# 場と粒子の双対性

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## 1．非平衡熱力学の数理モデル

### Results

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u \nabla v)$$

$$\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0$$

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

**Theorem B** $T < +\infty \longrightarrow$

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$m(x_0) \in 8\pi\mathbf{N}$ collapse mass quantization possibly with sub-collapse collision

blowup set

exclusion of boundary blowup

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0, \ t_k \uparrow T, \ u(x_k, t_k) \to +\infty\} \subset \Omega$$

$\sharp\mathcal{S} < +\infty$ finiteness of blowup points

$0 < f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$ measure theoretic regular part

**Theorem C**

$$T = +\infty, \ \limsup_{t \uparrow +\infty} \|u(\cdot, t)\|_\infty = +\infty$$

$\longrightarrow$ $\lambda \equiv \|u_0\|_1 = 8\pi\ell, \ \exists \ell \in \mathbf{N}$ initial mass quantization

$\exists x_* \in \Omega^\ell \setminus D, \ \nabla H_\ell(x_*) = 0$ recursive hierarchy

point vortex Hamiltonian, Robin function, Green function

$$H_\ell(x_1, \cdots, x_\ell) = \frac{1}{2}\sum_j R(x_j) + \sum_{i<j} G(x_i, x_j)$$

**Corollary 1** $T < +\infty$ if $\nexists$ stationary solution or $\mathcal{F}(u_0) \ll -1$

and $\lambda \notin 8\pi\mathbf{N}$ or

$\lambda \in 8\pi\ell, \ \ell \in \mathbf{N}, \ \nexists$ critical point of $H_\ell$

**Corollary 2** $\Omega$ convex $\lambda \neq 8\pi$

$\Rightarrow \ T < +\infty$ or $T = +\infty$ pre-compact orbit

$\exists$ stationary solution

c.f. Grossi-F. Takahashi (2010)

## Bounded free energy and simplicity

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1)dx - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u\otimes u\ dxdx'$$

free energy of Helmholtz

- (entropy)　temperature　　inner energy　(self-attractive)

$$\frac{d\mathcal{F}}{dt} = -\int_\Omega u|\nabla(\log u - v)|^2 \leq 0,\ v = (-\Delta)^{-1}u$$

**Simplicity of the blowup points**

**Emergence**

$$T < +\infty,\ \lim_{t\uparrow T}\mathcal{F}(u(t)) > -\infty \longrightarrow \forall x_0 \in \mathcal{S} \xrightarrow{\text{simple}} \lim_{t\uparrow T}\mathcal{F}_{x_0,b(T-t)^{1/2}}(u(\cdot,t)) = +\infty,\ \forall b > 0$$

**Blowup rate** ↓

$$\lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,b(T-t)^{1/2})} = +\infty,\ \forall b > 0$$

$$\mathcal{F}_{x_0,R}(u) = \int_{\Omega\cap B(x_0,R)} u(\log u - 1)dx$$
$$-\frac{1}{2}\iint_{\Omega\cap B(x_0,R)\times(\Omega\cap B(x_0,R))} G(x,x')u\otimes u\ dxdx'$$

## Remark

**Rate of blowup is always type II**

$$\exists b > 0, \forall x_0 \in \mathcal{S},\ \lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,b(T-t)^{1/2})} = +\infty$$

## Supplements

## Systems on the whole space

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

$$u_t = \Delta u - \nabla \cdot (u \nabla \Gamma * u) \text{ in } \mathbf{R}^2 \times (0,T)$$

$$u|_{t=0} = u_0(x) \in L^\infty \cap L^1(\mathbf{R}^2)$$

$$\int_{\mathbf{R}^2} |x|^2 u_0 \, dx < +\infty \Longrightarrow \not\exists$$

dichotomy

concentrated compactness

$$0 < \lambda < 8\pi \Longrightarrow T = +\infty, \ \|u(\cdot,t)\|_\infty \leq C$$

$$\lambda > 8\pi \Longrightarrow T < +\infty$$

$$\lambda = 8\pi \Longrightarrow T = +\infty$$

vanishing

$$\lim_{t \uparrow +\infty} \|u(\cdot,t)\|_\infty = 0$$

compact (concentration)

$$\lim_{t \uparrow +\infty} \|u(\cdot,t)\|_\infty = +\infty$$

## Chemotaxis system in biology

$$u_t = \nabla \cdot (\nabla u - u \nabla v), \ -\Delta v = u - \frac{1}{|\Omega|} \int_\Omega u$$

$$\left(\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}, \frac{\partial v}{\partial \nu}\right)\Big|_{\partial \Omega} = 0, \ \int_\Omega v = 0$$

$$n = 2, \ T = T_{\max} < +\infty$$

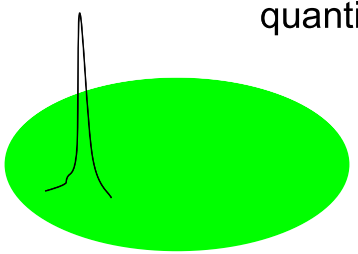
$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0) \delta_{x_0}(dx) + f(x)dx$$

existence of the boundary blowup

$$m(x_0) \in m_*(x_0)\mathbf{N}, \ m_*(x_0) = \begin{cases} 8\pi, & x_0 \in \Omega \\ 4\pi, & x_0 \in \partial\Omega \end{cases}$$

$$0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$$

quantized blowup mechanism



also in infinite time

Keller-Segel system

$$u_t = \nabla \cdot (d_1(u,v)\nabla u) - \nabla \cdot (d_2(u,v)\nabla v)$$
$$v_t = d_v \Delta v - k_1 vw + k_{-1} p + f(v)u$$
$$w_t = d_w \Delta w - k_1 vw + (k_{-1} + k_2)p + g(v,w)u$$
$$p_t = d_p \Delta p + k_1 vw - (k_{-1} + k_2)p$$

$u = u(x,t)$ cellular slime molds

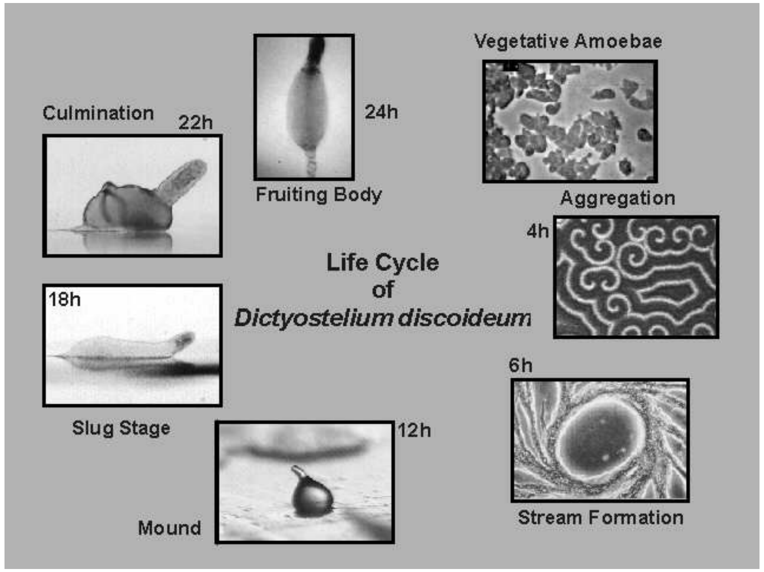
$v = v(x,t)$ attractant

$w = w(x,t)$ enzyme

$p = p(x,t)$ complex

1. transport, gradient
   (a) diffusion $u, v, w, p$
   (b) chemotaxis $v \to u$

2. production $u \to (v, w)$




moving clustered cells aggregating cells

3. chemical reaction $v, w, p$

$$V + W \underset{\leftarrow (k_{-1})}{\overset{(k_1) \rightarrow}{}} P \overset{(k_2)}{\rightarrow} W + A$$

$$v_t = -k_1 vw + k_{-1} p$$
$$w_t = -k_1 vw + (k_{-1} + k_2)p$$
$$p_t = k_1 vw - (k_{-1} + k_2)p$$

## Reductions

$$v_t = -k_1 vw + k_{-1} p$$
$$w_t = -k_1 vw + (k_{-1} + k_2) p$$
$$p_t = k_1 vw - (k_{-1} + k_2) p$$

$$k_1 vw - (k_{-1} + k_2) p = 0$$ quasi-static

$$w + p = c$$ total mass conservation

⟶

$$u_t = \nabla \cdot (d_1(u,v)\nabla u) - \nabla \cdot (d_2(u,v)\nabla v)$$
$$v_t = d_v \Delta v - k(v) v + f(v) u$$

$$k(v) = \frac{c k_1 k_2}{(k_{-1} + k_2) + k_1 v}$$

Nanjundiah 73

$$d_1(u,v), \ k(v), \ f(v)$$ constant

$$d_2(u,v) = u\chi'(v)$$ mass × velocity = flux (momentum)

sensitivity

$$u_t = d_u \Delta u - \nabla \cdot (u \nabla \chi(v))$$
$$v_t = d_v \Delta v - b_1 v + b_2 u$$

Childress-Percus 81, Jager-Luckhaus 92

Smoluchowski $$u_t = \nabla \cdot (\nabla u - u\nabla v), \ \left. \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0$$

Poisson $$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \ \int_\Omega v = 0, \ \left. \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0$$

## Spectral mechanics

Jager-Luckhaus model

$$u_t = \Delta u - \nabla\cdot(u\nabla v), -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega\times(0,T)$$

$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega\times(0,T)$$

$$u \geq 0, \ \frac{d}{dt}\|u(\cdot,t)\|_1 = 0$$

mass conservation

free energy decreasing

$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 dx$$

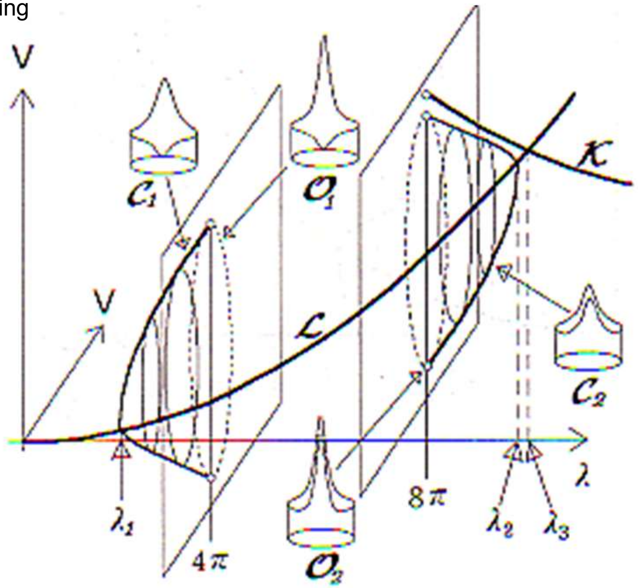
c.f.

$$-\Delta v = u, \ \ v|_{\partial\Omega} = 0$$

$$\Rightarrow \quad -\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \ \ v|_{\partial\Omega} = 0$$

Boltzmann-Poisson equation

Potentials of self-organization




Senba-S. 00

stationary state $\Rightarrow$

$$\log u - v = \text{constant}, \ \ \|u\|_1 = \lambda$$

$$\Rightarrow \quad u = \frac{\lambda e^v}{\int_\Omega e^v}$$

simplified system of chemotaxis

$$-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right), \ \int_\Omega v = 0$$

$$\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\lambda = 8\pi, \ 4\pi$$

interior boundary

Biler-Hilhorst-Nadieja 94, Nagai 95, Nagai-Senba-Yoshida 97, Gajewski-Zacharius 98, Biler 98

Higher-dimensional quantization $n > 2,\ m = \frac{n}{n-2}$

weak solution

$$u_t = \frac{m-1}{m}\Delta u^m - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^n \times (0,T)$$

$$u|_{t=0} = u_0(x) \geq 0 \in L^\infty \cap L^1(\mathbf{R}^n) \quad \int_{\mathbf{R}^n} |x|^2 u_0 dx < +\infty$$

$$\Gamma(x) = \frac{|x|^{2-n}}{|\partial B|},\ B = B(0,1) \quad \Longrightarrow \quad \frac{d}{dt}\int_{\mathbf{R}^n} u\ dx = 0$$

$$\frac{d}{dt}\mathcal{F}(u) = -\int_{\mathbf{R}^n} u|\nabla u^{m-1} - \Gamma * u|^2 dx \leq 0$$

$$\mathcal{F}(u) = \int_{\mathbf{R}^n} \frac{u^m}{m} - \frac{1}{2}\langle \Gamma * u, u\rangle$$

Tsallis entropy

**Theorem D** $T < +\infty \Longrightarrow \mathcal{S} \subset \mathbf{R}^n \quad \sharp\mathcal{S}_{II} < +\infty$

$$\mathcal{S} = \{x_0 \in \mathbf{R}^n \cup \{\infty\} \mid \exists x_k \to x_0,\ t_k \uparrow T,\ \lim_{k\to\infty} u(x_k,t_k) = +\infty\}$$

$$S_{II} = \{x_0 \in \mathcal{S} \mid \lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,r_0)} = +\infty,\ \forall r_0 > 0$$

S.-R. Takahashi 09a, 09b, 20, 12

Toward the theory of elliptic uniformization

$$-\Delta w = w_+^m \text{ in } \Omega,\ w = c \in \mathbf{R} \text{ on } \partial\Omega$$

$$\int_\Omega w_+^m = \lambda \quad (w_k, c_k, \lambda_k) \quad \lambda_k \to \lambda_0$$

solution sequence

$\Longrightarrow$ Subsequence alternatives

(a) $\|w_k\|_\infty \leq C$

(b) $\sup_\Omega w_k \to -\infty$

(c) $\lambda_0 = m_*\ell,\ \ell \in \mathbf{N}$

$$\mathcal{S} = \{x_1^*, \cdots, x_\ell^*\} \in \Omega$$

$$\nabla_{x_j} H_\ell(x)\big|_{x=x_*} = 0,\ 1 \leq j \leq \ell$$

$$x = (x_1, \cdots, x_\ell),\ x_* = (x_1^*, \cdots, x_\ell^*)$$

$$H_\ell(x_1, \cdots, x_\ell) = \frac{1}{2}\sum_j R(x_j) + \sum_{i<j} G(x_i, x_j)$$

## Systems with relaxation time

full system of chemotaxis

$$\varepsilon u_t = \nabla\cdot(\nabla u - u\nabla v)$$
$$\tau v_t = \Delta v + u \text{ in } \Omega\times(0,T)$$
$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, v\right)\Big|_{\partial\Omega} = 0$$

$\tau = 0$ Smoluchowski-Poisson quantized blowup mechanism

$$\varepsilon = 0 \quad v_t = \Delta v + \frac{\lambda e^v}{\int_\Omega e^v}, \ v|_{\partial\Omega} = 0$$

non-local parabolic equation

$$\Omega = B(0,1)\subset \mathbf{R}^2, \ v = v(|x|,t), \ \lambda \geq 8\pi$$

Wolansky 97

$$\longrightarrow \frac{\lambda e^v}{\int_\Omega e^v}dx \rightharpoonup \lambda\delta_0, \ t\uparrow T = T_{\max}\in(0,+\infty]$$

Kavallaris-S. 07

$$\lambda > 8\pi \ \Rightarrow \ T = T_{\max} < +\infty$$

**dis-quantized blowup mechanism**

## Duality between field and particles

Lagrangian in **Toland duality**

$$L(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}|\nabla v|^2 - vu \ dx$$

→ unfolding-minimality

$$\inf\{L(u,v) \mid u\geq 0, \|u\|_1 = 8\pi, \ v\in H_0^1(\Omega)\} > -\infty$$

**Model C equation**

$$\varepsilon u_t = \nabla\cdot(u\nabla L_u(u,v)), \ \tau v_t = -L_v(u,v)$$

## Models in non-equilibrium thermo-dynamics

**semi-unfolding-minimality**

infinitesimal stability → dynamical stabilitiy

local minimum of the analytic field functional → infinitesimal stable

S.-Tasaki 10

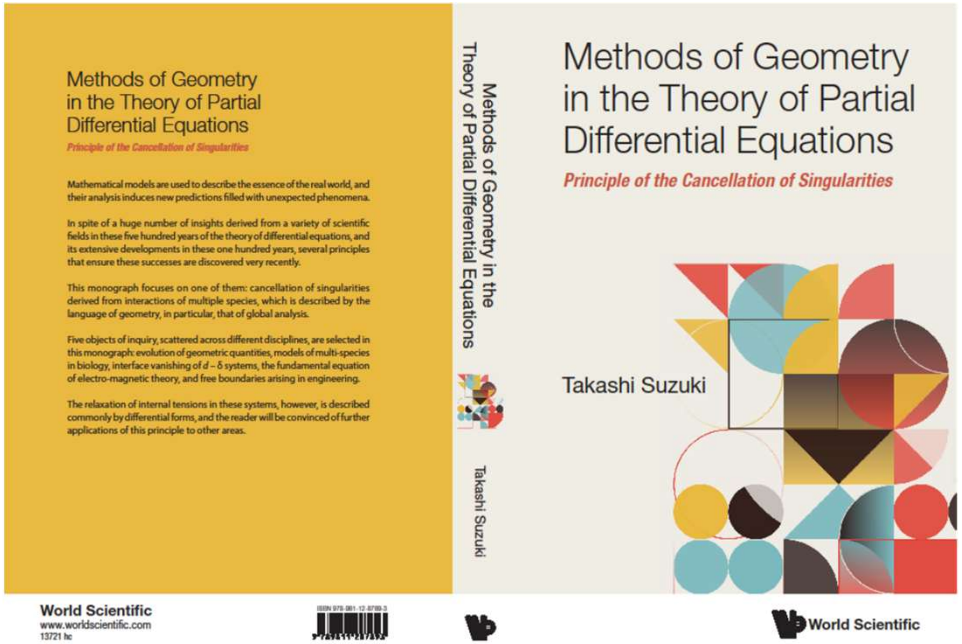
phase transition, phase separation, shape memory alloys

# Summary

1. Thermal equilibrium of the point vortex mean field is described by the Boltzmann Poisson equation.
2. Onsager's conjecture of the formation of an ordered structure in negative inverse temperature is realized as the Hamiltonian recurrence with quantized blowup mechanism.
3. Smoluchowski Poisson equation is the fundamental equation for canonical ensembles of Newtonian particles.
4. Its stationary state is the Boltzmann Poisson equation of which total set of solutions controls the global-in-time dynamics (potentials of self-organization).
5. As a consequence there is a quantization with Hamiltonian control in blowup in finite and infinite time.
6. If blowup in finite time occurs there is a formation of sub-collapses of normalized masses with a possible collision.
7. The residual part other than sub-collapses vanishes in the whole space of rescaled variables, called the parabolic envelope, while the motion of sub-collapses is controlled by the point vortex Hamiltonian in the rescaled variables.
8. As a consequence any blowup point is of type II, and if the free energy is bounded then it is simple, whereby the local free energy in the parabolic envelope diverges to plus infinity (emergence).
9. Blowup in infinite time, on the other hand, occurs only when the initial mass is quantized, whereby there is a formation of collapses with a normalized mass of which kinetics is subject to the anti-gradient system of Hamiltonian to create a clinic orbit of its critical points.
10. A relative of the Smoluchowski-Poisson equation is the simplified system of chemotaxis, where the Poisson part is modified.
11. The total set of stationary solutions, however, is quite different according to the form of the Poisson part.
12. There is a dis-quantized blowup mechanism if the model is provided with the relaxation time, which is nothing but the model B – model A equation derived from the Lagrangian associated with the Toland duality.
13. Several models in non-equilibrium thermo-dynamics are provided with the structure of semi-unfolding-minimality between the Lagrangian and the field functional, which induces a general criterion of the dynamical stability of the critical point if it is analytic.
14. Higher dimensional analogous of the 2D Smoluchowski Poisson equation is a degenerate parabolic equation associated with the Tsallis entropy, where the finiteness of type II blowup points is known.
15. It stationary state is realized as an elliptic free boundary problem provided with the quantized blowup mechanism controlled by the Hamiltonian.

# Open questions

1. Real rate of blowup (for the case of sub-collapse collision) partial answer
2. Any blowup point is of type II in the higher dimensional degenerate parabolic equation.
3. Hausdorff dimension of the blowup set of the higher dimensional Smoluchowski Poisson equation I less then or equal to (n-2). partial answer
4. Hamiltonian control of the blowup set is efficient to many elliptic problems.
5. There is a general view of dynamics in the models associated with the Kuhn Tucker duality.




World Scientific 2024

## 4．双対変分原理

### 統計集団と非平衡熱力学

S. Mean Field Theories and Dual Variation, 2nd edition, Atlantis Press, 2015

| system | consistency | dynamics | ensemble |
|---|---|---|---|
| isolated | energy | entropy | micro-canonical |
| closed | temperature | Helmholtz free energy | canonical |
| open | pressure | Gibbs free energy | grand-canonical |

### 場と粒子の双対性

**particle density**
Smoluchowski

**duality** ⟷

**field potential**
Poisson

$$v=(-\Delta)^{-1}u=\int_\Omega G(\cdot,x')u(x')dx'$$

symmetry

$$u_t=\nabla\cdot(\nabla u-u\nabla v)$$

$$\left.\frac{\partial u}{\partial \nu}-u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega}=0$$

$$-\Delta v=u \quad v|_{\partial\Omega}=0$$

**Helmholtz fee energy**

$$\mathcal{F}(u)=\int_\Omega u(\log u-1)-\frac{1}{2}\langle(-\Delta)^{-1}u,u\rangle$$

$$\delta\mathcal{F}(u)=\log u-(-\Delta)^{-1}u$$

**Model (B) equation**

$$u_t=\nabla u\cdot\nabla\delta\mathcal{F}(u),\ \left.\frac{\partial}{\partial\nu}\delta\mathcal{F}(u)\right|_{\partial\Omega}=0$$

total mass conservation, free energy decreaseing

$$\Longrightarrow\ \frac{d}{dt}\int_\Omega u=0,\ \frac{d\mathcal{F}}{dt}=-\int_\Omega u|\nabla\delta\mathcal{F}(u)|^2\le 0$$

## 双対変分原理

$X/\mathbf{R}$ 実バナッハ空間　　$F: X \to (-\infty, +\infty]$, prop. c'x, l.s.c. 適切, 凸, 下半連続

**ルジャンドル変換** $F^*(p) = \sup_{x \in X} \{\langle x, p\rangle - F(x)\}$　　**Fenchel-Moreau 双対** $F^{**} = F,\ F^{**}(x) = \sup_{p \in X^*} \{\langle x, p\rangle - F^*(p)\}$

**Toland 双対** $F, G: X \to (-\infty, +\infty]$, prop. c'x l.s.c.

$J(x) = G(x) - F(x),\ J^*(p) = F^*(p) - G^*(p)$

鈴木・田崎理論

$\frac{d}{dt} L(x(t), p(t)) \leq 0$ ⟶ 無限小安定（線形化安定より弱い）定常解は力学安定
非自明解析的非線形性のもとで極小臨界点は無限小安定

$L(x,p) = F^*(p) + G(x) - \langle x, p\rangle$ **ラグランジュ関数**

⟶

$\inf_{X \times X^*} L = \inf_X J = \inf_{X^*} J^*$ **ミニマリティ**

$L|_{p \in \partial G(x)} = J(x)$

$L|_{x \in \partial F^*(p)} = J^*(p)$ **アンフォールディング**

非平衡熱力学モデルの多くはセミ・アンフォールディング・ミニマリティをもつ

相転移・相分離・記憶形状　　Nonlinearity 2010

$p \in \partial G(x) \overset{\text{def}}{\Longleftrightarrow} G(y) - G(x) \geq \langle y - x, p\rangle,\ \forall y$

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$\inf\{\mathcal{F}(u) \mid u \geq 0,\ \|u\|_1 = 8\pi\} > -\infty$　　$n = 2$　　$\inf\left\{J_{8\pi}(v) \mid v \in H_0^1(\Omega)\right\} > -\infty$

双対

スモルコフスキー　**自由エネルギー** ⟷ **場の汎関数**　ポアソン

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle \qquad J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v + \lambda(\log\lambda - 1)$$

$u \geq 0,\ \|u\|_1 = \lambda$ **粒子密度**　　$v \in H_0^1(\Omega)$ **ポテンシャル分布**

# フォークモデル

マルテンサイティック相転移（NiTi金属）

変位 $u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x$

温度 $\theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt} \quad \text{in } 0 < x < 1,\ t > 0$

$(u_x, u_{xxx}, \theta_x)|_{x=0,1} = 0$ or $(u, u_{xx}, \theta_x)|_{x=0,1} = 0$

$(u, u_t, \theta)|_{t=0} = (u_0(x), u_1(x), \theta_0(x))$

臨界温度

$$f_i = F_i',\ i = 1, 2, \quad F_1(\varepsilon) = \alpha_1 \varepsilon^2, \quad F_2(\varepsilon) = \alpha_3 \varepsilon^6 - \alpha_2 \varepsilon^4 - \alpha_1 \theta_c \varepsilon^2$$

粘性項なし ⟹ 分散系

# 線形部分

ブシネスク

$$(\partial_t + \imath \partial_x^2)(\partial_t - \imath \partial_x^2) = \partial_t^2 + \partial_x^4$$

ストリッカーツ評価

⟷

熱

$$\partial_t - \partial_x^2$$

極大正則性

$$\left\| e^{\pm \imath t \partial_x^2} g \right\|_{L^4(0,T;L^4)} \le C \|g\|_2$$

$$\left\| \int_0^t e^{\pm \imath (t-1) \partial_x^2} f(\cdot, s) ds \right\|_{L^4(0,T;L^4)} \le C \|f\|_{L^{4/3}(0,T;L^{4/3})}$$

$$\left\| \int_0^t e^{-\imath (t-s) \partial_x^2} f(\cdot, s) ds \right\|_{L^\infty(0,T;L^2)} \le C \|f\|_{L^{4/3}(0,T;L^{4/3})}$$

$$\theta_t - \theta_{xx} = f(x,t),\ \ \theta_x|_{x=0,1} = 0,\ \ \theta|_{t=0} = 0$$

⟹

$$\int_0^T \|\theta_t\|_p^p + \|\theta_{xx}\|_p^p \, dt \le C(p,T) \int_0^T \|f(\cdot,t)\|_p^p \, dt$$

⟹ 局所適切性と大域存在 Yoshikawa 05

熱力学的構造

Falk model

$$u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x,\ \ \theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt} \qquad (u_x, u_{xxx}, \theta_x)|_{x=0,1} = 0 \qquad f_1(0) = f_2(0) = 0$$

エネルギー保存

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\|u_{xx}\|_2^2 = -\int_0^1 [f_1(u_x)\theta + f_2(u_x)]u_{xt}\ dx \quad \Longrightarrow \quad \frac{dE}{dt} = 0$$

$$= -\int_0^1 \theta_t - \theta_{xx}\ dx - \frac{d}{dt}\int_0^1 F_2(u_x)\ dx = -\frac{d}{dt}\int_0^t \theta + F_2(u_x)\ dx$$

$$E = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^t F_2(u_x) + \theta\ dx$$

$$\theta|_{t=0} = \theta_0 > 0 \quad \Longrightarrow \quad \theta(\cdot, t) > 0 \qquad \frac{1}{\theta}(\theta_t - \theta_{xx}) = f_1(u_x)u_{xt} = F_1(u_x)_t$$

エントロピー増大

$$\frac{dW}{dt} = -\int_0^1 \frac{\theta_{xx}}{\theta}\ dx = -\int_0^1 \left(\frac{\theta_x}{\theta}\right)^2\ dx \le 0 \qquad W = \int_0^1 F_1(u_x) - \log\theta\ dx$$

定常状態 $u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x,\ \theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt}$ $(u_x, u_{xxx}, \theta_x)|_{x=0,1} = 0$

$u_t = \theta_t = 0$ ⟶ $\theta = \overline{\theta} > 0$ constant associated with $u$ by $E(u, 0, \overline{\theta}) = b \equiv E(u_0, u_1, \theta_0)$

$u_{xxxx} = (\overline{\theta} f_1(u_x) + f_2(u_x))_x,\ (u_x, u_{xxx})|_{x=0,1} = 0$

$$E(u, u_t, \theta) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^1 F_2(u_x) + \theta\ dx$$

変分汎関数 $J_1(u_x) = \int_0^1 F_1(u_x)dx$

⟶ $\overline{\theta} + \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^1 F_2(u_x)dx = b$

$$J_2(u_x) = \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^1 F_2(u_x)dx$$

⟶ $\delta J_2(u_x) = \overline{\theta}\delta J_1(u_x),\ \overline{\theta} = b - J_2(u_x)$ ⟶ $\delta J_2(u_x) = (b - J_2(u_x))\delta J_1(u_x)$

⟶ $\delta J_b(u_x) = 0, \quad J_b(u_x) = J_1(u_x) - \log(b - J_2(u_x))$

$v = u_x$ ⟶ $J_b = J_b(v),\ v \in V_b, \quad V_b = \{v \in H_0^1 \mid J_b(v) < b\}$

$\delta J_b(\overline{v}) = 0,\ \overline{v} \in V_b$ 定常状態

セミ・ミニマリティ

$$b = E(u, u_t, \theta) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^1 F_2(u_x) + \theta \; dx \geq \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^1 F_2(u_x) + \theta \; dx$$

$$W(u_x, \theta) = \int_0^1 F_1(u_x) - \log\theta \; dx \geq \int_0^1 F_1(u_x)dx - \log\left(\int_0^1 \theta dx\right)$$

$$\Longrightarrow \quad \geq \int_0^1 F_1(u_x)dx - \log\left(b - \frac{1}{2}\|u_{xx}\|_2^2 - \int_0^1 F_2(u_x)dx\right) = J_b(u_x)$$

セミ・アンフォールディング $\overline{\theta} = b - J_2(u_x) > 0 \Longrightarrow W(u_x, \overline{\theta}) = J_b(u_x)$

$$\Longrightarrow \boxed{W(u_x, \theta) \geq W(u_x, \overline{\theta}) = J_b(u_x)}$$

$$J_b(u_x) - J_b(\overline{v}) \leq W(u_{0x}, \theta_0) - J_b(\overline{v}) = W(u_{0x}, \theta_0) - W(\overline{v}, \overline{\theta})$$

$$W(u_{0x}, \theta_0) - W(\overline{v}, \overline{\theta}) < \delta \Rightarrow J_b(u_x(\cdot, t)) - J_b(\overline{v}) < \delta$$

定義：無限小安定

$$\exists\varepsilon_0 > 0 \text{ s.t. } \forall\varepsilon \in (0, \frac{\varepsilon_0}{4}], \; \exists\delta > 0 \text{ s.t. } \|(v - \overline{v})_x\|_2 < \varepsilon_0, \; J_b(v) - J_b(\overline{v}) < \delta \Rightarrow \|(v - \overline{v})_x\|_2 < \varepsilon$$

$u_x \in C([0, +\infty); H_0^1) \Longrightarrow (\overline{v}, \overline{\theta})$ は力学系安定 S.-Yoshikawa 07

**実解析性**

$f_1(v) \neq 0, \quad v \neq 0$

$F_i$, $i = 1, 2$: real analytic

鈴木・田崎理論 $\Longrightarrow$ すべての極小は無限小安定

**分岐解析**

$\mathcal{C} = \{(b, v)\}$: total set of solutions

1. $v = 0$...the trivial solution, $^{\forall} b$

2. $b_k = \theta_c - \frac{k^2\pi^2}{2\alpha_1}$, $k = 1, 2, \cdots$
...bifurcation points
(for normalized physical constants)

3. $\forall v \neq 0$, stationary state, generates
a one-dimensional manifold $\subset \mathcal{C}$
(branch)

$(b(s), v(s))$, $|s| \ll 1$
the bifurcated branch
from the trivial solution at $b = b_1$
$\Rightarrow$

$$\ddot{b}(0) = -\alpha_1\theta_c + \frac{\pi^2}{2} + \frac{3\alpha_2}{\alpha_1}$$

$\ddot{b}(0) > 0$

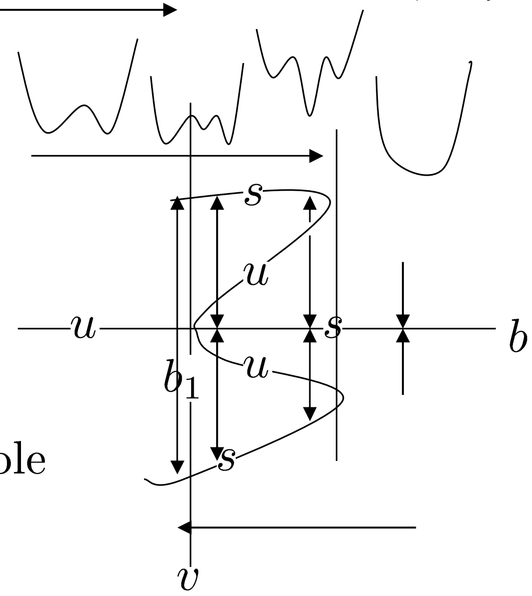
$\Rightarrow$ oscillation of $b$

$\Rightarrow$ hysteresis

$\Rightarrow$ hetero-clinic orbits

non-trivial global minimizer

極小の多重存在




$b < b_1 \Rightarrow v = 0$: linearly unstable

$^{\exists}$ non-trivial global minimum

## 気体分子の動力学

$v = v(x,t)$ velocity

$\{T(t,s)\}$ propagator

$x(t) = T(t,s)\xi$

$\Leftrightarrow$

$$\frac{dx}{dt} = v(x,t),\ x|_{t=s} = \xi$$

## Flux

$$\frac{d}{dt}\int_\omega \rho dx = -\int_{\partial\omega} \nu \cdot j\ ds$$

$j = v\rho$

$\rho_t + \nabla \cdot v\rho = 0$, mass conservation

## State Equation

$p = A\rho^\gamma$, $A > 0$, $1 < \gamma < 2$

## Liouville's Theorem

$x = x(\xi,t) \ \leftrightarrow \ x = T(t,s)\xi$

$$J_t = \det\left(\frac{\partial x_i}{\partial \xi_j}\right)$$

$\Rightarrow$

$$J_t(\xi) = 1 + (t-s)\nabla \cdot v(\xi,s) + o(t-s)$$

## Accelation

$$\left.\frac{d^2x}{dt^2}\right|_{t=s} = \left.\frac{d}{ds}v(x(t),t)\right|_{t=s} = \left.\frac{Dv}{Dt}\right|_{t=s}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla,$$ material derivative

$\rho$ density, $p$ pressure

$\Rightarrow$

$$\rho\frac{Dv}{Dt} = -\nabla p,$$ equation of motion

## Momentum Balance

$$\left.\frac{d}{dt}\int_{T(t,s)\omega} \rho(x,t)v(x,t)dx\right|_{t=s}$$
$$= \int_\omega \frac{D}{Dt}(\rho v) + \rho v \nabla \cdot v\ dx$$
$$= \int_\omega (\rho v)_t + \nabla \cdot (\rho v \otimes v)\ dx$$
$$= -\int_\omega \nabla p\ dx$$

## Mass Balance

$$0 = \left.\frac{d}{dt}\int_{T(t,s)\omega} \rho(x,t)dx\right|_{t=s}$$
$$= \left.\frac{d}{dt}\int_\omega \rho(x(\xi,t),t)|J_t(\xi)|d\xi\right|_{t=s}$$
$$= \int_\omega \frac{D\rho}{Dt} + \rho\nabla \cdot v\ d\xi$$
$$= \int_\omega \rho_t + \nabla \cdot v\rho\ d\xi$$

圧縮性オイラー方程式

$$\rho_t + \nabla \cdot v\rho = 0$$
$$\rho(v_t + v \cdot \nabla v) + \nabla p = 0$$
$$p = \rho^\gamma \text{ in } \mathbf{R}^n \times (0,T)$$
$$\gamma > 1$$

**Total Mass Conservation**

$$\frac{d}{dt}\int \rho = -\int \nabla \cdot v\rho = 0$$
$$\rho \geq 0$$
$$\|\rho\|_1 = \|\rho_0\|_1 = M$$

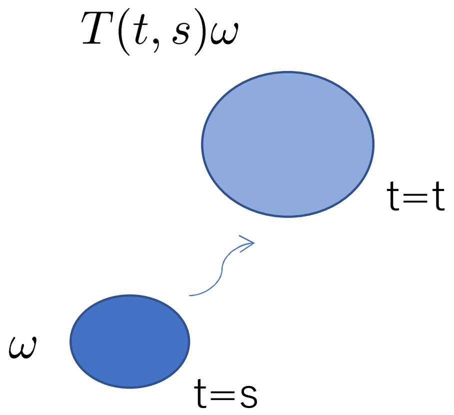
**Total Energy Conservation**

$$\left(\frac{1}{2}\rho|v|^2 + \frac{p}{\gamma - 1}\right)_t$$
$$+\nabla \cdot \left(\frac{1}{2}\rho|v|^2 + \frac{\gamma p}{\gamma - 1}\right) v = 0$$
$$\frac{d}{dt}\int \frac{1}{2}\rho|v|^2 + \frac{p}{\gamma - 1}\ dx = 0$$

**Total Energy Conservation - 2**

$$(\rho v)_t + \nabla \cdot (\rho v \otimes v) + \nabla p = 0$$

$$\int [\nabla \cdot (\rho v \otimes v)] \cdot v = -\int \rho v^j v^i \partial_j v^j$$
$$= -\frac{1}{2}\int \rho v^j \partial_j |v|^2 = \frac{1}{2}\int |v|^2 \nabla \cdot (\rho v)$$
$$= -\frac{1}{2}\int |v|^2 \rho_t$$
$$\int (\rho v)_t \cdot v = \int \rho_t |v|^2 + \frac{1}{2}\rho \partial_t |v|^2$$
$$\int [(\rho v)_t + \nabla \cdot (\rho v \otimes v)] \cdot v = \frac{1}{2}\frac{d}{dt}\int |v|^2 \rho$$




$$\int \nabla p \cdot v = \frac{\gamma}{\gamma - 1}\int \nabla \rho^{\gamma - 1} \cdot \rho v$$
$$= \frac{\gamma}{\gamma - 1}\int \rho^{\gamma - 1}\rho_t = \frac{d}{dt}\int \frac{p}{\gamma - 1}$$

$$\frac{d}{dt}\int \frac{1}{2}\rho|v|^2 + \frac{p}{\gamma - 1}\ dx = 0$$

カイネティック形式（ペルタム）

$$(|x-tv|^2\rho)_t + \nabla\cdot|x-tv|^2\rho v$$
$$= \nabla\cdot\left(2txp - \frac{2\gamma t^2}{\gamma-1}vp\right)$$
$$-\left(\frac{2t^2p}{\gamma-1}\right)_t - 2\left(n - \frac{2}{\gamma-1}\right)tp$$

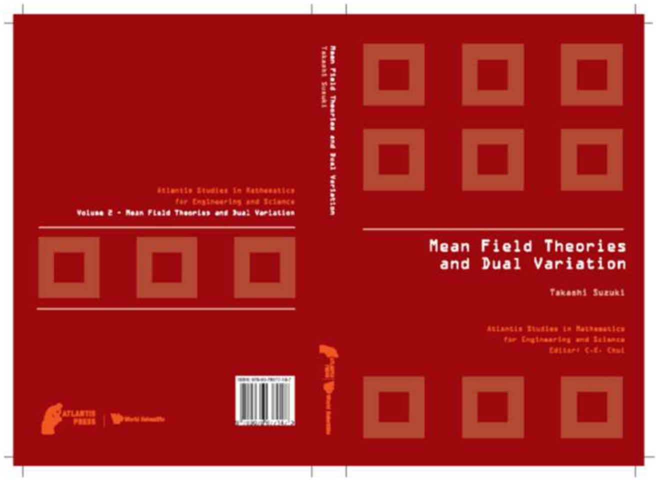
$$\frac{d}{dt}\int |x-tv|^2\rho + \frac{2t^2p}{\gamma-1}\ dx$$
$$+\int \frac{n(\gamma-1)-2}{t}\cdot\frac{2t^2p}{\gamma-1} = 0$$

$\Longrightarrow$

$$\int p dx = o(1)$$

$$\rho_t + \nabla\cdot v\rho = 0$$
$$\rho(v_t + v\cdot\nabla v) + \nabla p = 0$$
$$p = \rho^\gamma \text{ in } \mathbf{R}^n \times (0,T)$$

$$v\cdot\nabla v = \nabla\frac{1}{2}|v|^2 - v\times\omega$$
$$\omega = \nabla\times v$$




ハミルトン形式

$$E = \int \frac{\rho}{2}|v|^2 + \frac{\rho^\gamma}{\gamma-1}\ dx$$
$$\frac{\delta E}{\delta\rho} = \frac{1}{2}|v|^2 + \frac{\gamma\rho^{\gamma-1}}{\gamma-1}$$
$$\frac{\delta E}{\delta v} = \rho v$$

$$\rho_t + \nabla\cdot\frac{\delta E}{\delta v} = 0$$
$$v_t + \nabla\frac{\delta E}{\delta\rho} = \frac{\delta E}{\delta v}\times\omega/\rho$$
$$\Rightarrow$$
$$\frac{d}{dt}E(\rho,v) = \langle\frac{\delta E}{\delta\rho},\rho_t\rangle + \langle\frac{\delta E}{\delta v},v_t\rangle$$
$$= -\int\nabla\cdot(\frac{\delta E}{\delta\rho}\frac{\delta E}{\delta v}) = 0$$

渦なし流の有限時間消滅