偏微分方程式論における
幾何学的方法 V

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2D正規化リッチ流

ポアンカレ予想とリッチ流

Ricci flow approach to the Poincare-Thurston conjecture

bridge initial and expected ultimate metrics
geometric process of surgery at the formation of singularities

2D NRFについて

Hamilton 82

$$\frac{\partial g}{\partial t} = (r-R)g \quad \text{on } \Omega \times (0,T)$$

Ω compact Riemann surface

$g = g(t)$ metric

$R = R(t)$ scalar curvature

$r(t) = \frac{\int_\Omega R(t) d\mu_t}{\int_\Omega d\mu_t}$ average volume

$\mu = \mu_t$ volume element

Theorem (Hamilton 88)

$$g(t) \to g_\infty \text{ in } C^\infty \text{ as } t \uparrow \infty$$

metric with constant scalar curvature

Remark

1. No singularity in 2D-NRF
2. $$\frac{\partial R}{\partial t} = \Delta R + R(R-r), \ \Delta = \Delta_g$$

Henceforth, assume R>0

Gauss-Bonnet $R_0 > 0$ ⟹ genus=0

Chow 91 genus=0 $R > 0$ for $t \gg 1$

ハミルトンの定理

geometric part

$$\Delta f = R - r$$ curvature potential

$$M_g = \nabla\nabla f - \frac{1}{2}\Delta f \cdot g$$ trace free part

co-variant derivative

Lie derivate

→ $$2M_g = (r - R)g + \mathcal{L}_{\nabla f} g$$

→ modified Ricci flow

$$\frac{\partial \tilde{g}}{\partial t} = 2M_{\tilde{g}}, \ \tilde{g} = \tilde{g}(t) = T_t^* g(t), \ \{T_t\} \ \leftrightarrow \ \nabla f$$

$$|M_g|^2 = |\nabla\nabla f|^2 - \frac{1}{2}(\Delta f)^2$$

$$|M_g|^2 = |M_{\tilde{g}}|^2$$ invariant under the semigroup

analytic part

c.f. Smoluchowski-Poisson equation

$$\frac{d}{dt}\int_\Omega R \log R \, d\mu \leq 0 \ \Rightarrow \ \sup_{t\in(0,\infty)} \|R_t\|_\infty < +\infty$$

→ $$\inf_{0<t<\infty} \min_\Omega R_t > 0$$ Harnack inequality of Li-Yau type

$$\frac{\partial}{\partial t}|M_g|^2 = \Delta |M_g|^2 - 2|\nabla M_g|^2 - 2R|M_g|^2$$

→ comparison theorem

$$|M_g|^2 \leq Ce^{-\gamma t}, \ \gamma > 0$$

conclusion

$$\tilde{g}(t) = T_t^* g(t) \to \tilde{g}_\infty \text{ in } C^\infty \text{ as } t \uparrow \infty, \ M_{\tilde{g}_\infty} = 0$$

$$g(t) \to g_\infty \text{ in } C^\infty \text{ as } t \uparrow \infty, \ R_{g_\infty} = \text{constant}$$

力学系の観点から

Summary

1. Increase of the surface entropy provides an a priori estimate
2. Harnack inequality implies reverse inequality
3. Convergence of the transformed metric to the trivial state in infinite time
4. Convergence of the original metric
5. Geometric structure guarantees these transformations and a priori estimates

Achievement in the theory of dynamical systems
1. global in time existence of the solution
2. pre-compactness of the orbit
3. uniqueness of the omega-limit set
4. exponential rate convergence

Analytic proof
1. Trudinger-Moser-Fontana inequality
2. Benilan-Crandall inequality
3. Concentration compactness
4. Gradient inequality

Bartz-Struwe-Ye 94, Struwe 02
1. Modified flow by covariant and Lie derivative
2. Moving spheres based on the symmetry
3. Bochner-Weitzenbock, Harnack
4. Conformal transformation on the sphere

→

Analytic formulation

Gauss-Bonnet

genus

$$\int_\Omega R_g d\mu_g = 4\pi\chi(\Omega), \ \chi(\Omega) = 2 - 2g(\Omega)$$

$$R_g > 0 \ \Rightarrow \ g(\Omega) = 0$$

standard metric

uniformization theorem $\Omega = S^2, \ g = e^w g_0$

$$\Delta = \Delta_{g_0}, \ dx = d\mu_{g_0}, \ R_0 = R_{g_0}$$

⟹

$$R_g = e^{-w}(-\Delta w + R_0), \ |\Omega| R_0 = 8\pi$$

$$\int_\Omega R_g d\mu_g = 8\pi, \ r = \frac{\int_\Omega R_g d\mu_g}{\int_\Omega d\mu_g} = \frac{8\pi}{\int_\Omega e^w dx}$$

$$\frac{\partial g}{\partial t} = (r - R)g$$ ⟶

$$\boxed{\frac{\partial e^w}{\partial t} = \Delta w + 8\pi\left(\frac{e^w}{\int_\Omega e^w dx} - \frac{1}{|\Omega|}\right) \text{ in } \Omega \times (0, T)}$$

対数拡散

constant　compact Riemann surface without boundary

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda\left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|}\right) \text{ in } \Omega \times (0,T)$$

$$\longrightarrow \quad \frac{d}{dt}\int_\Omega e^w = 0$$

$$u = re^w, \ t \mapsto t' = r^{-1}t, \ r = \frac{\lambda}{\int_\Omega e^w}$$

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u \quad \|u_0\|_1 = \lambda$$

$$u_t = \Delta \log u \text{ in } \mathbf{R}^2 \times (0,T)$$ c.f. logarithmic diffusion

Hermholtz free energy

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle$$

$$\longrightarrow \quad \delta\mathcal{F}(u) = \log u - (-\Delta)^{-1}u$$

モデルB方程式

$$u_t = \Delta(\log u - v), \quad v = (-\Delta)^{-1}u$$

$$\longleftrightarrow \quad u_t = \Delta\delta\mathcal{F}(u)$$

$$\frac{d}{dt}\int_\Omega u = 0, \quad \frac{d}{dt}\mathcal{F}(u) = -\int_\Omega |\nabla\delta\mathcal{F}(u)|^2 \le 0$$

主定理 $0 < \lambda \le 8\pi \longrightarrow T = +\infty$

$$\|u(\cdot,t)\|_\infty + \|u(\cdot,t)^{-1}\|_\infty \le C$$

非退化であれば指数減衰

$\exists u_* = u_*(x) > 0$ 定常解 $\lim_{t\uparrow+\infty}\|u(\cdot,t) - u_*\|_\infty = 0$

c.f. Smoluchowski-Poisson

$$u_t = \nabla\cdot(\nabla u - u\nabla v), \ -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \ \int_\Omega v = 0$$

$$\longleftrightarrow \quad u_t = \nabla\cdot u\nabla\delta\mathcal{F}(u) \ \text{ in } \Omega \times (0,T)$$

possible blowup in infinite time for $\lambda = 8\pi$

有限時間爆発の条件

$$\int_\Omega u = \lambda \equiv \|u_0\|_1$$

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u, \ \ u|_{t=0} = u_0(x) > 0$$

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u \le \Delta \log u + u$$

$$T < +\infty \ \ \Rightarrow \ \ \lim_{t\uparrow T} \min_\Omega u(\cdot, t) = 0$$

$$w = \log u \ \longrightarrow \ \frac{\partial e^w}{\partial t} = \Delta w + \lambda\left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|}\right)$$

$$\begin{aligned} J_\lambda(w) &= \frac{1}{2}\|\nabla w\|_2^2 - \lambda \log \int_\Omega e^{w-\overline{w}} \\ &= \frac{1}{2}\|\nabla w\|_2^2 - \lambda(\log \int_\Omega e^w - \overline{w}) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}J_\lambda(w) &= \int_\Omega \nabla w \cdot \nabla w_t - \lambda\left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|}\right) w_t \ dx \\ &= -\int_\Omega e^w w_t^2 \le 0 \end{aligned}$$

$$\overline{w} = \frac{1}{|\Omega|}\int_\Omega w \overset{\text{Jensen}}{\le} \log\left(\frac{\lambda}{|\Omega|}\right), \ \int_\Omega e^w = \lambda$$

Trudinger-Moser-Fontana inequality (94)

$$\inf_E J_{8\pi} > -\infty, \ \ E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$$

劣臨界質量

$$\lambda < 8\pi \ \longrightarrow \ \|\nabla w\|_2 \le C, \ \overline{w} \ge -C$$

$$\xrightarrow{\text{Poincare-Wirtinger}} \ \|w\|_{H^1} \le C$$

$$\|\log u\|_{H^1(\Omega)} \le C \ \xrightarrow{\text{TMF inequality}}$$

$$\|e^{p \log u}\|_1, \ \|e^{-p\log u}\|_1 \le C_p, \ p \ge 1$$

$$\|u(\cdot,t)\|_p + \|u(\cdot,t)^{-1}\|_p \le C_p, \ 1 \le p < \infty$$

モーザーの反復スキーム

$$u_t = \Delta \log u + u - \alpha, \ \alpha = \frac{\lambda}{|\Omega|}$$

スモルコフスキー・ポアソンと同様

$$\frac{1}{p+1}\frac{d}{dt}\|u\|_{p+1}^{p+1} + \frac{4}{p}\|\nabla u^{\frac{p}{2}}\|_2^2 + \alpha\|u\|_p^p = \|u\|_{p+1}^{p+1} = \|u^{\frac{p}{2}}\|_{2\frac{p+1}{p}}^{2\frac{p+1}{p}}$$

ガリヤード・ニレンバーグ

$$\|z\|_q^q \leq C_{q_0}\|z\|_{H^1}^{q-1}\|z\|_1, \ 1 \leq q \leq q_0 < \infty \qquad n = 2$$

$$q = 2\frac{p+1}{p} \in [2, 4]$$

⟹ ヤング, ポアンカレ・ワーティンガー

$$\frac{d}{dt}\|u\|_{p+1}^{p+1} + 2\|\nabla u^{\frac{p}{2}}\|_2^2 \leq C \cdot p^2 \left(\|u^{\frac{p}{2}}\|_1^{\frac{2p}{p-1}} + 1\right) \quad p \geq 3$$

$$\|u\|_{p+1}^{p+1} \leq \|\nabla u^{\frac{p}{2}}\|_2^2 + C\left(\|u^{\frac{p}{2}}\|_1^{\frac{2p}{p-2}} + 1\right)$$

⟹
$$\frac{d}{dt}\|u\|_{p+1}^{p+1} + \|u\|_{p+1}^{p+1} \leq Cp^2\left(\|u^{\frac{p}{2}}\|_1^{\frac{2p}{p-2}} + 1\right) \leq Cp^2\left(\|u\|_{\frac{3}{4}(p+1)}^{p+1} + 1\right)$$

⟹

$$\|u(\cdot, t)\|_\infty + \|u(\cdot, t)^{-1}\|_\infty \leq C$$

$$T = +\infty$$

pre-compactness of the orbit

c.f. スモルコフスキー・ポアソン方程式

臨界質量 $\lambda = \int_\Omega e^w = 8\pi$ →

$$J_{8\pi}(w) = \frac{1}{2}\|\nabla w\|_2^2 - 8\pi \log(\int_\Omega e^w - \overline{w})$$
$$= \frac{1}{2}\|\nabla w\|_2^2 + 8\pi\overline{w} - 8\pi\log(8\pi)$$

$$-C \leq \frac{1}{2}\|\nabla w\|_2^2 + 8\pi\overline{w} \leq C \qquad \overline{w} \leq C$$

$T < +\infty$ → $\liminf_{t\uparrow T} \overline{w} = -\infty$

ベニランの不等式 $u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u$

$$u_{tt} = \Delta(\frac{u_t}{u}) + u_t$$

$$p_t = e^{-w}\Delta p + p - p^2, \ p = \frac{u_t}{u}$$

$$\overline{p}_t = \overline{p} - \overline{p}^2, \ \overline{p} = \frac{e^t}{e^t - 1}$$

→ $\frac{u_t}{u} \leq \frac{e^t}{e^t-1}$

→ $\frac{\partial}{\partial t}\left(\frac{u}{e^t - 1}\right) \leq 0$ → pointwise $\exists \lim_{t\uparrow T} u(x,t) = u(x,T)$

TMF inequality to $w/2$

$$\frac{1}{2}\int_\Omega |\nabla w|^2 \geq 4\cdot 8\pi \log\int_\Omega e^{w/2} - 16\pi\overline{w} - C$$

recall $-C \leq \frac{1}{2}\|\nabla w\|_2^2 + 8\pi\overline{w} \leq C$

→ $\overline{w} \geq 4\log\int_\Omega e^{w/2} - C$

→ $\liminf_{t\uparrow T}\int_\Omega e^{w/2} = \liminf_{t\uparrow T}\int_\Omega u(\cdot,t)^{1/2}$

dominated convergence theorem $= \int_\Omega u(\cdot,T)^{1/2} = 0$

$$u(\cdot,T) = 0 \text{ a.e. in } \Omega$$

$8\pi = \lim_{t\uparrow T}\int_\Omega u(\cdot,t) = \int_\Omega u(\cdot,T) = 0$ a contradiction

軌道のコンパクト性 $\lambda = 8\pi,\ T = +\infty$

$\overline{w} \geq -C \longrightarrow \|u(\cdot,t), u(\cdot,t)^{-1}\|_\infty \leq C$

assume $\liminf_{t\uparrow+\infty} \overline{w} = -\infty$ エントロピー $H = \int_\Omega e^w w$

$$\frac{dH}{dt} = -\|\nabla w\|_2^2 + H - 8\pi\overline{w} \leq H + 8\pi\overline{w} + C$$

$\exists t_k \uparrow +\infty,\ \exists\delta > 0$ Benilan-Crandall inequality

$$8\pi\overline{w}(t) + C \leq -k,\ t_k - \delta < t < t_k$$

$$H \geq -e|\Omega|$$

$\longrightarrow$

$$\lim_{k\to\infty} \inf_{t_k-\delta<t<t_k-\delta/2} H(t) = +\infty$$

TMF inequality

$$\sum_{k=1}^\infty \int_{t_k-\delta}^{t_k-\delta/2} dt \int_\Omega e^w w_t^2 \leq \int_0^\infty dt \int_\Omega e^w w_t^2 < +\infty$$

$$\|e^w w_t\|_1^2 \leq \int_\Omega e^w \cdot \int_\Omega e^w w_t^2 = 8\pi \int_\Omega e^w w_t^2$$

$$\longrightarrow \lim_{k\to\infty} \int_{t_k-\delta}^{t_k-\delta/2} \|e^w w_t(\cdot,t)\|_1^2\, dt = 0$$

$$t_k - \delta < \exists t_k' < t_k - \delta/2$$

$$\lim_{k\to\infty} \left\| \frac{\partial e^w}{\partial t}(\cdot, t_k') \right\|_1 = 0 \qquad \lim_{k\to\infty} \int_\Omega e^w w(\cdot, t_k') = +\infty$$

確率測度の集中

Ω コンパクトリーマン面 $\partial\Omega = \emptyset$

グリーン関数 $G = G(x,x') \longleftrightarrow -\Delta_D$

$$P(\Omega) = \{\rho \in L^1(\Omega) \mid \rho \geq 0,\ \|\rho\|_1 = 1\}$$

$$\mathcal{K}(\rho) = \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')\rho\otimes\rho\ dxdx'$$ 内部エネルギー

$$\mathcal{E}(\rho) = -\int_\Omega \rho(\log\rho - 1)\ dx$$ エントロピー

TM不等式

$$\mathcal{I}_\beta = \mathcal{K} + \mathcal{E}/\beta \Longrightarrow \sup_{P(\Omega)} \mathcal{I}_{8\pi} < +\infty$$

定理 $\rho_k \in P(\Omega),\ k = 1,2,\cdots$ $\quad \Omega \hookrightarrow \mathbf{R}^N$

$$\lim_{k\to\infty} \mathcal{K}(\rho_k) = +\infty,\ \lim_{k\to\infty} \mathcal{I}_{8\pi}(\rho_k) = I_\infty > -\infty$$

$$\lim_{k\to\infty}\int_\Omega x\rho_k dx = x_\infty \in \mathbf{R}^N \Longrightarrow x_\infty \in \Omega$$

$$\rho_k(x)dx \rightharpoonup \delta_{x_\infty}(dx) \text{ in } \mathcal{M}(\Omega)$$

補題 $\forall d > 0, \forall m \in (0,1),\ \exists C > 0, \exists\beta > 8\pi$

$A_i \subset \Omega,\ i = 1,2$ measurable $\mathrm{dist}(A_1, A_2) \geq d$

$$\rho \in P(\Omega),\ \int_{A_i} \rho\ dx \geq m,\ i = 1,2 \Longrightarrow \mathcal{I}_\beta(\rho) \leq C$$

補題 $\exists c_0 > 0,\ \forall\rho \in P(\Omega)$

concentration function

$$Q(r) \equiv \sup_{y\in\Omega}\int_{\Omega\cap B(y,r)} \rho\ dx \geq c_0 r^2,\ 0 < r \leq 1$$

$$u_k = u(\cdot, t_k)$$

$$\lim_{k\to\infty} \int_\Omega e^w w(\cdot, t_k') = +\infty \longrightarrow \lim_{k\to\infty} \int_\Omega u_k \log u_k = +\infty$$

$$\longrightarrow \lim_{k\to\infty} \langle (-\Delta)^{-1} u_k, u_k \rangle = +\infty$$

$$\mathcal{F}(u_k) \leq \mathcal{F}(u_0)$$

subsequence

$$\Omega \hookrightarrow \mathbf{R}^N \qquad \lim_{k\to\infty} \int_\Omega x u_k = x_\infty \in \mathbf{R}^N$$

補題

$$\|u_k\|_1 = 8\pi$$

$$\mathcal{F}(u_k) \leq C$$

$$\lim_{k\to\infty} \langle (-\Delta)^{-1} u_k, u_k \rangle = +\infty$$

$$\lim_{k\to\infty} \int_\Omega x u_k(x) dx = x_\infty \in \mathbf{R}^N$$

$$\longrightarrow \quad x_\infty \in \Omega, \ u_k \rightharpoonup 8\pi \delta_{x_\infty}$$

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda\left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|}\right)$$

elliptic estimate (Brezis-Strauss)

$$\lim_{k\to\infty} \int_\Omega e^{w(\cdot, t_k')} = +\infty \qquad \text{Fatou}$$

$$G(x, x') \approx \frac{1}{2\pi} \log \frac{1}{\mathrm{dist}\ (x, x')}$$

$$\lim_{k\to\infty} \left\| \frac{\partial e^w}{\partial t}(\cdot, t_k') \right\|_1 = 0$$

$$e^{w(\cdot, t_k')} = u_k \rightharpoonup 8\,\pi \delta_{x_\infty} \quad \text{in } \mathcal{M}(\Omega)$$

$$w(\cdot, t_k') \to 8\pi G(\cdot, x_\infty) \text{ in } W^{1,q}(\Omega), \ 1 \leq q < 2$$

$$\int_\Omega e^w = 8\pi \qquad \text{a contradiction}$$

定常解の一意性

stationary problem $\longleftrightarrow$

$$-\Delta \log u_* = u_* - \frac{1}{|\Omega|}\int_\Omega u_*\, dx, \quad x \in \Omega, \quad \int_\Omega u_*\ dx = \lambda$$

$$u^* = \frac{\lambda e^{v^*}}{\int_\Omega e^{v^*} dx} \quad \updownarrow \quad v_* = \log u_* - \frac{1}{|\Omega|}\int_\Omega \log u_*$$

$$-\Delta v_* = \lambda \left(\frac{e^{v_*}}{\int_\Omega e^{v_*}\, dx} - \frac{1}{|\Omega|} \right), \ \int_\Omega v_*\, dx = 0 \ \leftrightarrow \ v_* \in E_\lambda$$

$$\longleftrightarrow \quad J_\lambda(v) = \frac{1}{2}||\nabla v||_2^2 - \lambda \log \int_\Omega e^v\, dx$$

$$V_0 = \{v \in H^1(\Omega) \mid \int_\Omega v\ dx = 0\}$$

Trudinger-Moser-Fontana $\quad v \in V_0,\ \|\nabla v\|_2 \le 1 \quad \Rightarrow \quad \int_\Omega e^{4\pi v^2}\, dx \le C$

$$0 < \lambda \le 8\pi \qquad E_\lambda = \{0\}$$

$$\longrightarrow$$

$$u(t) \to u_* \equiv \frac{\lambda}{|\Omega|} \qquad \text{in } C^\infty \text{ topology}$$

elliptic theories

(a) $\lambda = 8\pi, \quad \Omega = S^2$ (Hamilton)

(b) $\lambda = 8\pi,\ \Omega = \mathcal{T} \equiv \mathbf{R}^2/a\mathbf{Z} \times b\mathbf{Z},\ \frac{b}{a} \ge \frac{\pi}{4}$

Lin-Lucia 06

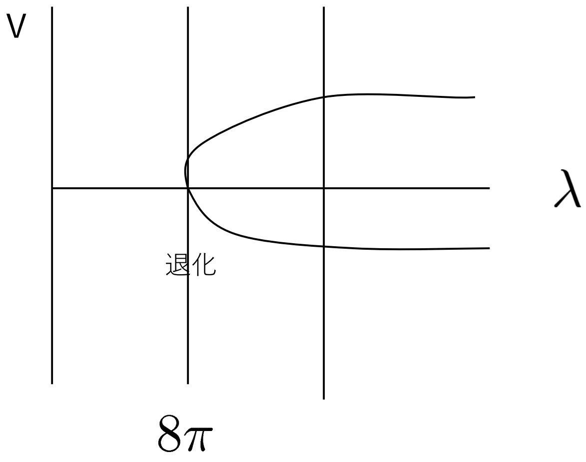
Remark $\quad E_\lambda$ may be a continuum elliptic theory

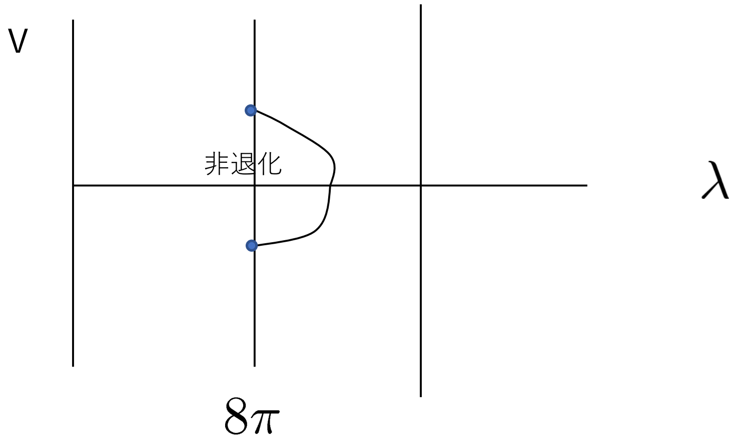
$$J_\lambda(v) = \frac{1}{2}||\nabla v||_2^2 - \lambda \log \int_\Omega e^v \, dx, \ V_0 = \{v \in H^1(\Omega) \mid \int_\Omega v \, dx = 0\}$$

$$\delta J_\lambda(v_*) = 0 \ \Leftrightarrow \ -\Delta v_* = \lambda(\frac{e^{v_*}}{\int_\Omega e^{v_*} \, dx} - \frac{1}{|\Omega|}), \ \int_\Omega v_* \, dx = 0$$

$\Omega = S^2$ (analytic proof)
Chanillo-Kiessling 95, Cheng-Lin 97, Lin 00

$\Omega = T^2 = \mathbf{R}^2/a\mathbf{Z} \times b\mathbf{Z}, \ \frac{b}{a} \geq \frac{\pi}{4}$ (Lin-Lucia 07)





定常解への収束

定理　$T=+\infty,\ \|u(t)\|_\infty + \|u(t)^{-1}\|_\infty \le C$ ⟶ $\exists u_* \in F_\lambda \quad u(t) \to u_* \quad \text{in } C^\infty \text{ topology}$

in algebraic order

u_* non-degenerate ⟶ in exponential order

注意

⟷ $v_* = \log u_* - \frac{1}{|\Omega|}\int_\Omega \log u_*$ non-degenerate critical point of $J_\lambda = J_\lambda(v),\ v \in V_0$

⟷ $\psi \in H^2(\Omega),\ -\Delta\psi = u^*\psi \text{ in } \Omega,\ \int_\Omega \psi u^*\, dx = 0 \quad \Rightarrow \quad \psi = 0$

$$u = e^w \qquad \frac{\partial e^w}{\partial t} = \Delta w + \lambda\Big(\frac{e^w}{\int_\Omega e^w\, dx} - \frac{1}{|\Omega|}\Big) \text{ in } \Omega \times (0,T)$$ ⟷ $$\frac{\partial e^w}{\partial t} = -\delta\mathcal{E}(w)$$

$$\mathcal{E}(w) = \int_\Omega \frac{1}{2}|\nabla w|^2 - e^w + \frac{\lambda}{|\Omega|} w\, dx, \quad w \in H^1(\Omega) = V$$

勾配不等式

$$w_* \in V,\ \delta\mathcal{E}(w_*) = 0 \ \Rightarrow\ 0 < \exists\theta \le \frac{1}{2},\ \exists\varepsilon_0 > 0$$

$$\forall w \in V,\ \|w - w_*\|_V < \varepsilon_0 \ \Rightarrow\ |\mathcal{E}(w) - \mathcal{E}(w_*)|^{1-\theta} \le C\|\delta\mathcal{E}(w)\|_{V^*}$$

u_* non-degenerate ⟶ $\theta = \frac{1}{2}$

勾配不等式

(Lojasiewicz 63, Simon 83)

補題 $E:\mathbf{R}^n\to\mathbf{R}$ real analytic at $x=0$

$$E(0)=0,\ \delta E(0)=0 \ \Longrightarrow\ 0<\exists\theta\le\frac{1}{2}$$

$$|E(x)|^{1-\theta}\le C|\delta E(x)|,\ |x|\ll 1$$

証明 $E(x)=\sum_{\alpha}C_\alpha x^\alpha$

$$\alpha=(\alpha_1,\cdots,\alpha_n),\ x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$$

suffices to assume $E\not\equiv 0 \ \Longrightarrow\ \exists m\ge 2$

$$C_\alpha=0,\ |\forall\alpha|\le m-1$$

$$C_\alpha\neq 0,\ |\exists\alpha|=m$$

$$E(x)=E_0(x)+E_1(x),\ E_0(x)=\sum_{|\alpha|=m}C_\alpha x^\alpha$$

$$|E_1(x)|=o(|x|^m),\ |E_1'(x)|=o(|x|^{m-1})$$

$$|E_0(x)|=r^m|E_0(\omega)|\le Cr^m,\ x=r\omega,\ r=|x|$$

$$\begin{aligned}|E_0'(x)| &= \{\sum_{i=1}^n(\frac{\partial E_0}{\partial x_i})^2\}^{1/2}\\ &= r^{m-1}\{\sum_{i=1}^n(\frac{\partial E_0}{\partial x_i}(\omega))^2\}^{1/2}\\ &\ge c_1r^{m-1},\ \exists c_1>0\end{aligned}$$

$$\dot{x}=-\delta E(x) \qquad \frac{d}{dt}E(x)=-|\dot{x}|^2\le 0$$

assume; global in time solution with pre-compact orbit

theory of dynamical systems. LaSalle principle

$$\Longrightarrow\ \exists t_k\uparrow+\infty,\ x(t_k)\to\exists x_*,\ \delta E(x_*)=0$$

$H=(E(x)-E(x_*))^\theta$ well-defined

$$\lim_{t\uparrow+\infty}H(x(t))=0$$

$\boxed{\dot{x} = -\delta E(x)}$

Haraux-Jendoubi 01

$$0 < \exists\theta \le \frac{1}{2} \qquad |E(x)|^{1-\theta} \le C|\delta E(x)|, \ |x| \ll 1$$

$$\exists t_k \uparrow +\infty, \ x(t_k) \to \exists x_*, \ \delta E(x_*) = 0$$

$$H = (E(x) - E(x_*))^\theta \qquad \lim_{t\uparrow+\infty} H(x(t)) = 0$$

$$\begin{aligned} -\frac{dH}{dt} &= \theta(E(x) - E(x_*))^{\theta-1}(-\dot{x}\cdot\delta E(x)) \\ &= \theta(E(x) - E(x_*))^{\theta-1}|\delta E(x)|^2 \\ &= \theta(E(x) - E(x_*))^{\theta-1}|\delta E(x)||\dot{x}| \\ &\ge c_2|\dot{x}|, \ \exists c_2 > 0 \end{aligned}$$

$$\Longrightarrow \int_0^\infty |\dot{x}| \, dt < +\infty$$

$$\exists \lim_{t\uparrow+\infty} x(t) = x_*$$

$$\begin{aligned} -\frac{dH}{dt} &= \theta(E(x) - E(x_*))^{\theta-1}|\delta E(x)|^2 \\ &\ge c_3(E(x) - E(x_*))^{1-\theta} \\ &= c_3 H^{\frac{1}{\theta}-1}, \ \exists c_3 > 0 \end{aligned}$$

$$H \le C\Phi, \ \Phi(t) = \begin{cases} t^{-\frac{\theta}{1-2\theta}}, & 0 < \theta < 1/2 \\ e^{-\delta_0 t}, & \theta = \frac{1}{2}, \ \delta_0 = c_3^{-1} \end{cases}$$

$$|x(s) - x(t)| \le C\Phi(t), \ s > t \quad \longleftarrow$$

$$s \uparrow +\infty \quad \Longrightarrow \quad |x(t) - x_*| \le C\Phi(t)$$

application to the NRF

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda(\frac{e^w}{\int_\Omega e^w\,dx} - \frac{1}{|\Omega|})$$

$$\frac{\partial e^w}{\partial t} = -\delta\mathcal{E}(w)$$

analytic

$$\mathcal{E}(w) = \int_\Omega \frac{1}{2}|\nabla w|^2 - e^w + \frac{\lambda}{|\Omega|} w\ dx$$

$$w \in H^1(\Omega) = V$$

realized as a self-adjoint operator in

$$X = L^2(\Omega)$$

$$w_* \in V,\ \ \delta\mathcal{E}(w_*) = 0$$

$$\mathcal{L} \equiv \delta^2\mathcal{E}(w^*) = -\Delta - e^{w^*} : V \to V^*$$

linearized operator

$$X_1 \equiv \mathrm{Ker}\ \mathcal{L} = \{v \in D(\mathcal{L}) \big| \mathcal{L}v = 0\} \subset V = H^1(\Omega)$$

$\mathcal{P} : V^* \to X_1$ orthogonal projection $\dim\ X_1 = n$

補題 (Chill 03, 06) $w_* \in \exists U \subset V$ neighborhood

$$\mathcal{S} = \{w \in U \big| (I - \mathcal{P})\delta\mathcal{E}(w) = 0\}$$

critical manifold

local analytic manifold around w_* with dimension n

$$\exists g : U_1 = U \cap X_1 \to U_2 = (I - \mathcal{P})U$$

analytic

$$g(w_1^*) = w_2^*,\ w^* = w_1^* + w_2^* \in U_1 \oplus U_2$$

$$\mathcal{S} = \{w_1 + g(w_1) \mid w_1 \in U_1\}$$

⟶ $0 < \exists\theta \leq \frac{1}{2}$

$$w \in V,\ \|w - w^*\|_V < \exists\varepsilon_0 \ \Rightarrow$$

$$|\mathcal{E}(w) - \mathcal{E}(w^*)|^{1-\theta} \leq C\|\delta\mathcal{E}(w)\|_{V^*}$$

$$\sup_{t_0 \leq t < t_0+T} \|w(\cdot,t) - w^*\|_V$$

parabolic regularity

$$\leq C(\|w(\cdot,t_0) - w^*\|_V + \sup_{t_0 \leq t < t_0+T} \|w(\cdot,t) - w^*\|_2)$$

⟶ convergence in algebraic rate

role of non-degeneracy $w^* = \log u^*$

$$-\Delta \log u_* = u_* - \frac{1}{|\Omega|}\int_\Omega u_*\, dx, \ \int_\Omega u_* \ dx = \lambda$$

non-degenerate

$\Longleftrightarrow$

$$v_* = \log u_* - \frac{1}{|\Omega|}\int_\Omega \log u_*$$

non-degenerate critical point of $J_\lambda = J_\lambda(v), \ v \in V_0$

$\Longleftrightarrow$ non-degeneracy of the linearized operator

$$\mathcal{B}\phi = -\Delta\phi - u^*\phi + \frac{1}{\lambda}(\phi, u^*)u^*$$

$$\phi \in D(\mathcal{B}) = H^2(\Omega) \cap V_0$$

$\Longleftrightarrow$ $\mathcal{M} = -\Delta - u^* : V = H^1(\Omega) \to V^*$

$$\phi \in V, \ \int_\Omega u^*\phi\, dx = 0 \quad \Rightarrow \quad ||\phi||_V \leq C||\mathcal{M}\phi||_{V^*}$$

$\Longleftrightarrow$ implicit function theorem

$$\exists \varepsilon_0 > 0, \ w \in V, \ \int_\Omega e^w\, dx = \lambda, \ \|w - w^*\|_V < \varepsilon_0$$

$$\Rightarrow \ \|w - w^*\|_V \leq C\|\mathcal{M}(w - w^*)\|_{V^*}$$

Lemma 3 u_* non-degenerate $\longrightarrow$ $\theta = \frac{1}{2}$

$$w \in V, \ \|w - w^*\|_V < \exists\varepsilon_0, \ \int_\Omega e^w = \lambda$$

$$\Rightarrow$$

$$|\mathcal{E}(w) - \mathcal{E}(w^*)|^{1-\theta} \leq C\|\delta\mathcal{E}(w)\|_{V^*}$$

中心多様体の消滅

$$\Omega = S^2,\ \lambda = 8\pi,\ u_* = \frac{\lambda}{|\Omega|} = 2$$

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u \text{ in } \Omega \times (0,+\infty)$$

$$\int_\Omega u(\cdot,t) = \lambda,\ \|u(\cdot,t)\|_\infty + \|u(\cdot,t)^{-1}\|_\infty \le C$$

$z = u - u_* \to 0$ in C^∞ at least in algebraic order

$$z_t = \Delta\varphi(z) + z$$

$$\begin{aligned}\varphi(z) &= \log u - \log u_* = \log(z+2) - \log 2\\ &= \log\left(\frac{z}{2}+1\right) = \frac{z}{2} + g(z)\end{aligned}$$

$$g(z) = -\frac{1}{2}\left(\frac{z}{2}\right)^2 + R(z) = -\frac{z^2}{8} + R(z)$$

$$|R(z)| \le C|z|^3$$

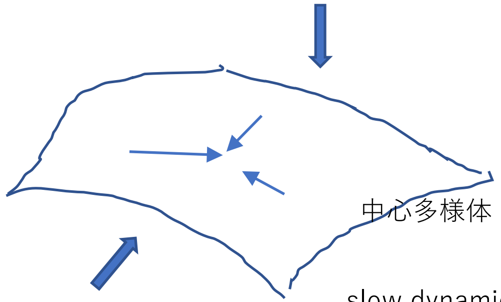
$$E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$$

$$H = \{v \in L^2(\Omega) \mid \int_\Omega v = 0\}$$

$$z_t + Lz = \boxed{\Delta g(z)},\ z \in E$$

$$L = -\frac{1}{2}(\Delta + 2) \text{ in } H$$

$$D(L) = H^2(\Omega) \cap E$$



$$\Omega = \{x \in \mathbf{R}^3 \mid |x| = 1\}$$

$$\sigma(-\Delta) = \{\mu_\ell \mid \ell = 0,1,2,\cdots\},\ \mu_\ell = \ell(\ell+1)$$

$$\sigma(L) = \{0\} \cup \{2 - \ell(\ell+1) \mid \ell = 2,3,\cdots,\}$$

multiplicity 3=2・1+1

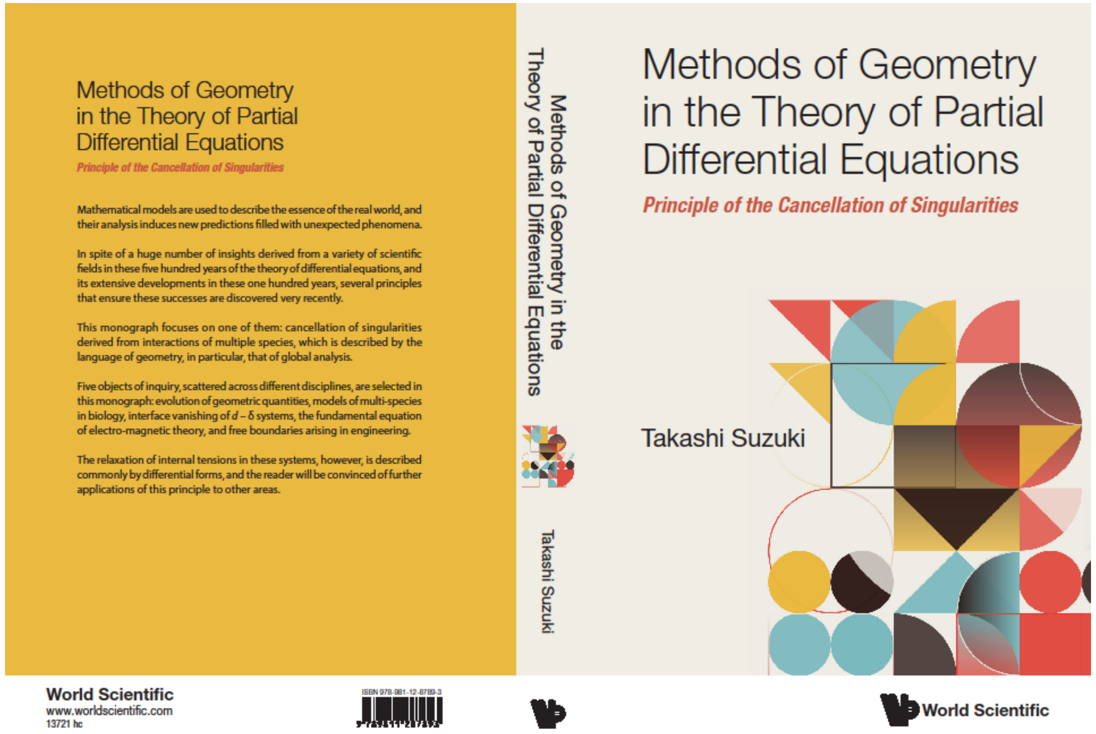
$$E_0 = \text{Ker } L,\ \dim E_0 = 3$$

まとめ

1. 熱平衡にある点渦平均場はBoltzmann-Poisson方程式で記述される
2. 負の逆温度での点渦秩序形成というOnsagerの予想は量子化する爆発機構として実現する
3. そこではさらにハミルトニアンが階層循環する
4. Smoluchowski-Poisson方程式はNewton粒子の正準集団が従う基礎方程式である
5. 定常状態はBoltzmann-Poisson方程式であり、その解集合が有限時間爆発も含めたダイナミクスを規定する（自己組織化のポテンシャル）
6. 結果として有限時間、無限時間爆発の量子化とハミルトニアンの循環が実現する
7. 有限時間爆発では規格化された質量をもつサブコラプスが衝突する
8. 有限時間爆発の無限大の放物包においてサブコラプス以外は真空となる（剰余項の消滅）
9. 同時にサブコラプスの運動は全空間の点渦Hamiltonianに関する反勾配系をスケール変換したものである.
10. 結果としてすべての爆発点の爆発レートはタイプIIとなり特に自由エネルギー有界の場合は単純である
11. すべての単純爆発点の放物包において局所自由エネルギーはプラス無限大となる（創発性）
12. 無限時間爆発は初期質量が量子化された場合のみに発生する。
13. そのとき生成されるコラプスはすべて単純でありその運動は点渦Hamiltonianに関する反勾配系に支配される、定常解を結ぶクリニック軌道が出現する
14. 動的循環的階層の原理は同一であってもポアソン部分によって定常状態の全体像は大きく変化する
15. 緩和時間がある場合には爆発機構は量子化しないが方程式はToland双対のラグランジュ関数を用いたモデルB-モデルA方程式として記述される
16. 非平衡熱力学のモデルの多くが半双対構造をもちラグランジュ関数と場の汎関数との間にsemi-unfolding-minimalityを実現している
17. 結果として場の汎関数の無限小安定臨界点は力学系安定である一方解析的な非線形項の下ですべての場の汎関数の極小点は無限小安定となる
18. ２D Smoluchowski-Poissonに対応する高次元モデルはTsallisエントロピーに付随する臨界指数の退化放物型方程式でありTypeII爆発点の有限性が成り立つ
19. その定常状態は非線形楕円型方程式の自由境界問題であり高次元の点渦ハミルトニアンはそこでも楕円型方程式の爆発状況を規定する（楕円型一様理論）
20. ２D-NRFは点渦ハミルトニアンに関するモデルB方程式であるが臨界質量においても爆発せず定常解に収束する
21. 定常解はそのレートは定常解が非退化であれば指数的であるがもとの幾何学的問題では退化している
22. R. Hamiltonによる幾何学的結果によりこの場合中心多様体は消滅していなければならない

未解決問題

1. 高次元退化放物型方程式において爆発点は常にtype IIであるか
2. 高次元Smoluchowski-Poisson方程式の爆発集合は n-2 次元以下であるか（部分的解決）
3. 2D NRFにおいて中心多様体が消滅するのはなぜか
4. 楕円型理論においてHamiltonianによる爆発点の制御はどこまで有効であるか
5. 平均場形式での解の非退化性はハミルトニアンと面積積分で定められるか
6. Kuhn-Tucker双対に由来する力学系はどのような描像を持つか



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