積分可能な多成分系

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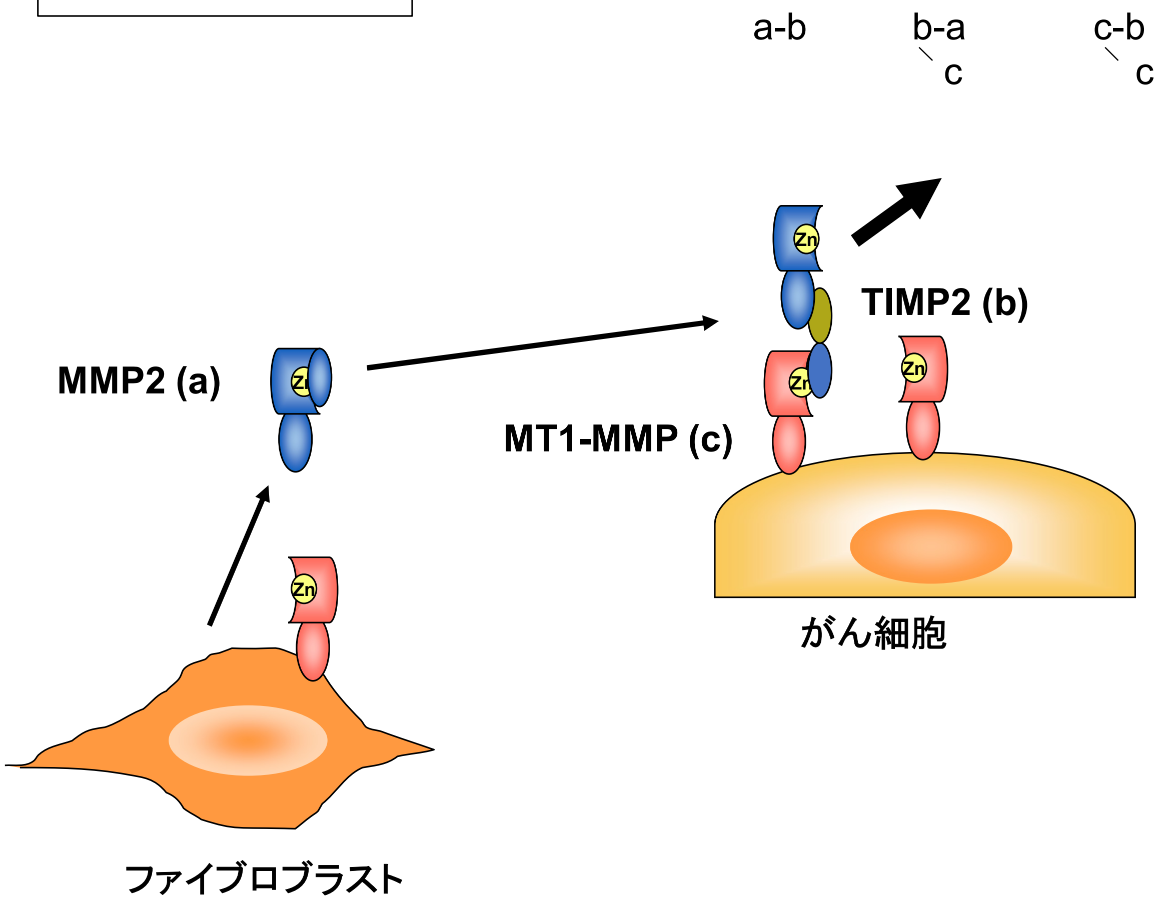
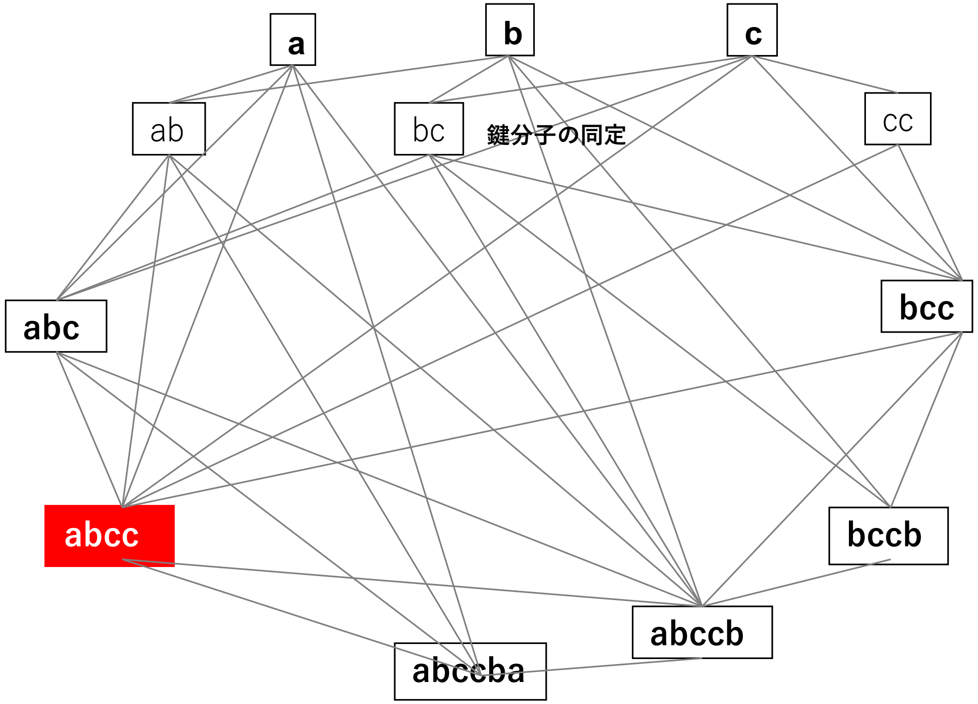
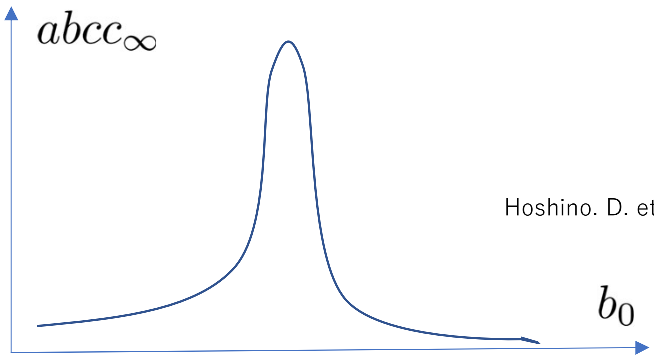
9． 積分可能な生命科学モデル

基底膜分解の機序







Hoshino. D. et. al. PLoS CB 2012

タンパク質構造 → 結合則 → 反応ネットワーク → 質量作用の法則 → 実測値 → シミュレーション

素過程の記述

1. 分子の衝突によって定率で化学反応が発生する
2. 分子の衝突確率は濃度の積に比例する

質量作用の法則

$$A + B \to AB\ (k)$$
$$AB \to A + B\ (\ell)$$

素過程（基本モデル）

$$\frac{d}{dt}[A] = -k[A][B] + \ell[AB]$$
$$\frac{d}{dt}[B] = -k[A][B] + \ell[AB]$$
$$\frac{d}{dt}[AB] = k[A][B] - \ell[AB]$$

質量保存

$$\frac{d}{dt}([A] + [AB]) = 0,\ [A] = \alpha - [AB]$$
$$\frac{d}{dt}([B] + [AB]) = 0,\ [B] = \beta - [AB]$$

$$\frac{dx}{dt} = k(\alpha - x)(\beta - x) - \ell x$$
$$x = [AB]$$

積分可能

$$A + AB \to ABA\ (k)$$
$$ABA \to A + AB\ (2\ell)$$

変形1

$$\frac{d}{dt}[A] = -k[A][AB] + 2\ell[ABA]$$
$$\frac{d}{dt}[AB] = -k[A][AB] + 2\ell[ABA]$$
$$\frac{d}{dt}[ABA] = k[A][BB] - 2\ell[ABA]$$

$$A + BB \to ABB\ (2k)$$
$$ABB \to A + BB\ (\ell)$$

変形2

$$\frac{d}{dt}[A] = -2k[A][BB] + \ell[ABB]$$
$$\frac{d}{dt}[BB] = -2k[A][BB] + \ell[ABB]$$
$$\frac{d}{dt}[ABB] = 2k[A][BB] - \ell[ABB]$$

$$A + AB \to AAB\ (k)$$
$$AAB \to A + AB\ (\ell)$$

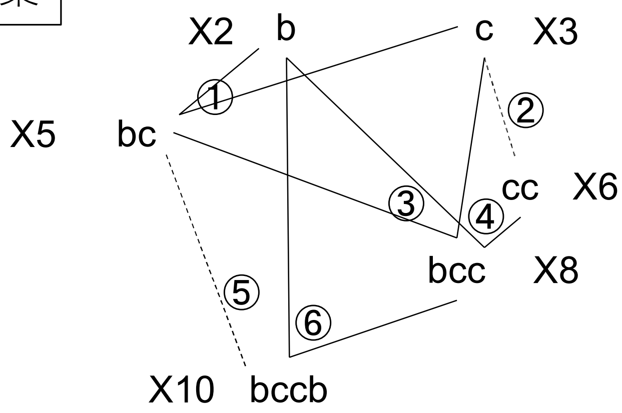
$$\frac{1}{2}N_A(N_A - 1) \approx \frac{N_A^2}{2}$$

変形3

$$A + A \to AA\ (k/2)$$
$$AA \to A + A\ (\ell)$$

$$\frac{d}{dt}[A] = 2\left(-\frac{k}{2}[A]^2 + \ell[AA]\right)$$
$$\frac{d}{dt}[AA] = \frac{k}{2}[A]^2 - \ell[AA]$$

反応ネットワークの構築



$$b + c \to bc \quad (k_2), \qquad bc \to b + c \quad (\ell_2)$$

$$c + c \to cc \quad (k_3), \qquad cc \to c + c \quad (\ell_3)$$

b - c

簡略化モデル

$$\frac{dX_2}{dt} = \text{①}\,-k_2X_2X_3 + \ell_2X_5 \;\text{④}\,- 2k_2X_2X_6 + \ell_2X_8 - k_2X_2X_8 + 2\ell_2X_{10}\,\text{⑥}$$

$$\frac{dX_3}{dt} = \text{①}\,-k_2X_2X_3 + \ell_2X_5 \;\text{②}\,- k_3X_3^2 + 2\ell_3X_6 \;\text{③}\,- k_3X_3X_5 + \ell_3X_8$$

$$\frac{dX_5}{dt} = \text{①}\,k_2X_2X_3 - \ell_2X_5 - k_3X_5X_3 + \ell_3X_8\,\text{③} - k_3X_5^2 + 2\ell_3X_{10}\,\text{⑤}$$

$$\frac{dX_6}{dt} = \frac{k_3}{2}X_3^2\,\text{②} - \ell_3X_6 - 2k_2X_6X_2 + \ell_2X_8\,\text{④}$$

$$\frac{dX_8}{dt} = 2k_2X_2X_6 - \ell_2X_8\,\text{④} + k_3X_3X_5\,\text{③} - \ell_3X_8 - k_2X_2X_8 + 2\ell_2X_{10}\,\text{⑥}$$

$$\frac{dX_{10}}{dt} = k_2X_2X_8\,\text{⑥} - 2\ell_2X_{10} + \frac{k_3}{2}X_5^2 - \ell_3X_{10}\,\text{⑤}$$

2X6.. ...b, c 結合
2X10…b, c 解離

単体粒子の質量保存
反応のグルーピング

1．有限個原子の結合解離
2．結合子は不変
3．結合解離定数は一定

反応のグルーピング

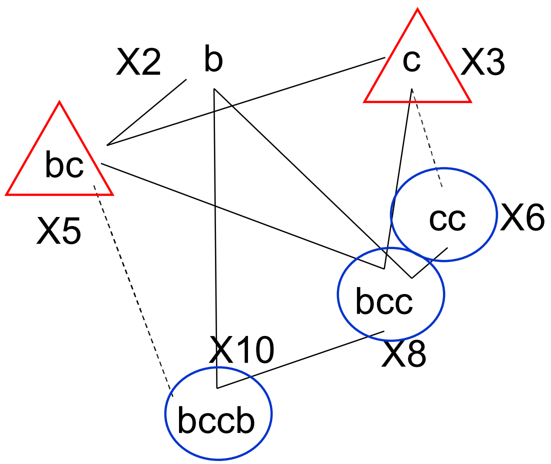
1. c polymerization at most once

c group ⇒ non-c group (uniform rate)

⇒

cg＋cg → ncg (X6, X8, X10)

cg… c-c active (X3, X5)



Axiom

1 cg attachment = 2 cg lost
1 cg detachment = 2 cg gain

$$2(X_3 + X_5) \to X_6 + X_8 + X_{10} \quad (k_3)$$
$$X_6 + X_8 + X_{10} \to 2(X_3 + X_5) \quad (\ell_3)$$

$$(X_3 + X_5)^2 = X_3^2 + 2X_3X_5 + X_5^2$$

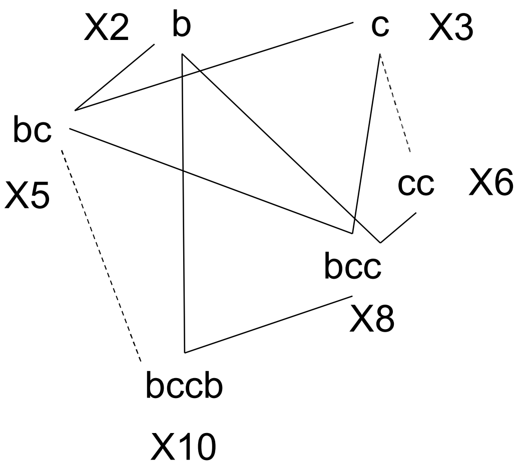
$$X_3 + X_3 \to X_6 \quad (k_3) \qquad X_5 + X_5 \to X_{10} \quad (k_3)$$
$$X_6 \to X_3 + X_3 \quad (\ell_3) \qquad X_{10} \to X_5 + X_5 \quad (\ell_3)$$

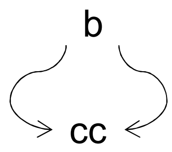
$$X_3 + X_5 \to X_8 \quad (2k_2)$$
meet twice
$$X_8 \to X_3 + X_5 \quad (\ell_2)$$

$$\frac{d}{dt}(X_3 + X_5) = -2k_3(X_3 + X_5)^2 + 2\ell_3(X_6 + X_8 + X_{10})$$
$$\frac{d}{dt}(X_6 + X_8 + X_{10}) = k_3(X_3 + X_5)^2 - \ell_3(X_6 + X_8 + X_{10})$$

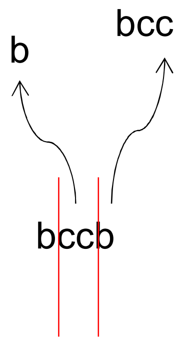
2. b, c attach-detach

$$X_2 + (X_3 + 2X_6 + X_8) \to X_5 + X_8 + 2X_{10} \quad (k_2)$$
$$X_5 + X_8 + 2X_{10} \to X_2 + (X_3 + 2X_6 + X_8) \quad (\ell_2)$$





X6…2spots attach

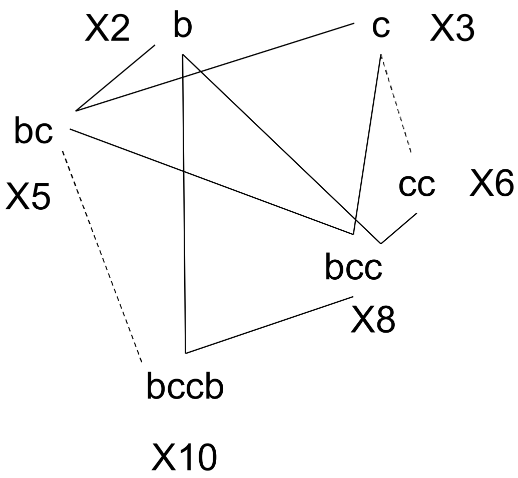


X10…2spots detach

$$\frac{d}{dt}X_2 = -k_2X_2(X_3 + 2X_6 + X_8) + \ell_2(X_5 + X_8 + 2X_{10})$$
$$\frac{d}{dt}(X_3 + 2X_6 + X_8) = -k_2X_2(X_3 + 2X_6 + X_8) + \ell_2(X_5 + X_8 + 2X_{10})$$
$$\frac{d}{dt}(X_5 + X_8 + 2X_{10}) = k_2X_2(X_3 + 2X_6 + X_8) - \ell_2(X_5 + X_8 + 2X_{10})$$

まとめ

mass conservation



b $X_2 + X_5 + X_8 + 2X_{10}$

c $X_3 + X_5 + 2X_6 + 2X_8 + 2X_{10}$

path classification

b-c

$$X_2 + (X_3 + 2X_6 + X_8) \to X_5 + X_8 + 2X_{10} \quad (k_2)$$

$$X_5 + X_8 + 2X_{10} \to X_2 + (X_3 + 2X_6 + X_8) \quad (\ell_2)$$

c-c

$$2(X_3 + X_5) \to X_6 + X_8 + X_{10} \quad (k_3)$$

$$X_6 + X_8 + X_{10} \to 2(X_3 + X_5) \quad (\ell_3)$$

⟹ $X_i,\ i = 2, 3, t, 6, 8, 10$ integrable

パターン形成

$$u_t = \varepsilon^2 \Delta u + f(u,v)$$

$$\tau v_t = D\Delta v + g(u,v) \quad \text{in } \Omega \times (0,T)$$

$$\frac{\partial}{\partial \nu}(u,v)\Big|_{\partial\Omega} = 0, \;\; (u,v)|_{t=0} = (u_0(x), v_0(x)) \geq 0$$

定常状態

$$\varepsilon^2 \Delta u_* + f(u_*, v_*) = 0, \; \frac{\partial u_*}{\partial \nu}\Big|_{\partial\Omega} = 0$$

$$D\Delta v_* + g(u_*, v_*) = 0, \; \frac{\partial v_*}{\partial \nu}\Big|_{\partial\Omega} = 0$$

Turing 52

拡散効果による定数
定常解の不安定化 ➡ パターン形成

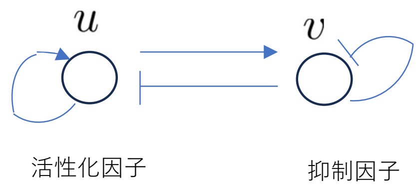
非定数定常解のスパイクパターン J. Wei

形態形成（ヒドラ）

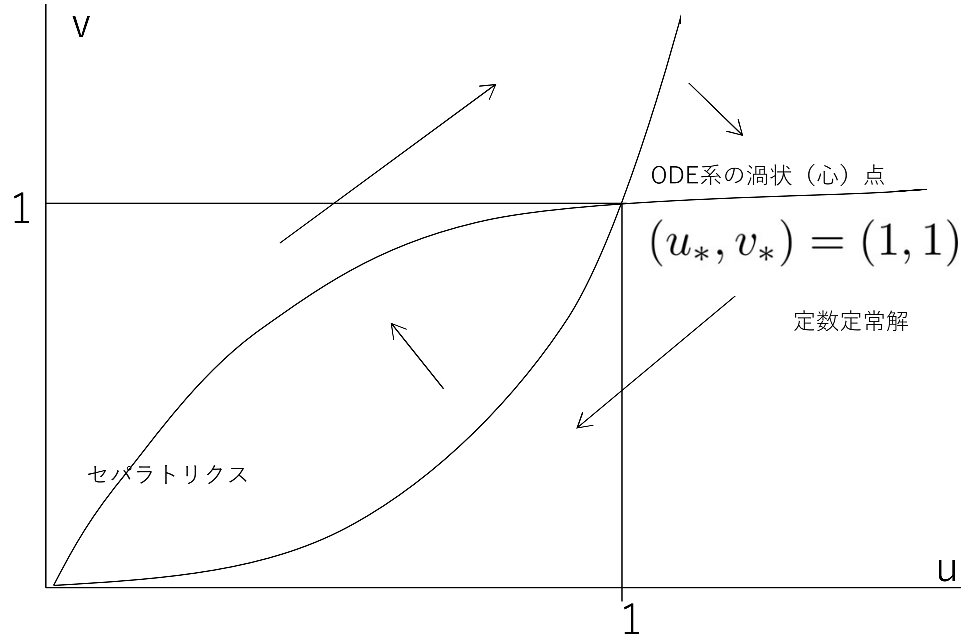
Gierer-Meinhardt 72

$$f(u,v) = -u + \frac{u^p}{v^q}, \;\; g(u,v) = -v + \frac{u^r}{v^s}$$

$$p > 1, \;\; q, r > 0, \;\; s > -1$$



$$0 < \frac{p-1}{r} < \frac{q}{s+1}$$



シャドー・システム

spiky stationary solutions Ni-Takagi 86

$$u_t = \varepsilon^2 \Delta u - u + \frac{u^p}{\xi^q}, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\tau \frac{d\xi}{dt} = -\xi + \frac{1}{|\Omega|}\int_\Omega \frac{u^r}{\xi^s} dx$$

一般系

$$u_t = \varepsilon^2 \Delta u + f(u,v), \ \tau v_t = D\Delta v + g(u,v)$$

$$\left.\frac{\partial}{\partial \nu}(u,v)\right|_{\partial\Omega} = 0 \Longrightarrow \tau \frac{d}{dt}\int_\Omega v = \int_\Omega g(u,v)$$

$$D \to +\infty \Longrightarrow \Delta v \to 0, \ v(\cdot,t) \to \xi(t)$$

$$u_t = \varepsilon^2 \Delta u + f(u,v), \ \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\tau \frac{d\xi}{dt} = \frac{1}{|\Omega|}\int_\Omega g(u,\xi)$$

遷移状態

GM系 $\xrightarrow{D \to +\infty}$ シャドー・システム

GM系:

$$u_t = \varepsilon^2 \Delta u - u + \frac{u^p}{v^q}$$

$$\tau v_t = D\Delta v - v + \frac{u^r}{v^s}$$

$$\left.\frac{\partial}{\partial \nu}(u,v)\right|_{\partial\Omega} = 0$$

$\downarrow$ $t \to +\infty$

定常解（１）

$$\varepsilon^2 \Delta u - u + \frac{u^p}{v^q} = 0$$

$$D\Delta v - v + \frac{u^r}{v^s} = 0$$

$$\left.\frac{\partial}{\partial \nu}(u,v)\right|_{\partial\Omega} = 0$$

シャドー・システム:

$$u_t = \varepsilon^2 \Delta u - u + \frac{u^p}{\xi^q}, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\tau \frac{d\xi}{dt} = -\xi + \frac{1}{|\Omega|}\int_\Omega \frac{u^r}{\xi^s} dx$$

$\downarrow$

$1 \ll t \ll +\infty$

$$\frac{du}{dt} = -u + \frac{u^p}{v^q}$$

$$\tau \frac{dv}{dt} = -v + \frac{u^r}{v^s}$$

$\downarrow$ $t \to +\infty$

定常解（2）

$$\varepsilon^2 \Delta u - u + \frac{u^p}{\xi^q} = 0, \ \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0$$

空間均質化　prey-predator系

$$u_t = \varepsilon^2 \Delta u + u(a - bv)$$

$$\tau v_t = D\Delta v + v(-c + du)$$

$$\frac{\partial}{\partial \nu}(u,v)\Big|_{\partial\Omega} = 0,\ \ (u,v)|_{t=0} = (u_0(x), v_0(x)) > 0$$

定理

$$\forall u_0, v_0 \in C^2(\overline{\Omega})$$

$$\exists 1 (u,v) = (u(\cdot,t), v(\cdot,t)),\ \ 0 \le t < +\infty$$

$$\exists 1 (\tilde{u},\tilde{v}) = (\tilde{u}(t), \tilde{v}(t))$$

$$\lim_{t\uparrow +\infty} \mathrm{dist}_{C^2}((u(\cdot,t), v(\cdot,t)), \mathcal{O}) = 0$$

$$\mathcal{O} = \{(\tilde{u}(t), \tilde{v}(t))\}_{t\ge 0}$$

$$\frac{d\tilde{u}}{dt} = \tilde{u}(a - b\tilde{v})$$

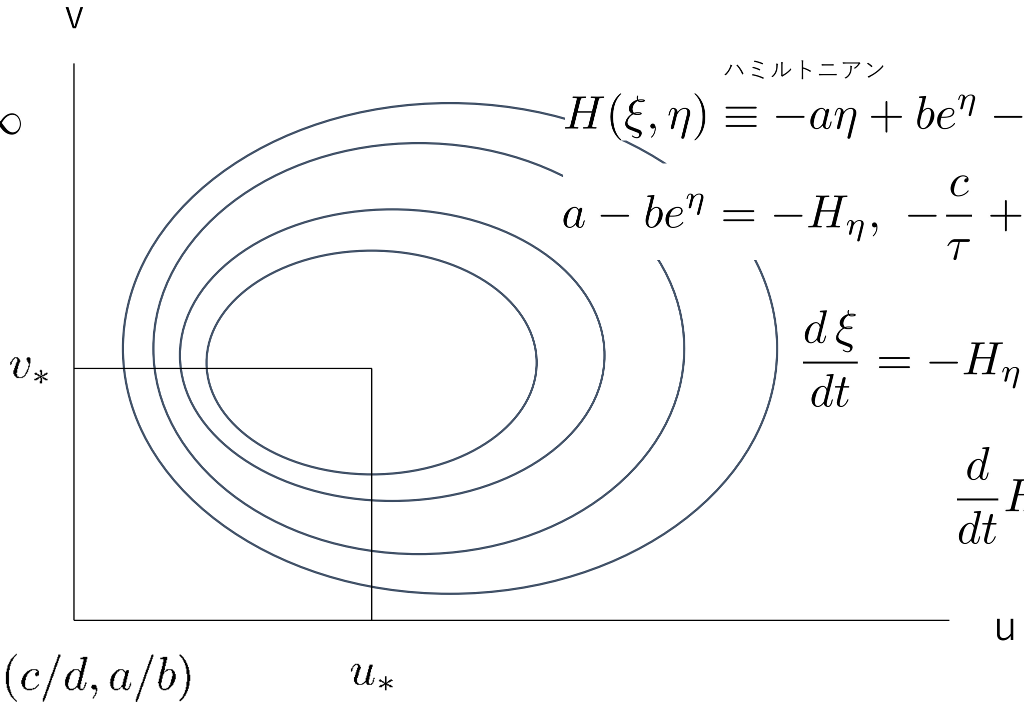
$$\tau \frac{d\tilde{v}}{dt} = \tilde{v}(-c + d\tilde{u})$$

ハミルトン構造

$$\frac{du}{dt} = u(a - bv),\ \ \tau\frac{dv}{dt} = v(-c + du)$$

成長率　$u_t/u,\ \ v_t/v$　変数変換　$\xi = \log u,\ \ \eta = \log v$

$$\frac{d\xi}{dt} = a - be^{\eta},\ \ \frac{d\eta}{dt} = -\frac{c}{\tau} + \frac{d}{\tau}e^{\xi}$$



ハミルトニアン

$$H(\xi,\eta) \equiv -a\eta + be^{\eta} - \tau^{-1}c\xi + \tau^{-1}de^{\xi}$$

$$a - be^{\eta} = -H_{\eta},\ \ -\frac{c}{\tau} + \frac{d}{\tau}e^{\xi} = H_{\xi}$$

$$\frac{d\xi}{dt} = -H_{\eta},\ \ \frac{d\eta}{dt} = H_{\xi}$$

$$\frac{d}{dt}H(\xi,\eta) = 0$$

$$(u_*, v_*) = (c/d, a/b)$$

アプリオリ評価

$$u_t = \varepsilon^2 \Delta u + u(a - bv)$$
$$\tau v_t = D\Delta v + v(-c + du)$$
$$\frac{\partial}{\partial \nu}(u, v)\Big|_{\partial\Omega} = 0$$

$$\xi_t = \varepsilon^2 e^{-\xi} \Delta e^{\xi} - H_\eta$$
$$\eta_t = \tau^{-1} D e^{-\eta} \Delta e^{\eta} + H_\xi$$
$$\frac{\partial}{\partial \nu}(\xi, \eta)\Big|_{\partial\Omega} = 0$$

$$\mathcal{H}(t) = \int_\Omega H(\xi(\cdot, t), \eta(\cdot, t))\ dx$$
$$\frac{d\mathcal{H}}{dt} = \int_\Omega H_\xi \xi_t + H_\eta \eta_t\ dx$$
$$= -\tau^{-1} \int_\Omega c\varepsilon^2 |\nabla \xi|^2 + aD|\nabla \eta|^2\ dx$$

$$H(\xi, \eta) \geq \delta_1 (e^{\xi} + e^{\eta}) - C_1$$
$$\int_\Omega e^{\xi} + e^{\eta}\ dx = \|u(\cdot, t)\|_1 + \|v(\cdot, t)\|_1 \leq C_2$$

$$H(\xi, \eta) \geq \delta_2 (\xi_- + \eta_-) - C_3$$
$$\|\log u(\cdot, t)\|_1 + \|\log v(\cdot, t)\|_1 \leq C_4$$

比較定理

$$\frac{\partial u}{\partial t} \leq \varepsilon^2 \Delta u + au = (\varepsilon^2 \Delta + a - \lambda) u + \lambda u$$
$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0,\ \ u|_{t=0} = u_0(x) > 0$$

$$\frac{\partial \overline{u}}{\partial t} = (\varepsilon^2 \Delta + a - \lambda)\overline{u} + \lambda u$$
$$\frac{\partial \overline{u}}{\partial \nu}\Big|_{\partial\Omega} = 0,\ \overline{u}|_{t=0} = u_0(x) \quad \Longrightarrow \quad 0 \leq u \leq \overline{u}$$

$$\|u(\cdot, t)\|_1 \leq C_2,\ \|\overline{u}(\cdot, t)\|_r \leq C_4(r)$$
$$\|u(\cdot, t)\|_r \leq C_4(r),\ 1 \leq r < \infty$$
$$\|\overline{u}(\cdot, t)\|_\infty \leq C_5,\ \|u(\cdot, t)\|_\infty \leq C_5$$

$$\tau \frac{\partial v}{\partial t} \leq D\Delta v + dC_5 v$$
$$\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0,\ \|v(\cdot, t)\|_1 \leq C \quad \Rightarrow \quad \|v(\cdot, t)\|_\infty \leq C_9$$

力学系の理論　ω-極限集合

$$\omega(u_0, v_0) = \{(u_*, v_*) \mid \exists t_k \uparrow +\infty,$$
$$\lim_{k\to\infty} \|(u(\cdot, t_k), v(\cdot, t_k)) - (u_*, v_*)\|_{C^2} = 0\}$$
$$\subset C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$$
連結コンパクト

流れ

$$u_t = \varepsilon^2 \Delta u + u(a - bv)$$
$$\tau v_t = D\Delta v + v(-c + du)$$
$$\frac{\partial}{\partial \nu}(u, v)\Big|_{\partial\Omega} = 0$$
で不変

$$\forall\ (u_*, v_*) \in \omega(u_0, v_0),\ u_* \geq 0,\ v_* \geq 0$$
$$\log u_* \in L^1(\Omega),\ \log v_* \in L^1(\Omega)$$
$$\mathcal{H}_* = \mathcal{H}_\infty \ \text{(Fatou's lemma)}$$

$$\mathcal{H}_* = \int_\Omega H(\log u_*, \log v_*)$$
$$\mathcal{H}_\infty = \lim_{t\uparrow+\infty} \mathcal{H}(t)$$

$$(\hat{u}, \hat{v})|_{t=0} = (u_*, v_*)$$
初期値

$$(\hat{u}, \hat{v}) = (\hat{u}(x,t), \hat{v}(x,t))$$
解

強最大原理 ⟹ $(\hat{u}(\cdot, t), \hat{v}(\cdot, t)) > 0$

$$\tau \frac{d}{dt} \int_\Omega H(\log \hat{u}, \log \hat{v})$$
$$= -\int_\Omega c\varepsilon^2 |\nabla \log \hat{u}|^2 + aD|\nabla \log \hat{v}|^2 = 0$$

⟹ $(u_*, v_*) \in \mathbf{R}^2_+$

ODE軌道はハミルトニアンの値で決まる

⟹ 定理 ■

GM系の場合 $\dfrac{du}{dt} = -u + \dfrac{u^p}{v^q},\ \tau\dfrac{dv}{dt} = -v + \dfrac{u^r}{v^s}$

$$u^{-p}u_t = u^{-p+1} + v^{-q} \qquad \xi = u^{-p+1}/(p-1)$$
$$v^s v_t = -\tau^{-1}v^{s+1} + \tau^{-1}u^r \qquad \eta = v^{s+1}/(s+1)$$

$$\xi_t = (p-1)\xi - \{(s+1)\eta\}^{-\frac{q}{s+1}}$$
$$\eta_t = -\tau^{-1}(s+1)\eta + \tau^{-1}\{(p-1)\xi\}^{-\frac{r}{p-1}}$$

$$\boxed{p-1 = \tau^{-1}(s+1)}$$

$$A(\xi) = \tau^{-1}(p-1)^{-\frac{r}{p-1}}\xi^{1-\frac{r}{p-1}}$$
$$B(\eta) = (s+1)^{-\frac{q}{s+1}}\eta^{1-\frac{q}{s+1}}$$

$$H = (p-1)\xi\eta + \left(\frac{r}{p-1} - 1\right)^{-1}A(\xi) + \left(\frac{q}{s+1} - 1\right)^{-1}B(\eta)$$

$$\Longrightarrow \quad \frac{d\xi}{dt} = H_\eta,\ \frac{d\eta}{dt} = -H_\xi$$

$$d_1 = \varepsilon^2,\ d_2 = \tau^{-1}D$$

$$u_t = d_1\Delta u - u + \frac{u^p}{v^q}$$
$$v_t = d_2\Delta v - \tau^{-1}\left(v + \frac{u^r}{v^s}\right)$$
$$\left.\frac{\partial}{\partial\nu}(u,v)\right|_{\partial\Omega} = 0$$

$$\Longrightarrow \quad \xi_t = -d_1(p-1)\xi^{\frac{p}{p-1}}\Delta\xi^{-\frac{1}{p-1}} + H_\eta$$
$$\eta_t = d_2(s+1)\eta^{\frac{s}{s+1}}\Delta\eta^{\frac{1}{s+1}} - H_\xi$$
$$\left.\frac{\partial}{\partial\nu}(\xi,\eta)\right|_{\partial\Omega} = 0$$

補題 $\dfrac{2\sqrt{d_1 d_2}}{d_1+d_2} \geq \sqrt{\dfrac{(s+1)(p-1)}{sp}}$ $\quad \alpha = \dfrac{r}{p-1} - 1 > 0$, $\quad \beta = \dfrac{q}{s+1} - 1 > 0$

⟹

$$\frac{d}{dt}\int_\Omega H(\xi,\eta) \leq -\int_\Omega c_1|\nabla \xi^{-\alpha/2}|^2 + c_2|\nabla \eta^{-\beta/2}|^2 \; dx$$

$$\mathcal{H}(\xi,\eta) \approx \int_\Omega u^{-p+1}v^{s+1} + u^{r-p+1} + v^{-q+s+1} \; dx$$

$\dfrac{q}{s+1} > 1$

$\ell > \max\{n/2, 1\}$

$a = \dfrac{\ell}{q-s-1} > 0$

$v = w^{-a} > 0$

⟹

$$\|w(\cdot,t)\|_\ell \leq C_1$$

$$w_t = d_2\Delta w - d_2(a+1)w^{-1}|\nabla w|^2 + a^{-1}\tau^{-1}(w - u^r w^{a(s+1)+1}) \leq d_2\Delta w + a^{-1}\tau^{-1}w$$

⟹ $\|v(\cdot,t)^{-1}\|_\infty \leq C_2$

注意 1（Jiang） $\dfrac{p-1}{r} < 1 \Rightarrow T = +\infty$

注意 2（Yanagida） $p+1=r,\ q+1=s$ ⟹ 歪勾配系

補題（Jiang）

$$0 < \frac{p-1}{r} < \min\{\frac{q}{s+1}, 1\},\quad a>1,\ b>0$$

$$\frac{2\sqrt{d_1 d_2}}{d_1+d_2} \geq \sqrt{\frac{ab}{(a-1)(b+1)}}$$

$$\frac{d}{dt}\int_\Omega u^a v^{-b} \leq (-a+\tau^{-1}b)\int_\Omega u^a v^{-b} + C_3(a,b)\left(\int_\Omega v^{-\theta/\varepsilon}\right)^\varepsilon \left(\int_\Omega u^a v^{-b}\right)^{1-\varepsilon}$$

定理 $\dfrac{2\sqrt{d_1 d_2}}{d_1+d_2} \geq \sqrt{\dfrac{(s+1)(p-1)}{sp}}$

$$\tau = \frac{s+1}{p-1},\ \frac{p}{r} < 1 < \frac{q}{s+1}$$

⟹ $\exists 1 (\tilde{u},\tilde{v}) = (\tilde{u}(t),\tilde{v}(t))$

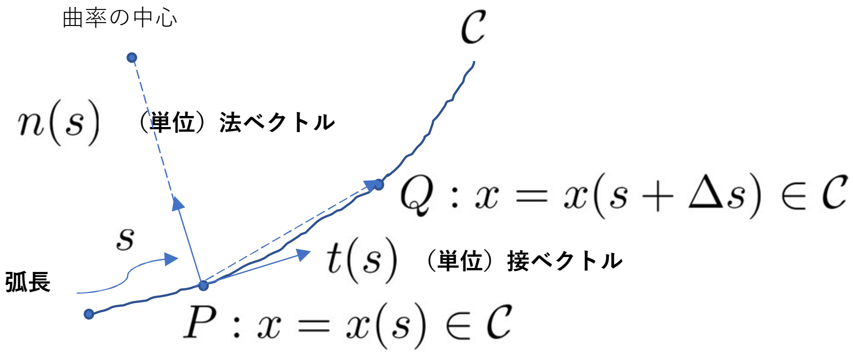
$$\lim_{t\uparrow+\infty} \mathrm{dist}_{C^2}((u(\cdot,t),v(\cdot,t)),\mathcal{O}) = 0$$

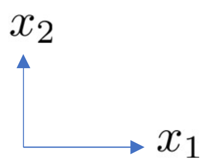
$\mathcal{O} = \{(\tilde{u}(t),\tilde{v}(t))\}_{t\geq 0}$ ODE軌道

10. 微分形式と消滅定理

幾何学の言葉

曲線のパラメータ表示





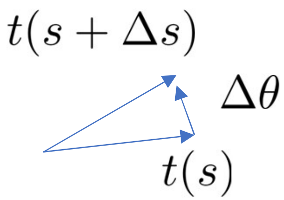
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^2$$

$$t = \lim_{\Delta s \to 0} \frac{1}{\Delta s}(x(s+\Delta s) - x(s)) = \frac{dx}{ds}$$

$$\left|\frac{dx}{ds}\right| = \lim_{\Delta s \to 0} \left|\frac{x(s+\Delta s) - x(s)}{\Delta s}\right| = 1$$

$$|t|^2 = t \cdot t = 1 \implies \frac{dt}{ds} \cdot t = 0$$

$$|t(s+\Delta s) - t(s)| = \Delta\theta + o(\Delta\theta)$$

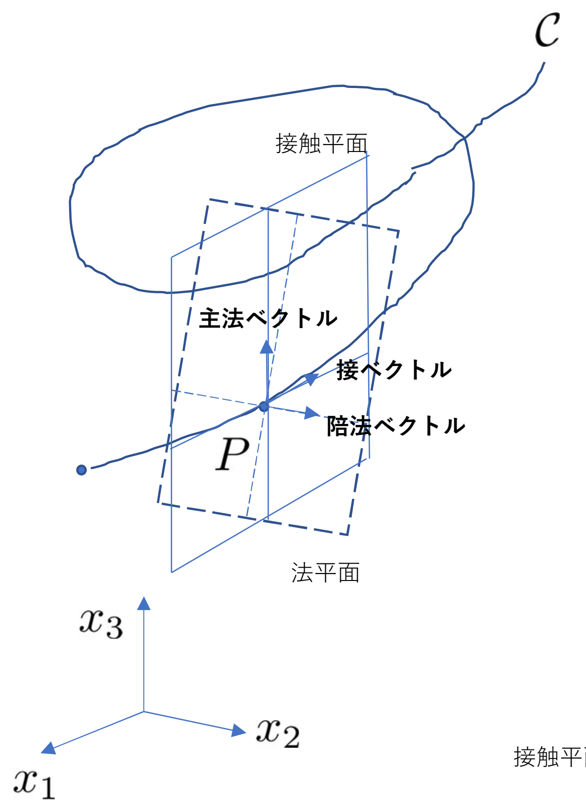


$$\left|\frac{dt}{ds}\right| = \lim_{\Delta \to 0} \left|\frac{t(s+\Delta s) - t(s)}{\Delta s}\right| = \lim_{\Delta s \to 0} \left|\frac{\Delta\theta}{\Delta s}\right| = \left|\frac{1}{\rho}\right|$$

$$\implies \frac{dt}{ds} = \frac{1}{\rho} n$$

$$\boxed{\frac{d}{ds}(t \cdot t) = \frac{dt}{ds} \cdot t + t \cdot \frac{dt}{ds} = 2\frac{dt}{ds} \cdot t}$$

空間曲線



$$x = (x_1, x_2, x_3)^T \in \mathbf{R}^3$$

$$P : x = x(s) \in \mathcal{C}$$

弧長パラメータ

$$t = \frac{dx}{ds}$$ 接ベクトル

n 主法ベクトル

b 陪法ベクトル

フレネ・セレの公式

$$\frac{dt}{ds} = \frac{1}{\rho} n$$ 接触平面上の運動

$$\frac{dn}{ds} = -\frac{1}{\rho} t + \tau b$$

$\frac{1}{\rho}$ 曲率

接触平面の変動

$$\frac{db}{ds} = -\tau n$$

τ 捩率

空間曲線を接触面上の円弧で近似すると曲率が表示される
接ベクトル方向に進むと接触平面が変動する

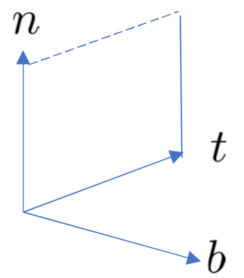
接触平面の変動は捩率で表示される

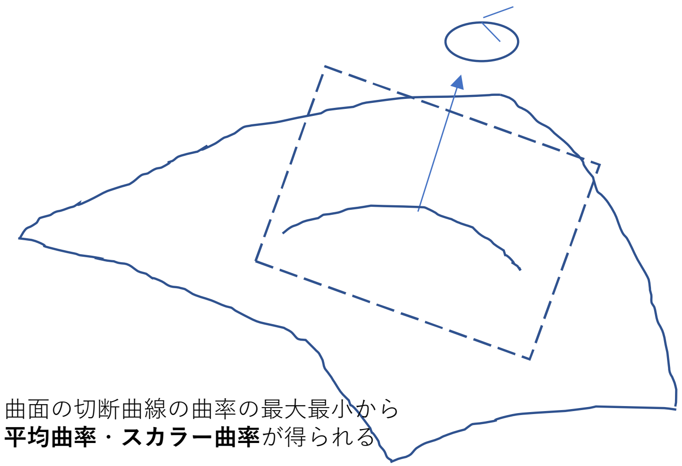
直線探索
勾配法
共役勾配法

$$b = t \times n, \ \ |b|^2 = 1$$

$$\frac{db}{ds} = \frac{dt}{ds} \times n + t \times \frac{dn}{ds} = t \times \frac{dn}{ds}$$

$$\frac{db}{ds} \cdot b = 0$$

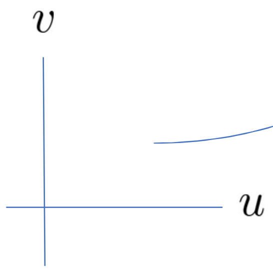


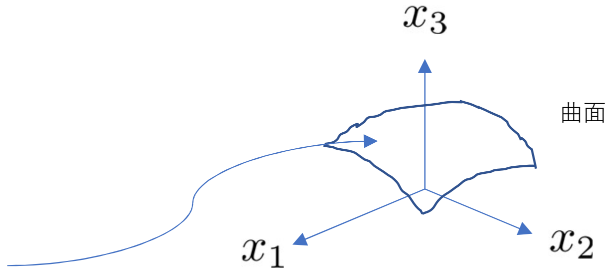


曲面の切断曲線の曲率の最大最小から
平均曲率・スカラー曲率が得られる

第2基本形式

曲面のパラメータ表示





$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3$$

$$x = x(u, v) \qquad dx = x_u du + x_v dv$$

$$x_u = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u} \right)^T$$

$$x_v = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v} \right)^T$$

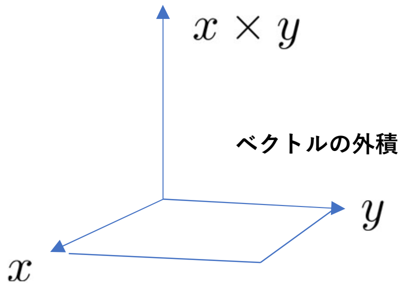
第1基本形式

曲面上2点の無限小距離

$$ds^2 = dx \cdot dx = Edu^2 + 2Fdudv + Gdv^2$$

第1基本量

$$E = |x_u|^2,\ F = 2x_u \cdot x_v,\ G = |x_v|^2$$



ベクトルの外積

ベクトルの成分表示
基底ベクトル間の演算
交換・結合・分配則

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

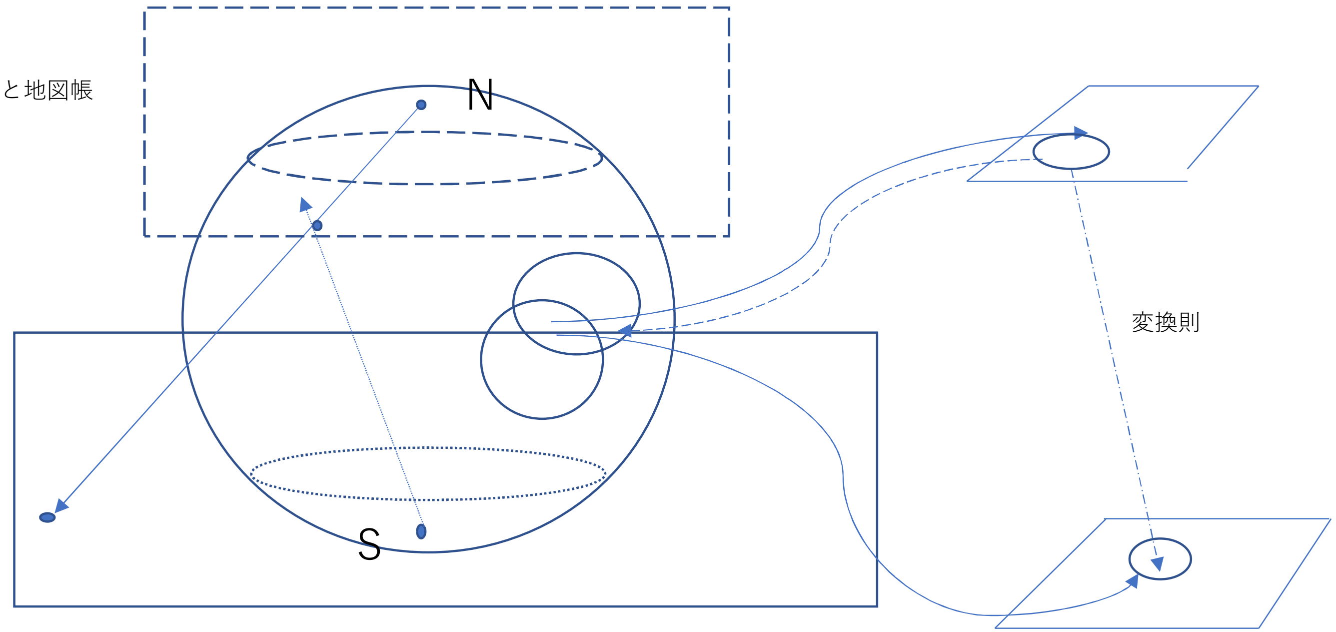
面積要素

$$dS = |x_u \times x_v| dudv = \sqrt{EG - F^2} dudv$$

曲面上平行四辺形の無限小面積

$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

地図と地図帳



多様体

ハウスドルフ空間 $\mathcal{M} = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ 開被覆

アトラス

局所座標
チャート

同相

$$\exists \psi_\alpha : U_\alpha \to \psi_\alpha(U_\alpha) = (x^1, \cdots, x^n) \subset \mathbf{R}^n,\ \alpha \in \mathcal{A}$$

$$U_\alpha \cap U_\beta \neq \emptyset$$

滑らか

$$\Longrightarrow \psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$$

$$p \in \mathcal{M}$$

$T_p\mathcal{M}$ 接空間　基底 $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$　接バンドル $T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}$

n 次ベクトル空間

$T_p^*\mathcal{M}$ 余接空間　基底 $dx^1, \cdots, dx^n$　余接バンドル $T^*\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p^*\mathcal{M}$

滑らか

$X : p \in \mathcal{M} \to X_p \in T_p\mathcal{M}$　ベクトル場　$\mathcal{X}(\mathcal{M})$

滑らか

$\omega : p \in \mathcal{M} \to \omega_p \in T_p^*\mathcal{M}$　1-form　$\mathcal{D}^1(\mathcal{M})$

接続

$$\nabla : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathcal{X}(\mathcal{M})$$
$$(X, Y) \mapsto \nabla_X Y$$

0-form

$$\forall X, Y, Z \in \mathcal{X}(\mathcal{M}),\ \forall f \in C^\infty(\mathcal{M})$$

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$$
$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$
$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$$
$$\nabla_{fX} Y = f\nabla_X Y$$

$(\mathcal{M}, g)$ リーマン多様体

計量

滑らか

$g : p \in \mathcal{M} \mapsto g_p$　$T_p\mathcal{M}$ 対称正定値 上の(0,2)テンソル

局所座標で $g = g_{ij} dx^i \otimes dx^j,\ g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x_j}\right)$

双対アフィン接続

(∇, ∇^*)　$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z)$

レビ・チビタ接続　$\nabla^* = \nabla$

情報幾何学 …双対平坦

解析力学

位置エネルギー　ニュートンの運動方程式

$$\frac{dp_i}{dt} = -\frac{\partial U}{\partial x_i}, \quad 1 \le i \le f$$

運動量　運動エネルギー

$$p_i = \frac{\partial K}{\partial \dot{x}_i},\ K = \frac{1}{2}\sum_i m\dot{x}_i^2$$

一般座標

$$x_i = x_i(q_1, \cdots, q_f; t),\ 1 \le i \le f$$

$$\Rightarrow \dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}$$

ラグランジュの運動方程式

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j},\ 1 \le j \le f$$

ラグランジアン

$$L = K(q, \dot{q}, t) - U(q, t)$$

最小作用の原理

ハミルトニアン　ルジャンドル変換

$$H(p, q, t) = \sup_{\dot{q}}(p \cdot \dot{q} - L(q, \dot{q}, t))$$

$$\Rightarrow p = \frac{\partial L}{\partial \dot{q}}$$

正準方程式　保存則

$$\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q} \Rightarrow \frac{d}{dt}H(p(t), q(t)) = 0$$

$$\Longleftrightarrow \dot{F} = \{F, H\},\ \forall F = F(p, q) \Longleftrightarrow \dot{z} = X_H z, \quad z = (q, p)^T$$

ポアソン括弧

$$\{F, G\} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q}$$

歪対称双線形形式

ハミルトンベクトル場

$$X_H = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)^T$$

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0$$ ヤコビの等式

$$\{FG, H\} = F\{G, H\} + \{F, H\}G$$ ライプニッツの等式

シンプレティック多様体

$$\mathcal{D}^1(\mathcal{M}) \xrightarrow{d} \Lambda^2 T^*\mathcal{M}$$

1-forms　exterior d derivative　$\wedge$ product　2-forms

$\mathcal{M}$ **シンプレティック多様体** $\overset{\text{def}}{\Longleftrightarrow}$ $\exists \omega \in \Lambda^2 T^*\mathcal{M},\ d\omega = 0$

non-degenerate　closed　3-form

定理（ダルブー）　$(\mathcal{M}, \omega) \cong (\mathbf{R}^{2n}, \omega_0)$　局所的に

$$\omega_0 = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$$

$$(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$$

シンプレティック多様体はポアソン多様体になる

定理　$(\mathcal{M}, \omega)$（シンプレティック多様体）　$H \in C^\infty(\mathcal{M})$

$X_H \in \mathcal{X}(\mathcal{M}) \overset{\text{def}}{\Longleftrightarrow} \omega(X_H, \cdot) = dH$　1-formとして

$\Longrightarrow \{F, G\} = \omega(X_F, X_G)$　ポアソン括弧

歪対称双線形形式, ヤコビの等式, ライプニッツの等式

一致

リー微分

$$X \in \mathcal{X}(\mathcal{M}),\ f \in C^\infty(\mathcal{M})$$

$$\mathcal{L}_X f = \frac{d}{dt} \Phi_{-t}^* f \Big|_{t=0}$$

引き戻し

力学系, 流れ

$$\frac{d}{dt}\Phi_t(x) = X(\Phi_t(x)),\ \ \Phi_t(x)|_{t=0} = x$$

ポアソン多様体　$\mathcal{P}$

$\exists \{\ ,\ \} : C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$　ポアソン括弧

$H \in C^\infty(\mathcal{P})$　given

$X_H \in \mathcal{X}(\mathcal{M}) \overset{\text{def}}{\Longleftrightarrow} \mathcal{L}_{X_H} F = \{F, H\},\ \forall F \in C^\infty(\mathcal{P})$

$\dot{x} = X_H(x) \overset{\text{def}}{\Longleftrightarrow} \mathcal{P}$ 上のハミルトン系

例　$\mathcal{P} = \mathbf{R}^3,\ C \in C^\infty(\mathcal{P})$

$$\{F, G\} = -\nabla C \cdot \nabla F \times \nabla G$$

$$\dot{X} = \nabla C \times \nabla H$$

ハミルトン・ヤコビ理論

$$\dot{q} = \frac{\partial H}{\partial q},\ \dot{p} = -\frac{\partial H}{\partial q}$$ ハミルトン系

$(p,q)^T \mapsto (\xi,\eta)^T$ 正準変換 $\overset{\text{def}}{\Longleftrightarrow}$ $d\xi \wedge d\eta = dp \wedge dq$

完全積分可能 $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ 正準変換 $\dfrac{\partial \tilde{H}}{\partial \eta} = 0,\ H(p,q) = \tilde{H}(\xi,\eta)$

$$\dot{\eta} = \tilde{H}_\xi,\ \dot{\xi} = 0 \Rightarrow \begin{array}{l} \xi(t) = \alpha \in \mathbf{R}^n \\ \eta(t) = \eta(0) + Et,\ E = \tilde{H}(\alpha,\eta(0)) \end{array}$$

$$\theta = pdq + \eta d\xi \Rightarrow d\theta = dp \wedge dq + d\eta \wedge d\xi = 0$$

$$\exists W = W(q,\eta),\ \theta = dW = W_\xi d\xi + W_q dq$$

$$W_q = p,\ W_\xi = \eta \qquad H(p,q) = H(W_q,q) = E$$

ハミルトン・ヤコビ方程式

$$S = W - Et \Rightarrow \frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0$$

逆に $S = S(\alpha,q,t) = W - Et,\ \alpha \in \mathbf{R}^n$

$$\Rightarrow \begin{array}{l} W = W(q,\xi),\ \xi = \alpha \\ p = W_q,\ \eta = W_\xi \end{array}$$

$$(p,q)^T \mapsto (\xi,\eta)^T$$

$$p = p(t),\ q = q(t)$$

理論生物学の数理モデル

{Hamilton structures} $\Subset$ {Poisson structures}

<u>prey predator system</u>

$$\frac{dx}{dt} = (a - by)x$$
$$\tau\frac{dy}{dt} = (-c + dx)y, \quad x, y > 0$$

$$\xi = \log x \qquad \frac{d\xi}{dt} = a - be^{\eta}$$
$$\eta = \log y \qquad \tau\frac{d\eta}{dt} = -c + de^{\xi}$$

$$H = -a\eta + be^{\eta} - \tau^{-1}c\xi + \tau^{-1}de^{\xi}$$

$$\frac{d\xi}{dt} = -H_\eta, \ \frac{d\eta}{dt} = H_\xi \qquad \frac{d}{dt}H(\xi,\eta) = 0$$

<u>Gierer-Meinhardt system</u>

$$\frac{du}{dt} = -u + \frac{u^p}{v^q}$$
$$\tau\frac{dv}{dt} = -v + \frac{u^r}{v^s}, \quad u, v > 0$$

$$u^{-p}(u_t + u) = v^{-q}$$
$$v^s(v_t + \tau^{-1}) = \tau^{-1}u^r$$

$$\xi = \frac{u^{-p+1}}{p-1} \qquad \frac{d\xi}{dt} = (p-1)\xi - \{(s+1)\eta\}^{-\frac{q}{s+1}}$$
$$\eta = \frac{v^{s+1}}{s+1} \qquad \frac{d\eta}{dt} = -\tau^{-1}(s+1)\eta + \tau^{-1}\{(p-1)\xi\}^{-\frac{1}{p-1}}$$

$$p - 1 = \tau^{-1}(s+1) \equiv a$$
$$H = a\xi\eta + (\frac{r}{p-1} - 1)^{-1}A(\xi) + (\frac{q}{s+1} - 1)^{-1}B(\eta)$$

$$A(\xi) = \tau^{-1}(p-1)^{-\frac{r}{p-1}}\xi^{1-\frac{r}{p-1}}$$
$$B(\eta) = (s+1)^{-\frac{q}{s+1}}\eta^{1-\frac{q}{s+1}}$$

$$\Longrightarrow \quad \frac{d\xi}{dt} = H_\eta, \ \frac{d\eta}{dt} = -H_\xi$$

three system $x_1, x_2, x_3 > 0$

$$\tau_1 \frac{dx_1}{dt} = (x_2 - x_3)x_2 \qquad \tau_1 \frac{d\xi_1}{dt} = e^{\xi_2} - e^{\xi_3}$$
$$\tau_2 \frac{dx_2}{dt} = (x_3 - x_1)x_2 \qquad \tau_2 \frac{d\xi_2}{dt} = e^{\xi_3} - e^{\xi_1}$$
$$\tau_3 \frac{dx_3}{dt} = (x_1 - x_2)x_3 \qquad \tau_3 \frac{d\xi_3}{dt} = e^{\xi_1} - e^{\xi_2}$$

$$\xi_1 = \log x_1$$
$$\xi_2 = \log x_2 \qquad M\frac{d\xi}{dt} = H(\xi) \times a$$
$$\xi_3 = \log x_3$$

$$0 = \frac{d}{dt} M\xi \cdot a = \frac{d}{dt}(\tau_1\xi_1 + \tau_2\xi_2 + \tau_3\xi_3$$
$$0 = \frac{d}{dt} M(\xi) \cdot H(\xi) = \frac{d}{dt}(\tau_1 e^{\xi_1} + \tau_2 e^{\xi_2} + \tau_3 e^{\xi_3})$$

$$M = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix},\ a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ H(\xi) = \begin{pmatrix} e^{\xi_1} \\ e^{\xi_2} \\ e^{\xi_3} \end{pmatrix},\ \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

SIR model

$$\frac{dx}{dt} = -\beta xy,\ \frac{dy}{dt} = \beta xy - \gamma y,\ \frac{dz}{dt} = \gamma y$$

$$\frac{d}{dt}(x + y + z) = 0 \qquad \rho = \frac{\gamma}{\beta}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\beta xy - \gamma y}{-\beta xy} = -1 + \frac{\rho}{x}$$

$$\int dy = \int -1 + \frac{\rho}{x}\ dx$$

$$y = -x + \rho \log x + C$$

three system $\quad x_1, x_2, x_3 > 0$

$$\tau_1 \frac{dx_1}{dt} = (x_2 - x_3)x_2 \qquad \tau_1 \frac{d\xi_1}{dt} = e^{\xi_2} - e^{\xi_3}$$
$$\tau_2 \frac{dx_2}{dt} = (x_3 - x_1)x_2 \qquad \tau_2 \frac{d\xi_2}{dt} = e^{\xi_3} - e^{\xi_1}$$
$$\tau_3 \frac{dx_3}{dt} = (x_1 - x_2)x_3 \qquad \tau_3 \frac{d\xi_3}{dt} = e^{\xi_1} - e^{\xi_2}$$

$$\xi_1 = \log x_1$$
$$\xi_2 = \log x_2 \qquad M\frac{d\xi}{dt} = H(\xi) \times a$$
$$\xi_3 = \log x_3$$

$$0 = \frac{d}{dt} M\xi \cdot a = \frac{d}{dt}(\tau_1\xi_1 + \tau_2\xi_2 + \tau_3\xi_3$$
$$0 = \frac{d}{dt} M(\xi) \cdot H(\xi) = \frac{d}{dt}(\tau_1 e^{\xi_1} + \tau_2 e^{\xi_2} + \tau_3 e^{\xi_3})$$

$$M = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix},\ a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ H(\xi) = \begin{pmatrix} e^{\xi_1} \\ e^{\xi_2} \\ e^{\xi_3} \end{pmatrix},\ \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

SIR model

$$\frac{dx}{dt} = -\beta xy,\ \ \frac{dy}{dt} = \beta xy - \gamma y,\ \ \frac{dz}{dt} = \gamma y$$

$$\frac{d}{dt}(x + y + z) = 0 \qquad \rho = \frac{\gamma}{\beta}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\beta xy - \gamma y}{-\beta xy} = -1 + \frac{\rho}{x}$$

$$\int dy = \int -1 + \frac{\rho}{x}\ dx$$

$$y = -x + \rho \log x + C$$

歪対称ロトカ・ボルテラ系

$$x_j > 0, \ 1 \le j \le N$$

$$\tau_j \frac{dx_j}{dt} = (-e_j + \sum_k a_{jk} x_k) x_j$$

$$\tau_j, e_j, a_{jk} \in \mathbf{R}$$

$$\tau = (\tau_j) > 0, \ e = (e_j) \quad A = (a_{jk})$$

(a1) irreducible $\not\exists P$ permutation

$$ {}^t PAP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

A_{11}, A_{22} square (no sub-system)

(a2) skew symmetric $A + A^T = 0$

$\Longrightarrow$ $\frac{d}{dt}\tau \cdot v = -e \cdot v \qquad T = +\infty$

(a3) sign condition

Each row of A contains both positive and negative components

(P) $E = L \cap \mathbf{R}_+^N, \ \exists L$ affine space of co-dimension 2

Any non-stationary solution is periodic-in-time with the orbit $\mathcal{O} \cong S^1$ contractible to a stationary solution in $\mathbf{R}_+^N \setminus E$

Any distinct two orbits $\mathcal{O}_1, \mathcal{O}_2 \cong S^1$ do not link in $\mathbf{R}_+^N$

$E = \{v = (v_j) \in \mathbf{R}_+^N \mid Av = e\}$ set of stationary solutions

Kobayashi-S.-Yamada 2019

Remark

$e = (e_j) = 0$ $\qquad \frac{d}{dt}\tau \cdot x = 0$ total mass

$$\xi_j = \log x_j \qquad \tau_j \frac{d\xi_j}{dt} = \sum_k a_{jk} e^{\xi_k}$$

$$b \in \mathrm{Ker}\ A^T \qquad \tilde{b} = (\tau_j b_j)$$

$\frac{d}{dt}\tilde{b} \cdot \xi = \sum_{jk} a_{jk} b_j e^{\xi_k} = 0$ entropy

Theorem 1 $(e_j) = 0,\ N \geq 3 \qquad A = (a_{jk}) \qquad$ (a1), (a2), (a3)

$$a_{ij}a_{kl} + a_{il}a_{jk} - a_{ik}a_{jl} = 0, \quad \forall i, j, k, l \in \{1, \ldots, N\} \Longrightarrow \text{(P)}$$

Remark

(a1)(a2) ~~(P)~~

$\Longrightarrow$ extinction in infinite time

Remark $a_{12} \neq 0$ $\Updownarrow$

$$\text{(Q)} \qquad a_{kl} = \frac{a_{1k}a_{2l} - a_{1l}a_{2k}}{a_{12}}, \ 3 \leq k < l \leq N$$

Example. N=3

$$A = \begin{pmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_2 & -c_1 & 0 \end{pmatrix}$$

Skew symmetric matrices

N(N-1)/2 dimension

$$\begin{matrix} 0 & a_{12} & a_{13} & a_{14} & \cdots & a_{1N} \\ a_{21} & 0 & a_{23} & a_{24} & \cdots & a_{2N} \\ a_{31} & a_{32} & 0 & a_{34} & \cdots & a_{3N} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & a_{N-1N} \\ a_{N1} & a_{N2} & \cdots & \cdots & a_{NN-1} & 0 \end{matrix}$$

$\longleftarrow$ free

2N-3 dimension

Theorem 2 $A = (a_{jk})$ (a1) (a2) (a3) $N \geq 3$ (e_j) has both positive and negative components

$$a_{jk}e_i - a_{ik}e_j + a_{ij}e_k = 0, \quad \forall i,j,k \in \{1,\dots,N\}$$ ⇒ (P)

Remark $e_1 \neq 0$ ⇕

$$a_{jk} = \frac{a_{1k}e_j - a_{1j}e_k}{e_1}, \quad 2 \leq j < k \leq N$$

$$\begin{array}{cccccc} 0 & a_{12} & a_{13} & \cdots & a_{1N} & e_1 \\ a_{21} & 0 & a_{23} & \cdots & a_{2N} & e_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & a_{N-1N} & e_{N-1} \\ a_{N1} & a_{N2} & \cdots & a_{NN-1} & 0 & e_N \end{array}$$

← free 2N-1 dimension

Skew symmetric matrices + vectors

N(N+1)/2 dimension

Not contain prey predator system

微分方程式と微分形式

$D \subset \mathbf{R}^n$ open set

1 interface vanishing

$\Lambda^p = \Lambda^p(D)$ p-forms

$\wedge$ wedge product

$d : \Lambda^p \to \Lambda^{p+1}$ outer derivative

inner product

$\alpha = \sum_\ell \alpha^\ell dx_\ell,\ \beta = \sum_\ell \beta^\ell dx_\ell$ 1-forms $\Longrightarrow$ $(\alpha, \beta) = \sum_\ell \alpha^\ell \beta^\ell$

$\lambda = \alpha_1 \wedge \cdots \wedge \alpha_p,\ \mu = \beta_1 \wedge \cdots \wedge \beta_p$ p-forms $\Longrightarrow$ $(\lambda, \mu) = \det\ ((\alpha_i, \beta_j))_{i,j}$

$* : \Lambda^p(D) \to \Lambda^{n-p}(D)$ Hodge operator

$$\omega \wedge \tau = (*\omega, \tau)\ dx_1 \wedge \cdots \wedge dx_n$$

$$\omega \in \Lambda^p(D),\ \tau \in \Lambda^{n-p}(\Lambda)$$

co-derivative

$$\delta = (-1)^p *^{-1} d* : \Lambda^p(D) \to \Lambda^{p-1}(D)$$

1 form: $$B = \sum_i B^i dx_i \ \Rightarrow\ \delta B = -\sum_i B^i_i$$

2 form: $$\omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j \ \Rightarrow\ \delta\omega = -\sum_{i,\ell} \tilde{\omega}^{\ell i}_\ell dx_i$$

$$\tilde{\omega}^{ij} = \begin{cases} \omega^{ij}, & i < j \\ 0, & i = j \\ -\omega^{ij}, & i > j \end{cases}$$

Laplacian

$$-\Delta = \delta d + d\delta : \Lambda^p \to \Lambda^p$$

$D \subset \mathbf{R}^n$ Lipschitz domain $\exists \gamma : H^1(D) \to H^{1/2}(\partial D)$ trace operator $H^{1/2}(\partial D) \cong H^1(D)/H^1_0(\Omega)$

$\longrightarrow$ $C^\infty(\overline{D}) \subset H^1(D)$ dense

write $\varphi|_{\partial D} = \gamma\varphi, \quad \varphi \in H^1(D)$

integral formulae

ν outer unit normal vector

$\nu ds = (*dx_1, \cdots, *dx_n)$ vector area element

Lemma 1 $B \in \Lambda^1(D),\, C \in \Lambda^2(D)$ $\longrightarrow$ $*B = (B,\nu)\ ds,\ B \wedge *C = (\nu \wedge B, C)\ ds$

write $\int_D \cdots\ dx_1 \wedge \cdots \wedge dx_n = \int_D, \qquad \int_{\partial D} \cdots\ ds = \int_{\partial D},$

Lemma 2
$\varphi \in H^1(\Lambda^0)$
$B \in H^1(\Lambda^1)$
$J \in H^1(\Lambda^2)$
$\longrightarrow$

$$\int_D (\delta B, \varphi) = \int_D (B, d\varphi) - \int_{\partial D} (B,\nu)\varphi \quad \text{Gauss}$$

$$\int_D (dB, J) = \int_D (B, \delta J) + \int_{\partial D} (\nu \wedge B, J) \quad \text{Stokes}$$

$$H^q(\Lambda^p) = H^q(\Lambda^p(\Omega)) = \{p\text{-forms} \mid \text{coefficients are in } H^q\}$$

$$L^2(\Lambda^0) = L^2(D)$$

Definition $\Omega \subset \mathbf{R}^n$ region with interface $\Longleftrightarrow$ $\exists \mathcal{M},\ \Gamma \equiv \Omega \cap \mathcal{M} \neq \emptyset$

smooth non-compact hyper-surface without boundary

$$\longrightarrow \quad \Omega = \Omega_+ \cup \Gamma \cup \Omega_-, \quad \Gamma_\pm = \partial\Omega_\pm \setminus \partial\Omega (= \Gamma)$$

Theorem 1 $\Omega \subset \mathbf{R}^n$ region with interface $\quad \nu$ outer unit normal on Γ_- $\quad \omega \in H^1(\Lambda^2(\Omega))$

$$d\omega = \theta,\ \delta\omega = 0,\ \delta\theta \in L^2(\Lambda^2(\Omega_\pm)) \ \Rightarrow\ \Delta(\nu, \hat{\omega}^i) \in L^2(\Lambda^0(\Omega)),\ 1 \leq i \leq n$$

$$\hat{\omega}^i = \sum_\ell \tilde{\omega}^{\ell i} dx_i, \quad \tilde{\omega}^{ij} = \begin{cases} \omega^{ij}, & i < j \\ 0, & i = j \\ -\omega^{ij}, & i > j \end{cases}, \quad \omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j$$

Application to the Minkowski space

Theorem 2 $\Omega \subset \mathbf{R}^4$ region with interface

$E, B \in H^1(\Omega)^3,\ J \in L^2(\Omega)^3,\ \rho \in L^2(\Omega)$

Sobolev space

$$\nabla \times B - \frac{\partial E}{\partial t} = J,\ \nabla \cdot E = \rho$$
$$\nabla \times E + \frac{\partial B}{\partial t} = 0,\ \nabla \cdot B = 0 \quad \text{in } \Omega$$

$$\nabla \times J \in L^2(\Omega_\pm)^3$$
$$\frac{\partial J}{\partial t} + \nabla \rho \in L^2(\Omega_\pm)^3$$

in the sense of distributions

$\Longrightarrow$

$$(-\partial_t^2 + \Delta_x)(\nu^0 B + \tilde{\nu} \times E) \in L^2(\Omega)^3$$
$$(-\partial_t^2 + \Delta_x)(\tilde{\nu} \cdot B) \in L^2(\Omega)$$

in the sense of distributions

$$\nu = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \\ \nu^0 \end{pmatrix}, \quad \tilde{\nu} = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix}$$

outer normal unit on Γ_-

S. 2021

H2 singularities of the above components of electric magnetic fields pass through the interface with light velocity

Remark

$$(-\partial_t^2 + \Delta_x)E \in L^2(\Omega_\pm)^3$$
$$(-\partial_t^2 + \Delta_x)B \in L^2(\Omega_\pm)^3$$

Interface vanishing does not occur to all components

2. Hadamard's variational formula (S.-Tsuchiya)

$\Omega \subset \mathbf{R}^n$ bounded Lipschitz domain

$\gamma_0, \gamma_1 \subset \partial\Omega$ relatively open

$$\gamma_0 \cup \gamma_1 = \partial\Omega, \ \gamma_0 \cap \gamma_1 = \emptyset$$

$$-\Delta z = f, \ z|_{\gamma_0} = \varphi, \ \left.\frac{\partial z}{\partial \nu}\right|_{\gamma_1} = \psi$$

$N = N(x,y)$ Green function

$$T_t : \Omega \to \Omega_t, \ |t| \ll 1, \ T_0 = I$$

family of bi-Lipschitz homeomorphism

$T_t x$ continuously differentiable twice in $t, \ \forall x \in \Omega$

$$\partial_t^k DT_t, \ \partial_t^k (DT_t)^{-1} : \Omega \to \mathbf{R}^n, \ k = 1,2$$

uniformly bounded

DT_t Jacobi matrix

$$T_t = I + tS + \frac{t^2}{2}R + o(t^2)$$

$$\delta\rho = S \cdot \nu$$
$$\delta^2\rho = R \cdot \nu$$

Ex. dynamical deformation

$v = v(y)$ locally Lipschitz continuous vector field

$$T_t x = y(x,t), \quad \frac{dy}{dt} = v(y), \ v|_{t=0} = x$$

$\Longrightarrow$ $S = v, \quad R = (v \cdot \nabla) v$ if $v \in C^{0,1}$

Jacobian

$$\det DT_t = 1 + t\nabla \cdot S + \frac{t^2}{2}\{\nabla \cdot R + (\nabla \cdot S)^2 - (DS)^T : DS\} + o(t^2)$$

Hodge operator Transformation of the vector area element

$$\nu ds = (*dx_1, \cdots, *dx_n)$$

$$y^i = x^i + tS^i + \frac{t^2}{2}R^i + o(t^2)$$

$$dy^i = \sum_j (\delta_{ij} + tS^i_j + \frac{t^2}{2}R^i_j) dx^j + o(t^2)$$

Green's formula $\quad \Omega$ Lipschitz domain

$$V = \{v \in H^1(\Omega) \mid v|_{\gamma_0} = 0\}$$

$$\varphi \in V \;\Rightarrow\; \varphi|_{\gamma_1} \in H^{1/2}(\gamma_1)$$

$$v \in V,\; \Delta v \in V' \;\Rightarrow\; \left.\frac{\partial v}{\partial \nu}\right|_{\gamma_1} \in H^{-1/2}(\gamma_1)$$

$$\langle \varphi, \frac{\partial v}{\partial \nu}\rangle_{\gamma_1} = (\nabla v, \nabla \varphi)_\Omega + \langle \varphi, \Delta v\rangle,\; \forall \varphi \in V$$

Theorem 1. first variation

$$\delta N(x,y) = \langle \delta\rho \frac{\partial N}{\partial \nu}(\cdot,x), \frac{\partial N}{\partial \nu}(\cdot,y)\rangle_{\gamma_0} - \langle \delta\rho \nabla_\tau N(\cdot,x), \nabla_\tau N(\cdot,y)\rangle_{\gamma_1} \;\; x,y \in \Omega$$

Theorem 2. second variation

$$\delta^2 N(x,y) = -2(\nabla \delta N(\cdot,x), \nabla \delta N(\cdot,y)) + \langle \chi \frac{\partial N}{\partial \nu}(\cdot,x), \frac{\partial N}{\partial \nu}(\cdot,y)\rangle_{\gamma_0} - \langle \sigma \nabla_\tau N(\cdot,x), \nabla_\tau N(\cdot,y)\rangle_{\gamma_1},\; x,y \in \Omega$$

$$\chi = \delta^2\rho - 2(S_\tau \cdot \nabla_\tau)\delta\rho - \mathcal{B}(S_\tau, S_\tau) - (\delta\rho)^2 \nabla \cdot \nu = [R - (S\cdot\nabla)S]\cdot \nu - (\delta\rho)^2 \nabla\cdot\nu + (\delta\rho \frac{\partial}{\partial \nu} - (S_\tau \cdot \nabla_\tau))\delta\rho$$

$$\sigma = \delta^2\rho - 2(S_\tau \cdot \nabla_\tau)\delta\rho - \mathcal{B}(S_\tau, S_\tau) = [R - (S\cdot\nabla)S]\cdot\nu + (\delta\rho\frac{\partial}{\partial\nu} - (S_\tau\cdot\nabla_\tau))\delta\rho$$

$$\mathcal{B}(\xi,\eta) = -(\nabla\nu)[\xi,\eta], \quad \xi,\; \eta \in \mathbf{R}^n$$ second fundamental form

3. 可積分系と微分形式

$$\xi = \begin{pmatrix} \xi_1 \\ \cdot \\ \cdot \\ \xi_n \end{pmatrix}, \ \xi = \sum_j \xi_j(t) d\xi_j, \ \frac{d\xi}{dt} = \sum_j \dot{\xi}_j d\xi_j$$

Theorem 1 $H(\xi) \in \Lambda^0(\mathbf{R}^n)$ $\quad h \in \Lambda^{n-2}(\mathbf{R}^n)$

$$*\frac{d\xi}{dt} = dH \wedge h \quad \Longrightarrow \quad \frac{dH}{dt} = 0$$

Proof

$$\begin{aligned} 0 &= dH \wedge *\frac{d\xi}{dt} \\ &= \sum_{jk} \frac{\partial H}{\partial \xi_j} \wedge (-1)^{k+1} \frac{d\xi_k}{dt} d\xi_1 \wedge \cdots \wedge d\hat{\xi}_k \wedge \cdots \wedge d\xi_n \\ &= \sum_j (-1)^{j+1} \frac{\partial H}{\partial \xi_j} d\xi_j \wedge \frac{d\xi_j}{dt} \wedge \cdots \wedge d\hat{\xi}_j \wedge \cdots \wedge d\xi_n \\ &= \sum_j \frac{\partial H}{\partial \xi_j} \frac{d\xi_j}{dt} d\xi_1 \wedge \cdots \wedge d\xi_n = \frac{dH}{dt} d\xi_1 \wedge \cdots \wedge d\xi_n \end{aligned}$$

Tensor representation of p-form

$$\theta = \sum_{j_1 \cdots j_p} \theta_{j_1 \cdots j_p} d\xi_{j_1} \wedge \cdots \wedge d\xi_{jp} \in \Lambda^p(\mathbf{R}^n)$$

$$\theta_{j_1 \cdots j_p} = \mathrm{sgn}\sigma \cdot \theta_{\sigma(j_1) \cdots \sigma(j_p)}, \ \sigma \in S_p$$

Corollary $1 \le r \le n-1$

$H^1, \cdots, H^r \in \Lambda^0(\mathbf{R}^n)$ $\quad h \in \Lambda^{n-r-1}(\mathbf{R}^n)$

$$*\frac{d\xi}{dt} = dH^1 \wedge \cdots \wedge dH^r \wedge h$$

$$\Longrightarrow \quad \frac{dH^i}{dt} = 0, \ 1 \le i \le r$$

… rank r

Exercise 1 $H \in \Lambda^{p-1}(\mathbf{R}^n)$ In Theorem 1

$$\xi = (\xi_j(t)) \in \mathbf{R}^n, \ \xi = \sum_j \xi_j(t) d\xi_j, \ \frac{d\xi}{dt} = \sum_j \dot{\xi}_j(t) d\xi_j$$

Theorem 2 rank 1 $\longleftrightarrow$ $\frac{d\xi}{dt} = h\nabla H, \ h^T + h = 0$

$H \in \Lambda^0(\mathbf{R}^n)$ $h = (h_{jk}) \in M_n(\mathbf{R})$

general Hamilton system

Exercise 2 $\xi = (\xi_j), \ \xi_j \in \mathbf{R}^m, \ 1 \leq j \leq n$

Theorem 3 rank 2 $\longleftrightarrow$

$$\frac{d\xi_i}{dt} = \sum_{jk} \frac{\partial H^1}{\partial \xi_j} \frac{\partial H^2}{\partial \xi_k} h_{ijk}, \ 1 \leq i \leq n$$

$$h_{\sigma(i)\sigma(j)\sigma(k)} = \mathrm{sgn}\ \sigma \cdot h_{ijk}, \ \forall \sigma \in S_3$$

Theorem 4 rank n-1 $\longleftrightarrow$

$$\frac{d\xi_j}{dt} = \det A_j, \ 1 \leq j \leq n$$

$$A_j = \begin{pmatrix} h_1^1 & \dots & \overset{\overset{j}{\vee}}{\cdot} & \dots & h_n^1 \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ h_1^{n-1} & \dots & \cdot & \dots & h_n^{n-1} \end{pmatrix}$$

$$h_j^i = (-1)^{j+1} \frac{\partial H^i}{\partial \xi_j}$$

Theorem 5 rank r $\longleftrightarrow$

$$\frac{d\xi_i}{dt} = \sum_{i_1 \cdots i_r} \frac{\partial H^1}{\partial \xi_{i_1}} \frac{\partial H^2}{\partial \xi_{i_2}} \cdots \frac{\partial H^r}{\partial \xi_{i_r}} h_{i, i_1 \cdots i_r}, \ 1 \leq i \leq n$$

$$h_{\sigma(i)\sigma(i_1)\sigma(i_2)\sigma(i_r)} = \mathrm{sgn}\ \sigma \cdot h_{i i_1 i_2 \cdots i_r}, \ \forall \sigma \in S_{r+1}$$

Example   rank 2 $n = 3$

$$\frac{d\xi_i}{dt} = \sum_{jk} \frac{\partial H^1}{\partial \xi_j} \frac{\partial H^2}{\partial \xi_k} h_{ijk},\ 1 \le i \le 3$$

$$h_{\sigma(i)\sigma(j)\sigma(k)} = \mathrm{sgn}\ \sigma \cdot h_{ijk},\ \forall \sigma \in S_3$$

$$h_{ijk} = \mathrm{sgn}\ \sigma \cdot h, \quad \sigma : (123) \mapsto (ijk)$$

$$\tau_1 \frac{d\xi_1}{dt} = e^{\xi_2} - e^{\xi_3}$$
$$\tau_2 \frac{d\xi_2}{dt} = e^{\xi_3} - e^{\xi_1}$$
$$\tau_3 \frac{d\xi_3}{dt} = e^{\xi_1} - e^{\xi_2}$$

$$\frac{d\xi_i}{dt} = \sum_{jk} \frac{\partial H^1}{\partial \xi_j} \frac{\partial H^2}{\partial \xi_k} \alpha_{ijk} h,\ 1 \le i \le 3$$

$$\alpha_{ijk} = \mathrm{sgn}\ \sigma,\ \sigma : (123) \mapsto (ijk)$$

$$H^1 = \tau_1\xi_1 + \tau_2\xi_2 + \tau_3\xi_3$$
$$H^2 = \tau_1 e^{\xi_1} + \tau_2 e^{\xi_2} + \tau_3 e^{\xi_3}$$
$$h = -(\tau_1\tau_2\tau_3)^{-1}$$