta \qquad \ell \uparrow +\infty$$

canonical statistics

Boltzmann constant

$$\mathbf{R}^{6N}/\{T\}, \ \beta = 1/(kT)$$

inverse temperature

$$d\mu^{\beta,N} = \frac{e^{-\beta H} dx}{Z(\beta, N)}, \ Z(\beta, N) = \int_{\mathbf{R}^{6N}} e^{-\beta H} dx$$

canonical measure

weight factor

thermo-dynamical relation

$$\beta = \frac{\partial}{\partial E} \log W(E)$$

order structure in negative temperature

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}$$

vorticity

duality

$$\psi = \int_\Omega G(\cdot, x')\rho(x')dx'$$

stream function

Boltzmann-Poisson equation

⟺

$$-\Delta v = u, \ \ v|_{\partial\Omega} = 0$$

$$u = \frac{\lambda e^v}{\int_\Omega e^v dx}, \ \lambda = \|u\|_1$$

Hamiltonian is recursive

$$u=\frac{\lambda e^v}{\int_\Omega e^v dx} \qquad -\Delta v = u, \ \ v|_{\partial\Omega}=0$$

Boltzmann Poisson

$$\longrightarrow \quad -\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \ \ v|_{\partial\Omega}=0$$

Nonlinear eigenvalue problem

point vortices ~ negative inverse temperature

L. Onsager 49

$$G(x,x')=G(x',x)$$

$$R(x)=\left[G(x,x')+\frac{1}{2\pi}\log|x-x'|\right]_{x'=x}$$

Theorem A (Nagasaki-S. 90)

$$\{(\lambda_k, v_k)\}, \ \lambda_k \to \lambda_0 \in (0,\infty), \ \|v_k\|_\infty \to \infty$$

$$\longrightarrow \quad \lambda_0 = 8\pi\ell, \ \ell \in \mathbf{N}, \ \exists \mathcal{S} \subset \Omega, \ \sharp\mathcal{S}=\ell$$ stationary quantization

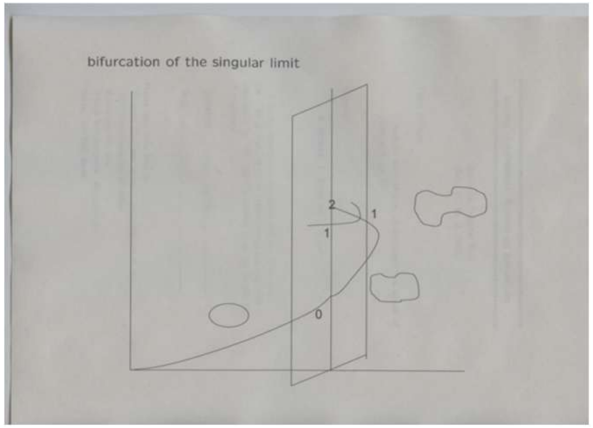
(sub-sequence) $v_k \to v_0$ loc. unif. in $\overline{\Omega}\setminus\mathcal{S}$

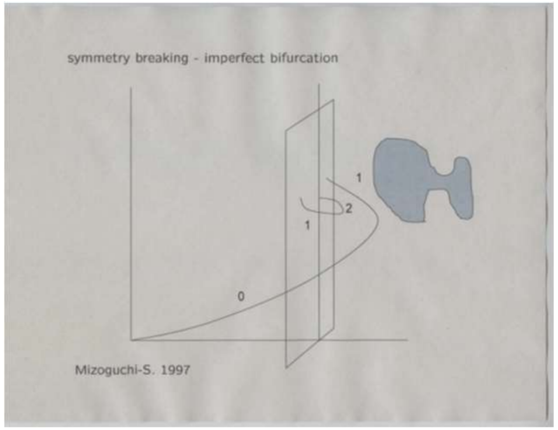
$$v_0(x)=8\pi\sum_{x_0\in\mathcal{S}} G(x,x_0), \ \ \mathcal{S}=\{x_1^*,\ldots,x_\ell^*\}$$

singular limit blowup set

$$\nabla H_\ell|_{(x_1,\cdots,x_\ell)=(x_1^*,\cdots,x_\ell^*)}=0$$

$$H_\ell(x_1,\ldots,x_\ell)=\frac{1}{2}\sum_i R(x_i)+\sum_{i<j}G(x_i,x_j)$$





Geometric background

$$-\Delta v = \sigma e^v, \ \sigma = \frac{\lambda}{\int_\Omega e^v dx}$$

$$\Leftrightarrow \ \exists F = F(z), \ z \in \Omega \subset \mathbf{R}^2 \cong \mathbf{C}$$ meromorphic function

$$\rho(F) = \left(\frac{\sigma}{8}\right)^{1/2} e^{v/2} = \frac{|F'|}{1+|F|^2}$$ spherical derivative

$$-\Delta v = \sigma e^v, \ v|_{\partial\Omega} = 0 \ \Leftrightarrow \ \rho(F)|_{\partial\Omega} = \left(\frac{\sigma}{8}\right)^{1/2}$$

$$\hat{F} = \sqrt{8} \circ F : \Omega \to S^2$$ conformal

$$\left.\frac{d\Sigma}{ds}\right|_{\partial\Omega} = \sigma^{1/2}$$

$(S^2, d\Sigma)$ round sphere

$|S^2| = 8\pi$

$$\int_{\partial\Omega} \frac{d\Sigma}{ds} ds = |\partial\Omega| \, \sigma^{1/2}$$

immersed length of $\hat{F}(\partial\Omega)$

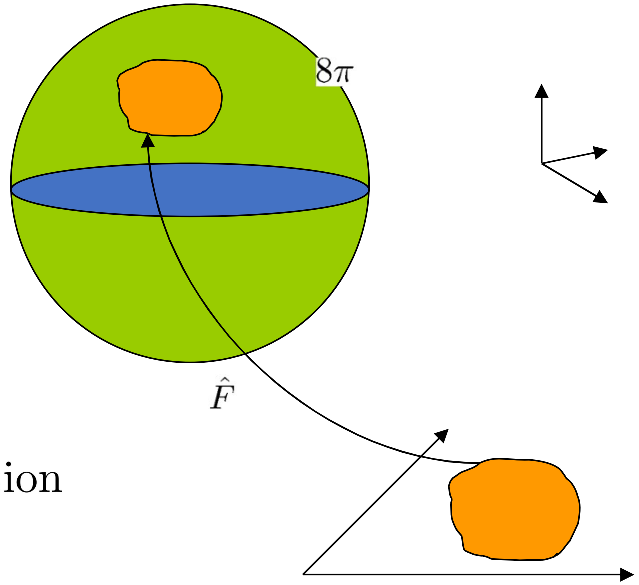
Proof of Theorem A

1. Liouville integral
2. boundary reflection
3. elliptic regularity

4. complex function theory
4-1. maximum principle
4-2. Montel's theorem
4-3. theorem of coincidence
4-4. residue analysis

$$\int_\Omega \left(\frac{d\Sigma}{ds}\right)^2 dx = 8\int_\Omega \rho(F)^2 dx = \int_\Omega \sigma e^v$$

immersed area of $\hat{F}(\Omega)$



$$\lambda = \int_\Omega \sigma e^v \to 8\pi\ell$$

$\Leftrightarrow$ total mass quantization

due to ℓ-covering

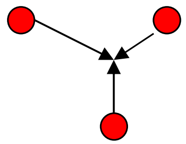
Blowup analysis

$\Omega \subset \mathbf{R}^2$: open set, $V \in C(\overline{\Omega})$

$-\Delta v = V(x)e^v, \ \ 0 \leq V(x) \leq b \quad \text{in } \Omega$

$\int_\Omega e^v \leq C$

Theorem a （Li-Shafrir 94）



$\{(V_k, v_k)\}$ solution sequence

$V_k \to V$ loc. unif. in Ω

$\Rightarrow {}^\exists$ sub-sequence with the alternatives;

1. $\{v_k\}$: loc. unif. bdd in Ω

2. $\exists \mathcal{S} \subset \Omega$, $\sharp \mathcal{S} < +\infty$

$v_k \to -\infty$ loc. unif. in $\Omega \setminus \mathcal{S}$

$\mathcal{S} = \{x_0 \in \Omega \mid \exists x_k \to x_0, \ v_k(x_k) \to +\infty$

$V_k(x)e^{v_k}dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx)$ in $\mathcal{M}(\Omega)$

$m(x_0) \in 8\pi\mathbf{N}$

3. $v_k \to -\infty$ loc. unif. in Ω

Comments

1. mass quantization for variable coefficients without boundary condition
2. possible collapse collision
3. many applications together with the proof

<u>prescaled analysis</u> ...Brezis-Merle 91

linear theory $\Rightarrow$

1, 2 with $m(x_0) \geq 4\pi$ (rough estimate), 3

2... localized to $B = B(0,R)$

$-\Delta v_k = V_k(x)e^{v_k}$, $V_k(x) \geq 0$ in B

$V_k \to V$ unif. in $\overline{B}$, $\max_{\overline{B}} v_k \to +\infty$

$\max_{\overline{B}\setminus B_r} v_k \to -\infty$, $\forall r \in (0,R)$

$\lim_k \int_B V_k e^{v_k} = \alpha$, $\int_B e^{v_k} \leq C$

$\Rightarrow \alpha \in 8\pi\mathbf{N}$

Scaling

$$v_k(x_k) = \|v_k\|_\infty, \ x_k \to 0$$

$$\tilde{v}_k(x) = v_k(\delta_k x + x_k) + 2\log \delta_k, \ \delta_k = e^{-v_k(x_k)/2} \to 0$$

$\longrightarrow$

$$-\Delta \tilde{v}_k = V_k(\delta_k x + x_k)e^{\tilde{v}_k}, \ \tilde{v}_k \leq \tilde{v}_k(0) = 0 \text{ in } B(0, R/2\delta_k)$$

$$\int_{B(0,R/2\delta_k)} e^{\tilde{v}_k} \leq C$$

Tail control in the scaled variable

Theorem c (Shafrir 91)

$$-\Delta v = V(x)e^v, \ 0 < a \leq V(x) \leq b \text{ in } \Omega$$

$K \subset \Omega$ compact $\longrightarrow$

$$\sup_K v + c_1 \inf_\Omega v \leq \exists c_2, \ \exists c_1 \geq 1$$

$\longrightarrow$ **residual vanishing**

Liouville property

Theorem b (Chen-Li 1991) **Liouville property**

$$-\Delta v = e^v \text{ in } \mathbf{R}^2, \ \int_{\mathbf{R}^2} e^v < +\infty$$

$\longrightarrow$

$$v(x) = \log\left\{\frac{8\mu^2}{(1+\mu^2|x-x_0|^2)^2}\right\}, \ x_0 \in \mathbf{R}^2, \ \mu > 0$$

$$\int_{\mathbf{R}^2} e^v = 8\pi$$

Simplicity

Theorem d (Y.Y. Li 99)

$$\max_{\partial B} v_k - \min_{\partial B} v_k \leq C, \ \|\nabla V_k\|_\infty \leq C$$

$\longrightarrow$ $\alpha = 8\pi$

$$\left| v_k(x) - \log \frac{e^{v_k(0)}}{\left(1 + \frac{V_k(0)}{8}e^{v_k(0)}|x|^2\right)^2} \right| \leq C, \ x \in B$$

1. non-radial bifurcation on annulus (S.S. Lin 89 Nagasaki-S. 90b)

2. effective bound of blowup points for simply-connected domain (S.-Nagasaki 89 Grossi-F.Takahashi 10)

3. classification of singular limits (Nagasaki-S. 90a)

4. spherical mean value theorem (S. 90)

5. localization (Brezis-Merle 91)

6. entire solution (W. Chen-C. Li 91)

7. sup + inf inequality (Shafrir 92)

8. uniqueness (S. 92)

9. field-particle duality (S. 92 Wolansky 92)

10. singular perturbation (Weston 78 Moseley 83 S. 93 Baraket-Pacard 98 Esposito-Grossi-Pistoia 05 del Pino-Kowarzyk-Musso 05)

11. blowup analysis (Li-Shafrir 94)

12. Chern-Simons theory (Tarantello 96)

13. global bifurcation (S.-Nagasaki 89 Mizoguchi-S. 97 Chang-Chen-Lin 03)

14. min-max solution (Ding-Jost-Li Wang 99)

15. local uniform esitmate (Y.Y. Li 99)

16. variable coefficient (Ma-Wei 01)

17. refined asymptotics (Chen-Lin 02)

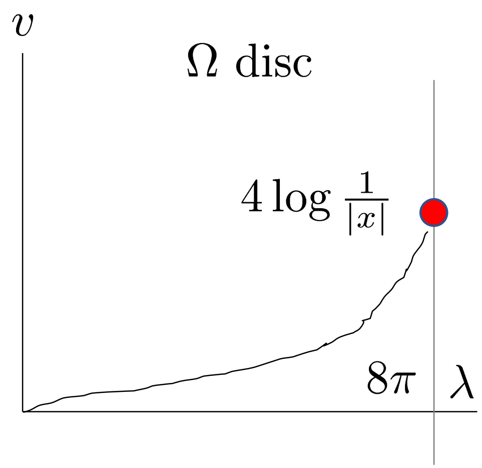
18. topological degree (Li 99 C.C. Chen-C.S. Lin 03 Malchiodi 08)

19. asymptotic non-degeneracy (Gladiali-Grossi 04 Grossi-Ohtsuka-S. 11)

20. isoperimetric profile (Lin-Lucia 06)

21. deformation lemma (Lucia 07)

22. Morse index (Gladiali-Grossi 09)



$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega \subset \mathbf{R}^2$$

$$v|_{\partial\Omega} = 0$$

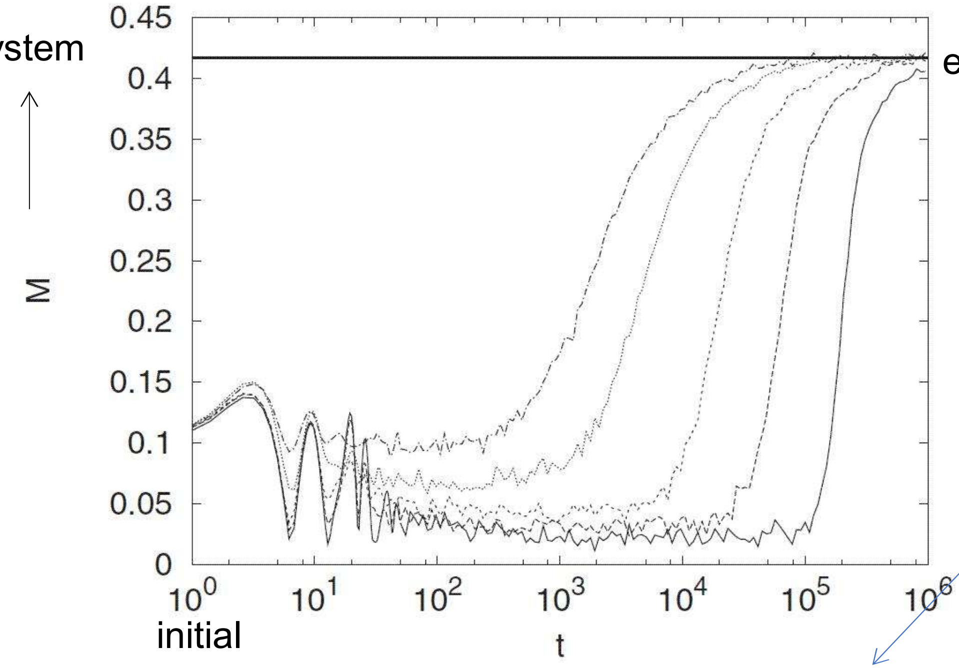
2．動的な階層の循環

From quasi-equilibrium to equilibrium

Hamilton system of many particles with inner interaction of long range

Staniscia-Chavanis-Ninno-Fanelli 09

state of the system



equilibrium

initial

time

quasi-equilibrium

relaxation

relaxation

kinetic

static

Poisson

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

Chavanis 08 relaxation to the equilibrium in the point vortices, kinetic equation + maximum entropy production

Sire-Chavanis 02 motion of the mean field of many self-gravitating Brownian particles, BBGKY hierarchy + factorization

$$u_t = \Delta u - \nabla \cdot u \nabla v, \quad \left. \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0$$

Smoluchowski → Boltzmann

$$u = \frac{\lambda e^v}{\int_\Omega e^v dx}, \quad \lambda = \|u\|_1$$

Results

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0$$

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

Theorem B $T < +\infty$ ⟶

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$m(x_0) \in 8\pi\mathbf{N}$ collapse mass quantization possibly with sub-collapse collision

blowup set; exclusion of boundary blowup

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0,\ t_k \uparrow T,\ u(x_k,t_k) \to +\infty\} \subset \Omega$$

$\sharp\mathcal{S} < +\infty$ finiteness of blowup points

$0 < f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$ measure theoretic regular part

Theorem C

$$T = +\infty, \ \limsup_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = +\infty$$

⟶ $\lambda \equiv \|u_0\|_1 = 8\pi\ell, \ \exists \ell \in \mathbf{N}$ initial mass quantization

$\exists x_* \in \Omega^\ell \setminus D, \ \nabla H_\ell(x_*) = 0$ recursive hierarchy

point vortex Hamiltonian; Robin function; Green function

$$H_\ell(x_1,\cdots,x_\ell) = \frac{1}{2}\sum_j R(x_j) + \sum_{i<j} G(x_i,x_j)$$

Corollary 1 $T < +\infty$ if $\nexists$ stationary solution or $\mathcal{F}(u_0) \ll -1$

and $\lambda \notin 8\pi\mathbf{N}$ or

$\lambda \in 8\pi\ell, \ \ell \in \mathbf{N}, \ \nexists$ critical point of H_ℓ

Corollary 2 Ω convex $\lambda \neq 8\pi$

$\Rightarrow T < +\infty$ or $T = +\infty$ pre-compact orbit

c.f. Grossi-F. Takahashi (2010) $\exists$ stationary solution

Preliminary ~ Green's function and the weak form

$$\varphi \in C^2(\overline{\Omega}), \ \frac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} = 0 \quad \Longrightarrow \quad \frac{d}{dt}\int_\Omega \varphi u(\cdot,t) = \int_\Omega \Delta\varphi \cdot u(\cdot,t) + \frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\varphi(x,x')u\otimes u$$

$$\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x') \in L^\infty(\Omega\times\Omega)$$

method of symmetrization

action reaction law

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$G(x,x') = \Gamma(x-x') + K(x,x'), \ K = K(x,x') \in C^{1+\theta,\theta}(\Omega\times\overline{\Omega}) \cap C^{\theta,1+\theta}(\overline{\Omega}\times\Omega)$$

$$x_0 \in \partial\Omega \longrightarrow G(x,x') = E(X,X') + K(x,x'), \ K = K(x,x') \in (C^{1+\theta,\theta}\cap C^{\theta,1+\theta})(\overline{\Omega\cap B(x_0,R)}\times\overline{\Omega\cap B(x_0,R)})$$

conformal diffeomorphism

$$X : \overline{\Omega\cap B(x_0,2R)} \to \overline{\mathbf{R}_+}^2 = \{(X_1,X_2) \mid X_2 \geq 0\} \qquad X(\partial\Omega\cap B(x_0,2R)) \subset \partial\mathbf{R}^2_+$$

$$E(X,X') = \Gamma(X-X') - \Gamma(X-X'_*) \qquad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mapsto X_* = \begin{pmatrix} X_1 \\ -X_2 \end{pmatrix}$$

Proof of Theorem B

ε regularity

$$\frac{1}{p+1}\frac{d}{dt}\|u\|_{p+1}^{p+1} = -\frac{4p}{(p+1)^2}\|\nabla u^{\frac{p+1}{2}}\|_2^2 + \frac{p}{p+1}\|u\|_{p+3}^{p+3}$$

$$\|z\|_{p+1}^{p+1} \leq C_p\|z\|_1\|z\|_{H^1}^p$$

$$n=2$$

Gagliardo-Nirenberg inequality

$$\exists\varepsilon_0>0,\ \|u_0\|_1<\varepsilon_0 \ \Rightarrow\ T=+\infty,\ \|u(\cdot,t)\|_\infty \leq C$$

elliptic regularity

semi-group estimate

Moser's iteration scheme
maximal regularity

localization

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists\varepsilon_0 \Rightarrow x_0 \notin \mathcal{S}$$

nice cut-off function

$$x_0\in\overline{\Omega},\ 0<R\ll 1,\ \boxed{\varphi=\varphi_{x_0,R}\in\mathcal{Y}}$$

$$0\leq\varphi\leq 1,\ \varphi=\begin{cases}1, & x\in B(x_0,\frac{R}{2})\\ 0, & x\in\mathbf{R}^2\setminus B(x_0,R)\end{cases}$$

$$|\nabla\varphi|\leq CR^{-1}\varphi^{\frac{5}{6}}$$

$$|\nabla^2\varphi|\leq CR^{-2}\varphi^{\frac{2}{3}}$$

$$\mathcal{Y}=\{\varphi\in C^2(\overline{\Omega})\mid \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega}=0\}$$

Formation of collapses

symmetry of the Green function → weak form (symmetrization)

$$\varphi \in C^2(\overline{\Omega}), \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\frac{d}{dt}\int_\Omega \varphi u(\cdot,t) = \int_\Omega \Delta\varphi \cdot u(\cdot,t) + \frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\varphi(x,x')u\otimes u$$

$$\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x')$$

$$\|\rho_\varphi\|_\infty \leq C\|\nabla\varphi\|_{C^1}$$

boundary behavior of the Green function
singularity cancellation by the symmetry

monotonicity formula $\lambda = \|u(\cdot,t)\|_1$

$$\left|\frac{d}{dt}\int_\Omega u\varphi\right| \leq C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1}$$

→ **weak continuation**

$$0 \leq \exists \mu(dx,t) \in C_*([0,T],\mathcal{M}(\overline{\Omega}))$$

$$u(x,t)dx = \mu(dx,t), \ 0 \leq t < T$$

ε -regularity

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists\varepsilon_0 \Rightarrow x_0 \notin \mathcal{S}$$

→

$$x_0 \in \mathcal{S} \Rightarrow \lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0$$

↔ monotonicity formula

$$\lim_{R\downarrow 0}\liminf_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0$$

→ $\sharp\mathcal{S} < +\infty$

formation of collapse

$$\mu(dx,T) = \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0} + f(x)dx, \ m(x_0) \geq \varepsilon_0, \ 0 \leq f = f(x) \in L^1(\Omega)$$

singular part regular part

Generation of the weak solution

$0 \leq \mu = \mu(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$ weak solution

$\overset{\text{def}}{\Longleftrightarrow}$ $0 \leq \exists \mathcal{N} = \mathcal{N}(\cdot,t) \in L^\infty_*([0,T], \mathcal{X}')$ multiplicated operator

1. $t \in [0,T] \mapsto \langle \varphi, \mu(dx,t) \rangle, \ \varphi \in \mathcal{Y}$ a.c.
2. $\frac{d}{dt}\langle \varphi, \mu \rangle = \langle \Delta\varphi, \mu \rangle + \frac{1}{2}\langle \rho_\varphi, \mathcal{N}(\cdot,t) \rangle$ a.e. $t \in [0,T]$
3. $\mathcal{N}|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu \otimes \mu$

separable

$$\mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0\} \quad \mathcal{X} = [\mathcal{X}_0]^{L^\infty(\Omega\times\Omega)}$$

$$\mathcal{X}_0 = \{\rho_\varphi + \psi \mid \varphi \in \mathcal{Y}, \ \psi \in C(\overline{\Omega}\times\overline{\Omega})\}$$

$\Rightarrow$

$$\mu(\overline{\Omega},t) = \mu(\overline{\Omega},0) \equiv \lambda, \ 0 \leq t \leq T$$

$$\left| \frac{d}{dt}\langle \varphi, \mu(dx,t) \rangle \right| \leq C(\lambda + \lambda^2)\|\nabla\varphi\|_{C^1}$$

$u = u(x,t)$ classical solution

$\Rightarrow$ $\mathcal{N}(\cdot,t) = u(x,t) \otimes u(x',t) \ dx dx'$

$$\|\mathcal{N}(\cdot,t)\|_{\mathcal{X}'} = \lambda^2, \ \lambda = \|u_0\|_1$$

Theorem 1 $\mu_k(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$

$\mathcal{N}_k \in L^\infty_*([0,T], \mathcal{X}')$ weak solutions

$0 \leq \mu_k(\overline{\Omega},t) \leq C$

$\|\mathcal{N}_k(\cdot,t)\|_{\mathcal{X}'} \leq C$ $\Rightarrow$ sub-sequence

$\mu_k(dx,t) \rightharpoonup \mu(dx,t)$ in $C_*([0,T], \mathcal{M}(\overline{\Omega}))$

$\mathcal{N}_k(\cdot,t) \rightharpoonup \mathcal{N}(\cdot,t)$ in $L^\infty_*([0,T], \mathcal{X}')$ weal solution

Backward self-similar transformation $x_0 \in \mathcal{S}$

$$y = (x - x_0)/(T-t)^{1/2}, \ s = -\log(T-t)$$
$$z(y,s) = (T-t)u(x,t)$$

weak limit $s_k \uparrow +\infty$ subsequence

$$z(y, s+s_k)dy \rightharpoonup \exists \zeta(dy, s) \text{ in } C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2))$$

Put $z(y,s) = 0$ where it is not defined.

$$\mathcal{M}(\mathbf{R}^2) = C_\infty(\mathbf{R}^2)'$$
$$C_\infty(\mathbf{R}^2) = \{z \in C(\mathbf{R}^2 \cup \{\infty\}), \ z(\infty) = 0\}$$

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

First parabolic envelope

$$\left| \frac{d}{dt} \int_\Omega u(\cdot, t)\varphi_{x_0,R} \right| \leq C_\lambda R^{-2}, \ 0 < R \ll 1$$

$$|\langle \varphi_{x_0,R}, u(\cdot,t)dx\rangle - \langle \varphi_{x_0,R}, \mu(dx,T)\rangle| \leq C_\lambda (T-t)/R^2$$

$$s_k + s = -\log(T-t), \ R = b(T-t)^{1/2}$$

$\longrightarrow$

$$|\langle \varphi_{0,b}, z(\cdot, s+s_k)dy\rangle - \langle \varphi_{x_0, be^{-(s+s_k)/2}}, \mu(dx,T)\rangle|$$
$$\leq C_\lambda / b^2$$

$$\mu(dx,T) = \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$k \to \infty, \ b \uparrow +\infty$ $\longrightarrow$ $m(x_0) = \zeta(\mathbf{R}^2, s)$

Second parabolic envelope $\langle |y|^2, \zeta(dy,s)\rangle \leq C$

Limit equation $x_0 \in \mathcal{S}$

$$y = (x - x_0)/(T-t)^{1/2}, \ s = -\log(T-t)$$
$$z(y,s) = (T-t)u(x,t)$$

$$y \in (T-t)^{-1/2}(\Omega - \{x_0\}) = \Omega_s$$
$$-\log T \leq s < +\infty, \ \|z(\cdot,s)\|_1 = \lambda$$

$$z_s = \nabla \cdot (\nabla z - z\nabla(w + |y|^2/4))$$
$$\frac{\partial z}{\partial \nu} - z\frac{\partial}{\partial \nu}(w + |y|^2/4)\Big|_{\partial\Omega_s} = 0$$

$$w(\cdot,s) = \int_{\Omega_s} G_s(\cdot,y')z(y',s)dy'$$
$$G_s(y,y') = G(x,x')$$

Theorem 2 $x_0 \in \Omega$ ⟹ $s_k \uparrow +\infty$ subsequence

$$z(y, s+s_k)dy \rightharpoonup \exists\zeta(dy,s) \text{ in } C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$
$$\zeta_s = \nabla \cdot (\nabla\zeta - \zeta\nabla(\Gamma * \zeta + |y|^2/4)) \ \text{ in } \mathbf{R}^2 \times (-\infty,+\infty)$$

Proof $\varphi \in C_0^2(\mathbf{R}^2), \ s \gg 1$

$$\frac{d}{ds}\int_{\mathcal{O}_s} z\varphi = \int_{\mathcal{O}_s}(\partial_s\varphi + y\cdot\nabla\varphi + \Delta\varphi)z$$
$$+\frac{1}{2}\int_{\mathcal{O}_s\times\mathcal{O}_s}\rho_\varphi^s(y,y')z\otimes z$$

$$\mathcal{O}_s = \Omega_s \times \{s\}$$
$$\rho_\varphi^s(y,y') = \nabla\varphi(y)\cdot\nabla_y G_s(y,y')$$
$$+\nabla\varphi(y')\cdot\nabla_{y'}G_s(y,y')$$

$$G(x,x') = \Gamma(x-x') + K(x,x')$$
$$(x,x') \in (\overline{\Omega}\times\Omega)\cup(\Omega\times\overline{\Omega})$$
$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$G_s(y,y') = \Gamma(y-y') - \frac{s}{4\pi}$$
$$+K(e^{-s}y + x_0, e^{-s}y' + x_0)$$

□

Exclusion of boundary blowup

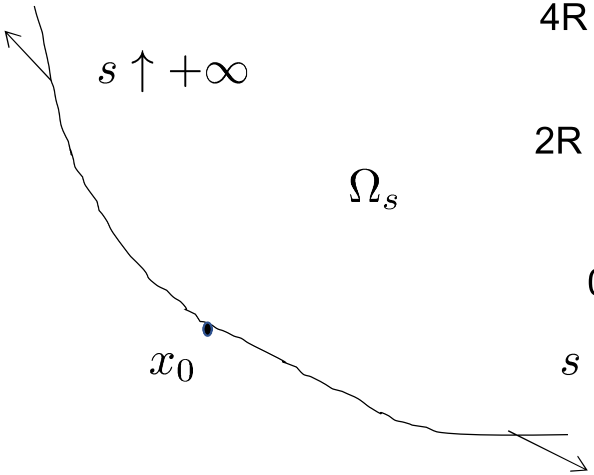
$$x_0 \in \partial\Omega$$

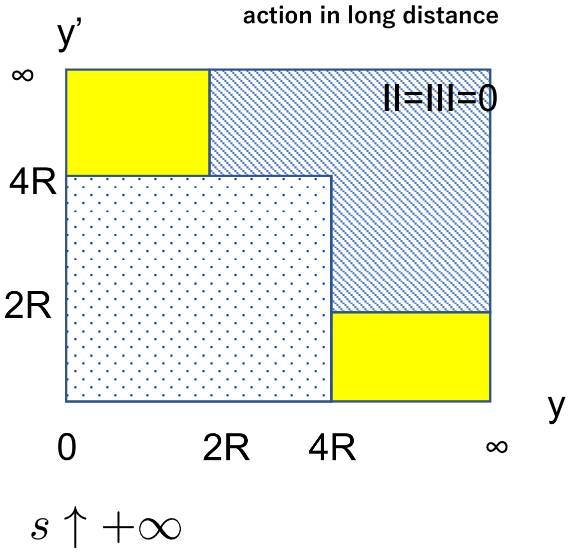
$$0 \le \zeta(dy,s),\ \zeta(\mathbf{R}^2,s) \le \lambda \equiv \|u_0\|_1,\ \mathrm{supp}\ \zeta(dy,s) \subset \overline{\mathbf{R}^2_+}$$

$$\zeta_s = \nabla\cdot(\nabla\zeta - \zeta\nabla(E*\zeta + |y|^2/4)) \text{ in } \mathbf{R}^2_+ \times (-\infty,+\infty)$$

$$\frac{\partial z}{\partial\nu} - z\frac{\partial}{\partial\nu}(E*\zeta + \frac{|y|^2}{4})\Big|_{\partial\mathbf{R}^2_+} = 0$$

$$E(y,y') = \Gamma(y-y') - \Gamma(y-y'_*)$$





$$\varphi = |y|^2\psi_R,\ \psi_R(y) = \psi(y/R)$$

$$\psi = \varphi_{0,2}(|y|)$$

$$\Delta\varphi = 4\psi_R + 4\frac{y}{R}\cdot\nabla\psi(\frac{y}{R}) + \frac{|y|^2}{R^2}\Delta\psi(\frac{y}{R})$$

$$y\cdot\nabla\varphi = 2|y|^2\psi_R + |y|^2\frac{y}{R}\cdot\nabla\psi(\frac{y}{R})$$

$$\langle 1+|y|^2, \zeta(dy,s)\rangle \le C$$

$\Rightarrow$ (dominated convergence theorem)

$$\lim_{R\uparrow+\infty}\int_{s_1}^{s_2}\langle\Delta\varphi + \frac{y}{2}\cdot\nabla\varphi, \zeta(dy,s)\rangle$$

$$= 4(s_2-s_1)m(x_0) + \int_{s_1}^{s_2} I(s)ds$$

$$I(s) = \langle |y|^2, \zeta(dy,s)\rangle$$

$$\frac{dI}{ds} = 4m(x_0) + I(s) \text{ a.e. } s \Rightarrow \lim_{R\uparrow+\infty} I(s) = +\infty$$

contradiction

□

Proof of Theorem B (continued) $x_0 \in \mathcal{S}$

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

backward self-similar transformation

$$y = (x-x_0)/(T-t)^{1/2}, \ s = -\log(T-t)$$
$$z(y,s) = (T-t)u(x,t)$$

parabolic envelope

$$m(x_0) = \zeta(\mathbf{R}^2, s) \qquad \langle |y|^2, \zeta(dy,s)\rangle \leq C$$

weak limit $s_k \uparrow +\infty$ subsequence

$$z(y, s+s_k)dy \rightharpoonup \exists \zeta(dy,s) \text{ in } C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$

scaling back

$$\zeta(dy,s) = e^{-s}A(dy', s'), \ y' = e^{-s/2}y, \ s' = -e^{-s}$$

limit equation exclusion of boundary blowup $x_0 \in \Omega$

$$\zeta_s = \nabla \cdot (\nabla\zeta - \zeta\nabla(\Gamma * \zeta + |y|^2/4)) \ \text{ in } \mathbf{R}^2 \times (-\infty,+\infty)$$

$$A_s = \nabla \cdot (\nabla A - A\nabla\Gamma * A) \text{ in } \mathbf{R}^2 \times (-\infty, 0)$$
$$A = A(dy,s) \geq 0, \ A(\mathbf{R}^2, s) = m(x_0)$$

ϵ − **regularity** ➡ singular part, sub-collapse

$$\zeta^s(dy,s) = \sum_{j=1}^{m(s)} \tilde{m}_j(s)\delta_{y_j(s)}(dy)$$

$$m(s) \leq m(x_0)/\varepsilon_0, \ |y_j(s)| \leq C, \ \tilde{m}_j(s) \geq \varepsilon_0$$

second parabolic envelope

singular part

$$A^s(dy', s') = \sum_{j=1}^{m(s')} \tilde{m}_j(s')\delta_{y_j'(s)}(dy')$$

$$A_s = \nabla \cdot (\nabla A - A\nabla\Gamma * A) \text{ in } \mathbf{R}^2 \times (-\infty, 0)$$
$$A = A(dy, s) \geq 0, \ A(\mathbf{R}^2, s) = m(x_0)$$

singular part

$$A^s(dy', s') = \sum_{j=1}^{m(s')} \tilde{m}_j(s')\delta_{y'_j(s)}(dy')$$

scaling limit $s'_0 < 0, \ 1 \leq j \leq m(s'_0)$ fix

$$\tilde{A}_\beta(dy', s') = \beta^2 A(dy, s)$$
$$y = \beta y' + y'_j(s'_0), \ s = \beta^2 s' + s'_0$$

$\beta_k \downarrow 0$ subsequence

$$\tilde{A}_{\beta_k}(dy', s') \rightharpoonup \tilde{A}(dy', s') \in C_*(-\infty, s'_0; \mathcal{M}(\mathbf{R}^2))$$

$$\tilde{A}(dy', s') = m'_j(s'_0)\delta_0(dy')$$ weak solution

translation limit

$s'_k \uparrow +\infty, \ \hat{A}_k(dy', s') = \tilde{A}(dy', s' + s'_k)$ subsequence

$$\hat{A}_k(dy', s') \rightharpoonup \hat{A}(dy', s') \in C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2))$$

$$\hat{A}(dy', s') = m'_j(s'_0)\delta_0(dy')$$

weak Liouville property weak solution (measure valued)

$$a_s = \nabla \cdot (\nabla a - a\nabla\Gamma * a) \text{ in } \mathbf{R}^2 \times (-\infty, +\infty)$$
$$\Rightarrow \ a(\mathbf{R}^2, s) = 0 \text{ or } 8\pi$$ Kurokiba-Ogawa03

$\longrightarrow$ $\tilde{m}_j(s'_0) = \tilde{A}(\mathbf{R}^2, 0) = 8\pi$

residual vanishing

if $\boxed{A^{ac}(dy', s') = 0}$ $\longrightarrow$ $A(dy', s') = 8\pi \sum_{j=1}^{\ell} \delta_{y'_j(s')}(dy')$ sub-collapse

absolutely continuous part

$\longrightarrow$ $m(x_0) = 8\pi\ell$ collapse mass quantization

Improved regularity

$\exists(\varepsilon_0, R) > 0$

Lemma $u_t = \Delta u - \nabla \cdot (u \nabla \Gamma * u)$ in $\mathbf{R}^2 \times (0,T)$

$u|_{t=0} = u_0(x) \geq 0$

$\exists \varepsilon_0,\ R,\ t_0 > 0,\ \forall x_0 \qquad \|u_0\|_{L^1(B(x_0,8R))} < \varepsilon_0/2$

→ $\forall \tau \in (0, t_0)$

$$\sup_{\tau \leq t < t_0} \|u(\cdot,t)\|_{L^\infty(B(x_0,R))} < +\infty$$

Proof $\exists t_1 \in (0,T),\ \|u_0\|_{L^1(B(x_0,8R))} < \varepsilon_0/2$

→ $$\sup_{t \in (0,t_1)} \|u(\cdot,t)\|_{L^1(B(x_0,4R))} < \varepsilon_0$$

→ $$\frac{d}{dt}\int_{\mathbf{R}^2} u(\log u - 1)\varphi\ dx + \frac{1}{8}\int_{\mathbf{R}^2} u^{-1}|\nabla u|^2 \varphi\ dx \leq C_\varphi,\ 0 \leq t < t_1,\ \varphi = \varphi_{x_0,R}$$

$\|u(\cdot,t)\|_{L \log L(B(x_0,2R))}$ Involved by the initial value!

Gagliardo-Nirenberg

→ $$\frac{dJ}{dt} + 3J^{3/2} \leq C_R,\ J = \int_\Omega u(\log u - 1) + 1\ dx$$

$$\frac{d}{dt} t^{-2} + 3(t^{-2})^{3/2} = t^{-3}$$

$J(t) \leq t^{-2},\ 0 < t \leq \min\{t_1, t_0\},\ t_0^{-3} = C_R$ □

parabolic regularity

scaling

Theorem 3 $\exists \varepsilon_0,\ \sigma_0,\ C$

$u_t = \Delta u - \nabla \cdot (u \nabla \Gamma * u)$ in $\mathbf{R}^2 \times (-T,T)$

$u_0 = u|_{t=0}$ weak solution generated by classical solutions

$\|u_0\|_{L^1(B(x_0,2R))} < \varepsilon_0,\ u_0 = u|_{t=0} \Rightarrow$

$$\sup_{t \in [-\sigma_0 R^2, \sigma_0 R^2] \cap (-T,T)} \|u(\cdot,t)\|_{L^\infty(B(x_0,R))} \leq CR^{-2}$$

scaling invariant regularity (inverse scaling back)

$\zeta(B(y_0,2r),s) < \varepsilon_0 \Rightarrow \|\zeta(\cdot,s)\|_{L^\infty(B(y_0,r))} \leq Cr^{-2}$

residual vanishing $A^{ac}(dy', s') = 0 \Leftrightarrow \zeta^{ac}(dy, s) = 0$

$$A_{s'} = \nabla' \cdot (\nabla' A - A \nabla' \Gamma * A)$$

1st envelope

$$m(x_0) = \zeta(\mathbf{R}^2, s)$$

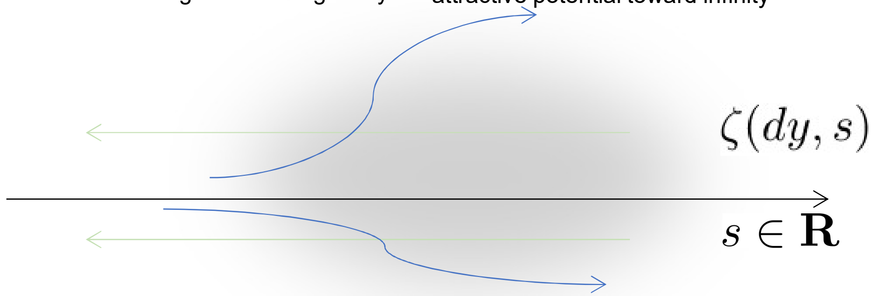
2nd envelope

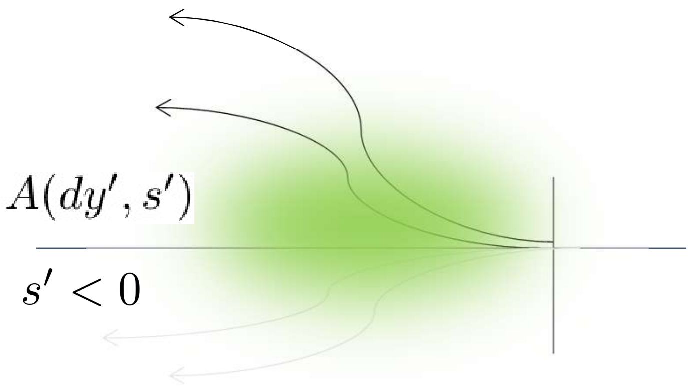
$$\langle |y|^2, \zeta(dy, s) \rangle \leq C$$

$$\zeta_s = \nabla \cdot (\nabla \zeta - \zeta \nabla (\Gamma * \zeta + |y|^2/4))$$

scaling invariant regularity

attractive potential toward infinity





Collision of sub-collapses

outer second moment

$$\frac{d}{ds}\langle \varphi, \zeta \rangle \geq \langle \Delta\varphi - C\varphi_r + \frac{1}{2} r \varphi_r, \zeta \rangle, \ \varphi = \varphi(r)$$

$$\varphi(r) = \xi(r/R), \ \xi(r) = r^2 - 1$$

$$R \gg 1 \ \Rightarrow \ \Delta\varphi + \frac{1}{2} r \varphi_r \geq C\varphi_r, \ r \geq R$$

$$\frac{d}{ds}\langle (\frac{|y|^2}{R^2} - 1)_+, \zeta(dy, s) \rangle \geq 0 \ \Longrightarrow \ \zeta(dy, s) = \zeta^s(dy, s)$$

uniform estimate of self-interaction part ⟵

improved regularity

$$\zeta(B(y_0, 2r), s) < \varepsilon_0$$

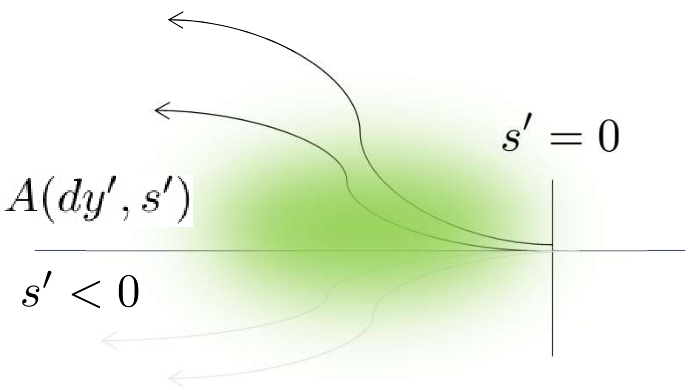
$$\Longrightarrow \ \|\zeta(\cdot, s)\|_{L^\infty(B(y_0, r))} \leq C r^{-2}$$

Sub-collapse dynamics

$$\zeta^s(dy,s)=\sum_{j=1}^{\ell}8\pi\delta_{y_j(s)}(dy)$$ → **scaling back**

$$A(dy',s')=\sum_{j=1}^{\ell}8\pi\delta_{y'_j(s')}(dy')$$

$$A_{s'}=\nabla'\cdot(\nabla' A-A\nabla'\Gamma*A)\quad \text{in } \mathbf{R}^2\times(-\infty,0)$$



Collision of sub-collapses

Tracing by the local second moment

simple blowup point

$$\ell=1 \ \Rightarrow\ \zeta(dy,s)=8\pi\delta_0(dy)$$

in dynamics

recursive hierarchy $\ell\geq 2$

$$\frac{dy'_j}{ds'}=8\pi\nabla_j H^0_\ell(y'_1,\cdots,y'_\ell)$$

$$H^0_\ell(y'_1,\cdots,y'_\ell)=\sum_{1\leq j<k\leq\ell}\Gamma(y'_j-y'_k)$$

$$\Gamma(y')=\frac{1}{2\pi}\log\frac{1}{|y'|}$$

Senba-S. 01	weak formulation monotonicity formula	formation of collapse weak solution generation
Senba-S. 02a	weak solution	instant blowup for over mass concentrated initial data
Kurokiba-Ogawa 03	scaling invariance	non-existence of over mass entire solution without concentration
Senba-S. 04	backward self-similar transformation scaling limit	
S. 05	parabolic envelope (1) scaling invariance of the scaling limit a local second moment	sub-collapse quantization collapse mass quantization
Senba-Ohtsuka-S. 07	defect moment (1)	radially symmetric dynamics
Senba 07, Naito-S. 08	parabolic envelope (2)	type II blowup rate
S. 08	scaling back	limit equation simplification
Senba-S. 11 (2nd ed.)	translation limit	concentration-cancelation simplification
S. 13a	limit equation classification	boundary blowup exclusion
S. 13b	improved regularity concentration compactness	cloud formation collision of sub-collapses
S. 14	tightness	residual vanishing
S. 18	defect moment (2)	quantization of BUIT
S. 22	outer second moment	residual vanishing in finite time