偏微分方程式論における幾何学的方法 IV

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スモフコフスキー・ポアソン方程式（続き）

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

$u = u(x,t) \geq 0$ density

$j = -\nabla u + u\nabla v$ flux (diffusion v.s. chemotaxis)

$u_t + \nabla \cdot j = 0$ conservation law

$v = (-\Delta)^{-1} u$ potential

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$\left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \ \ u|_{t=0} = u_0(x) > 0$$

2. Poisson Part

$$-\Delta v = u, \ \ v|_{\partial\Omega} = 0$$

Green's function

$$G(x,x') = G(x',x)$$

attractive (chemotaxis, gravitation)
action at a distance (long range potential)
symmetry (action-reaction)

Chavanis 08 relaxation to the equilibrium in the point vortices, kinetic equation + maximum entropy production

Sire-Chavanis 02 motion of the mean field of many self-gravitating Brownian particles, BBGKY hierarchy + factorization

canonical ensemble

1. total mass conservation $\frac{d}{dt}\|u(t)\|_1 = 0$

2. free energy decreasing

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u \otimes u$$

$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 \leq 0$$

self-similar transformation

$$u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t), \ \mu > 0$$

$$u_\mu(x) = \mu^2 u(\mu x), \ \ \mu > 0$$

$$\|u\|_1 = \|u_\mu\|_1 \equiv \lambda \ \Leftrightarrow \ n = 2$$ *critical dimension*

$$\mathcal{F}(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2}\langle \Gamma * u, u\rangle, \ \ \Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$\mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi}\right)\log\mu + \mathcal{F}(u)$$ *critical mass* $\lambda = 8\pi$

定理 A (有限時間爆発） $T < +\infty$

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$m(x_0) \in 8\pi\mathbf{N}$ collapse mass quantization possibly with sub-collapse collision

blowup set

exclusion of boundary blowup

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0,\ t_k \uparrow T,\ u(x_k,t_k) \to +\infty\} \subset \Omega$$

$\sharp\mathcal{S} < +\infty$ finiteness of blowup points

measure theoretic regular part

$$0 < f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$$

定理 B（無限時間爆発)

$$T = +\infty,\ \limsup_{t \uparrow +\infty} \|u(\cdot,t)\|_\infty = +\infty$$

⟶ $\lambda \equiv \|u_0\|_1 = 8\pi\ell,\ \exists \ell \in \mathbf{N}$ initial mass quantization

$\exists x_* \in \Omega^\ell \setminus D,\ \nabla H_\ell(x_*) = 0$ recursive hierarchy

point vortex Hamiltonian, Robin function, Green function

$$H_\ell(x_1,\cdots,x_\ell) = \frac{1}{2}\sum_j R(x_j) + \sum_{i<j} G(x_i,x_j)$$

自己組織化のポテンシャル

系 1 $T < +\infty$ if

(1) $\lambda \notin 8\pi\mathbf{N},\ \not\exists$ stationary solution or $\mathcal{F}(u_0) \ll -1$

(2) $\lambda \in 8\pi\ell,\ \ell \in \mathbf{N},\ \not\exists$ critical point of H_ℓ

系 2 Ω convex $\lambda \neq 8\pi$

c.f. Grossi-F. Takahashi $\Rightarrow T < +\infty$ or $T = +\infty$ pre-compact orbit

$\exists$ stationary solution

定理 A (有限時間爆発） $T < +\infty$

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$$m(x_0) \in 8\pi \mathbf{N}$$

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0, \ t_k \uparrow T, \ u(x_k,t_k) \to +\infty\} \subset \Omega$$

$$\sharp \mathcal{S} < +\infty$$

$$0 < f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$$

symmetry of the Green function → weak form (symmetrization)

$$\frac{d}{dt}\int_\Omega \varphi u(\cdot,t) = \int_\Omega \Delta\varphi \cdot u(\cdot,t) + \frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\varphi(x,x')u \otimes u$$

$$\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x')$$

boundary behavior of the Green function
singularity cancellation by the symmetry

$$\varphi \in C^2(\overline{\Omega}), \ \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0 \qquad \|\rho_\varphi\|_\infty \leq C\|\nabla\varphi\|_{C^1}$$

→ **monotonicity formula** $\lambda = \|u(\cdot,t)\|_1$

$$\left|\frac{d}{dt}\int_\Omega u\varphi\right| \leq C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1}$$

→ **weak continuation**

$$0 \leq \exists \mu(dx,t) \in C_*([0,T],\mathcal{M}(\overline{\Omega}))$$

$$u(x,t)dx = \mu(dx,t), \ 0 \leq t < T$$

Gagliard-Nirenberg inequality

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists\varepsilon_0$$

$$\longrightarrow x_0 \notin \mathcal{S}$$

→ **formation of collapse**

$$\mu(\cdot,T) = \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0} + f(x)$$

$$m(x_0) \geq \varepsilon_0, \ 0 \leq f \in L^1(\Omega), \ \sharp\mathcal{S} < +\infty$$

弱解の生成

$0 \le \mu = \mu(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$ weak solution

$\Longleftrightarrow \quad 0 \le \exists \mathcal{N} = \mathcal{N}(\cdot,t) \in L^\infty_*([0,T], \mathcal{X}')$

1. $t \in [0,T] \mapsto \langle \varphi, \mu(dx,t) \rangle, \ \varphi \in \mathcal{Y}$ a.c.
2. $\frac{d}{dt}\langle \varphi, \mu \rangle = \langle \Delta\varphi, \mu \rangle + \frac{1}{2}\langle \rho_\varphi, \mathcal{N}(\cdot,t) \rangle$ a.e. $t \in [0,T]$
3. $\mathcal{N}|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu \otimes \mu$

定理 1 $\mu_k(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$

$\mathcal{N}_k \in L^\infty_*([0,T], \mathcal{X}')$ weak solutions

$0 \le \mu_k(\overline{\Omega}, t) \le C$

$\|\mathcal{N}_k(\cdot,t)\|_{\mathcal{X}'} \le C$

$\Rightarrow$ sub-sequence

$\mu_k(dx,t) \rightharpoonup \mu(dx,t) \quad \text{in } C_*([0,T], \mathcal{M}(\overline{\Omega}))$

$\mathcal{N}_k(\cdot,t) \rightharpoonup \mathcal{N}(\cdot,t) \quad \text{in } L^\infty_*([0,T], \mathcal{X}')$ weal solution

$$\mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0\} \qquad \mathcal{X} = [\mathcal{X}_0]^{L^\infty(\Omega\times\Omega)}$$

$$\mathcal{X}_0 = \{\rho_\varphi + \psi \mid \varphi \in \mathcal{Y}, \ \psi \in C(\overline{\Omega}\times\overline{\Omega})\}$$

$\Rightarrow$

$$\mu(\overline{\Omega}, t) = \mu(\overline{\Omega}, 0) \equiv \lambda, \ 0 \le t \le T$$

$$\left| \frac{d}{dt}\langle \varphi, \mu(dx,t) \rangle \right| \le C(\lambda + \lambda^2)\|\nabla\varphi\|_{C^1}$$

$u = u(x,t)$ classical solution

$\Rightarrow$

$$\mathcal{N}(\cdot,t) = u(x,t) \otimes u(x',t) \ dxdx'$$

$$\|\mathcal{N}(\cdot,t)\|_{\mathcal{X}'} = \lambda^2, \ \lambda = \|u_0\|_1$$

Proof of Theorem A (continued) $x_0 \in \mathcal{S}$

backward self-similar transformation

$$y = (x - x_0)/(T-t)^{1/2}, \ s = -\log(T-t)$$
$$z(y,s) = (T-t)u(x,t)$$

weak limit $s_k \uparrow +\infty$ subsequence

$$z(y, s+s_k)dy \rightharpoonup \exists \zeta(dy,s) \text{ in } C_*(-\infty,+\infty; \mathcal{M}(\mathbf{R}^2))$$

limit equation exclusion of boundary blowup $x_0 \in \Omega$

$$\zeta_s = \nabla \cdot (\nabla \zeta - \zeta \nabla(\Gamma * \zeta + |y|^2/4)) \ \text{ in } \mathbf{R}^2 \times (-\infty, +\infty)$$

ϵ − **regularity** ➡ $$\zeta^s(dy,s) = \sum_{j=1}^{m(s)} \tilde{m}_j(s)\delta_{y_j(s)}(dy)$$ (sub-collapse)

$$m(s) \leq m(x_0)/\varepsilon_0, \ |y_j(s)| \leq C, \ \tilde{m}_j(s) \geq \varepsilon_0$$

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

parabolic envelope

$$m(x_0) = \zeta(\mathbf{R}^2, s) \qquad \langle |y|^2, \zeta(dy,s) \rangle \leq C$$

scaling back

$$\zeta(dy,s) = e^{-s}A(dy',s'), \ y' = e^{-s/2}y, \ s' = -e^{-s}$$

$$A_s = \nabla \cdot (\nabla A - A\nabla\Gamma * A) \text{ in } \mathbf{R}^2 \times (-\infty, 0)$$
$$A = A(dy,s) \geq 0, \ A(\mathbf{R}^2, s) = m(x_0)$$

$$A^s(dy',s') = \sum_{j=1}^{m(s')} \tilde{m}_j(s')\delta_{y'_j(s)}(dy')$$

$$A_s = \nabla \cdot (\nabla A - A\nabla\Gamma * A) \text{ in } \mathbf{R}^2 \times (-\infty, 0)$$

$$A = A(dy, s) \geq 0, \ A(\mathbf{R}^2, s) = m(x_0)$$

$$A^s(dy', s') = \sum_{j=1}^{m(s')} \tilde{m}_j(s')\delta_{y'_j(s)}(dy')$$

scaling limit $s'_0 < 0, \ 1 \leq j \leq m(s'_0)$ fix

$$\tilde{A}_\beta(dy', s') = \beta^2 A(dy, s)$$

$$y = \beta y' + y'_j(s'_0), \ s = \beta^2 s' + s'_0$$

$\beta_k \downarrow 0$ subsequence

$$\tilde{A}_{\beta_k}(dy', s') \rightharpoonup \tilde{A}(dy', s') \in C_*(-\infty, s'_0; \mathcal{M}(\mathbf{R}^2))$$

$\tilde{A}(dy', s') = m'_j(s'_0)\delta_0(dy')$ weak solution

translation limit

$s'_k \uparrow +\infty, \ \hat{A}_k(dy', s') = \tilde{A}(dy', s' + s'_k)$ subsequence

$$\hat{A}_k(dy', s') \rightharpoonup \hat{A}(dy', s') \in C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2))$$

$$\hat{A}(dy', s') = m'_j(s'_0)\delta_0(dy')$$

weak Liouville property

$$a_s = \nabla \cdot (\nabla a - a\nabla\Gamma * a) \text{ in } \mathbf{R}^2 \times (-\infty, +\infty)$$

$$\Rightarrow \ a(\mathbf{R}^2, s) = 0 \text{ or } 8\pi$$

$\longrightarrow$ $\tilde{m}_j(s'_0) = \tilde{A}(\mathbf{R}^2, 0) = 8\pi$

residual vanishing

if $A^{ac}(dy', s') = 0$ $\longrightarrow$ $A(dy', s') = 8\pi \sum_{j=1}^{\ell} \delta_{y'_j(s')}(dy')$ サブコラプス

$\longrightarrow$ $m(x_0) = 8\pi\ell$ collapse mass quantization

residual vanishing $A^{ac}(dy', s') = 0 \Leftrightarrow \zeta^{ac}(dy, s) = 0$

$$A_{s'} = \nabla' \cdot (\nabla' A - A \nabla' \Gamma * A)$$

1st envelope

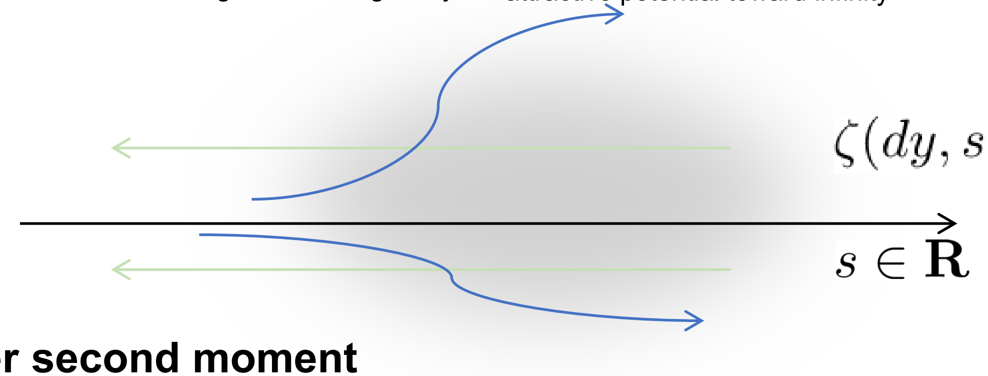
$$m(x_0) = \zeta(\mathbf{R}^2, s)$$

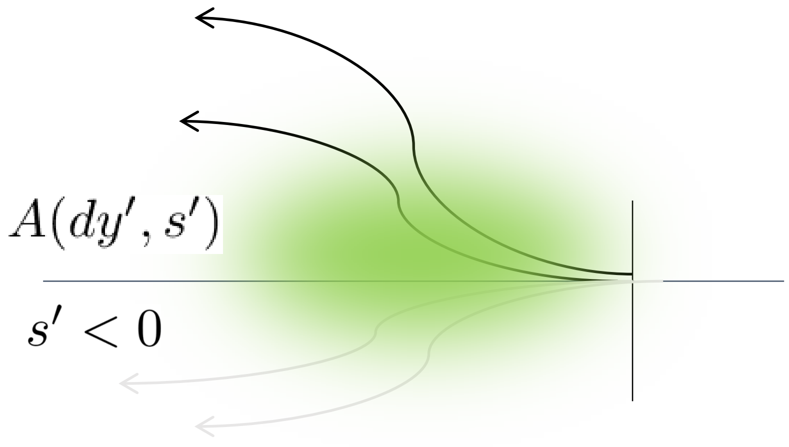
2nd envelope

$$\langle |y|^2, \zeta(dy, s) \rangle \leq C$$

$$\zeta_s = \nabla \cdot (\nabla \zeta - \zeta \nabla (\Gamma * \zeta + |y|^2/4))$$

scaling invariant regularity attractive potential toward infinity





Collision of sub-collapses

outer second moment

$$\frac{d}{ds}\langle \varphi, \zeta \rangle \geq \langle \Delta \varphi - C\varphi_r + \frac{1}{2} r \varphi_r, \zeta \rangle, \ \varphi = \varphi(r)$$

$$\varphi(r) = \xi(r/R), \ \xi(r) = r^2 - 1$$

$$R \gg 1 \ \Rightarrow \ \Delta \varphi + \frac{1}{2} r \varphi_r \geq C\varphi_r, \ r \geq R$$

$$\frac{d}{ds}\langle (\frac{|y|^2}{R^2} - 1)_+, \zeta(dy, s) \rangle \geq 0 \longrightarrow \zeta(dy, s) = \zeta^s(dy, s)$$

ポテンシャル項の一様評価

scaling invariant regularity

$$\zeta(B(y_0, 2r), s) < \varepsilon_0$$

$$\longrightarrow \ \|\zeta(\cdot, s)\|_{L^\infty(B(y_0, r))} \leq Cr^{-2}$$

改良 ε 正則性 $\exists(\varepsilon_0, R) > 0$

補題 $$u_t = \Delta u - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^2\times(0,T)$$
$$u|_{t=0} = u_0(x) \geq 0$$

$\exists\varepsilon_0,\ R,\ t_0>0,\ \forall x_0 \quad \|u_0\|_{L^1(B(x_0,8R))} < \varepsilon_0/2$

⟶ $\forall\tau\in(0,t_0)$
$$\sup_{\tau\leq t<t_0} \|u(\cdot,t)\|_{L^\infty(B(x_0,R))} < +\infty$$

Proof $\exists t_1\in(0,T),\ \|u_0\|_{L^1(B(x_0,8R))} < \varepsilon_0/2$

⟶ $$\sup_{t\in(0,t_1)} \|u(\cdot,t)\|_{L^1(B(x_0,4R)} < \varepsilon_0$$

⟶ $$\frac{d}{dt}\int_{\mathbf{R}^2} u(\log u - 1)\varphi\ dx + \frac{1}{8}\int_{\mathbf{R}^2} u^{-1}|\nabla u|^2\varphi\ dx \leq C_\varphi,\ 0\leq t<t_1,\ \varphi=\varphi_{x_0,R}$$

$\|u(\cdot,t)\|_{L\log L(B(x_0,2R))}$ 初期値に依存

Gagliardo-Nirenberg

⟶ $$\frac{dJ}{dt} + 3J^{3/2} \leq C_R,\ J = \int_\Omega u(\log u - 1) + 1\ dx$$

$$\frac{d}{dt}t^{-2} + 3(t^{-2})^{3/2} = t^{-3}$$

$J(t)\leq t^{-2},\ 0<t\leq\min\{t_1,t_0\},\ t_0^{-3} = C_R$ □

parabolic regularity

scaling

定理 2 $\exists\varepsilon_0,\ \sigma_0,\ C$

$$u_t = \Delta u - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^2\times(-T,T)$$
$$u_0 = u|_{t=0}$$ weak solution generated by classical solutions

$$\|u_0\|_{L^1(B(x_0,2R))} < \varepsilon_0,\ u_0 = u|_{t=0} \Rightarrow$$
$$\sup_{t\in[-\sigma_0 R^2,\sigma_0 R^2]\cap(-T,T)} \|u(\cdot,t)\|_{L^\infty(B(x_0,R))} \leq CR^{-2}$$

scaling invariant regularity (inverse scaling back)

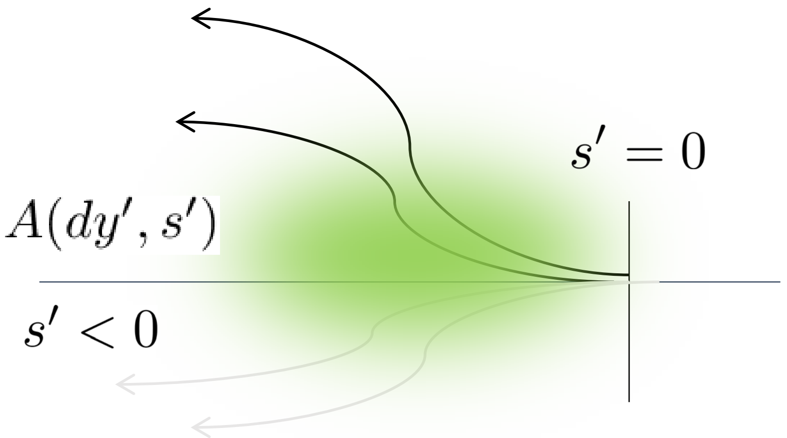
$$\zeta(B(y_0,2r),s) < \varepsilon_0 \Rightarrow \|\zeta(\cdot,s)\|_{L^\infty(B(y_0,r))} \leq Cr^{-2}$$

サブコラプスの運動 〜 再び循環するハミルトニアン

$$\zeta^s(dy,s)=\sum_{j=1}^{\ell}8\pi\delta_{y_j(s)}(dy)$$ ⟶ **scaling back**

$$A(dy',s')=\sum_{j=1}^{\ell}8\pi\delta_{y'_j(s')}(dy')$$

$$A_{s'}=\nabla'\cdot(\nabla' A-A\nabla'\Gamma * A)\quad \text{in } \mathbf{R}^2\times(-\infty,0)$$



Collision of sub-collapses

局所2次モーメントによる追跡

simple blowup point

$$\ell=1 \;\Rightarrow\; \zeta(dy,s)=8\pi\delta_0(dy)$$

recursive hierarchy $\ell\geq 2$

$$\frac{dy'_j}{ds'}=8\pi\nabla_j H_\ell^0(y'_1,\cdots,y'_\ell)$$

$$H_\ell^0(y'_1,\cdots,y'_\ell)=\sum_{1\leq j<k\leq\ell}\Gamma(y'_j-y'_k)$$

$$\Gamma(y')=\frac{1}{2\pi}\log\frac{1}{|y'|}$$

自由エネルギーと単純性

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1)dx - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u\otimes u\ dxdx'$$

$$\frac{d\mathcal{F}}{dt} = -\int_\Omega u|\nabla(\log u - v)|^2 \le 0,\ v = (-\Delta)^{-1}u$$

Simplicity of the blowup points

Emergence （創発性）

$$T < +\infty,\ \lim_{t\uparrow T}\mathcal{F}(u(t)) > -\infty \longrightarrow \forall x_0 \in \mathcal{S}$$

simple $\longrightarrow$

$$\lim_{t\uparrow T}\mathcal{F}_{x_0,b(T-t)^{1/2}}(u(\cdot,t)) = +\infty,\ \forall b > 0$$

$$\mathcal{F}_{x_0,R}(u) = \int_{\Omega\cap B(x_0,R)} u(\log u - 1)dx - \frac{1}{2}\iint_{\Omega\cap B(x_0,R)\times(\Omega\cap B(x_0,R))} G(x,x')u\otimes u\ dxdx'$$

Blowup rate $\downarrow$

$$\lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,b(T-t)^{1/2})} = +\infty,\ \forall b > 0$$

爆発レートは衝突する場合も含めて常にタイプII

$$\exists b > 0, \forall x_0 \in \mathcal{S},\ \lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,b(T-t)^{1/2})} = +\infty$$

無限時間爆発

無限時間爆発の量子化

定理 B (無限時間爆発)

$$T=+\infty,\ \limsup_{t\uparrow+\infty}\|u(\cdot,t)\|_\infty=+\infty$$

⟶

$$\lambda\equiv\|u_0\|_1=8\pi\ell,\ \exists\ell\in\mathbf{N}$$ initial mass quantization

$$\exists x_*\in\Omega^\ell\setminus D,\ \nabla H_\ell(x_*)=0$$ recursive hierarchy

point vortex Hamiltonian, Robin function, Green function

$$H_\ell(x_1,\cdots,x_\ell)=\frac{1}{2}\sum_j R(x_j)+\sum_{i<j}G(x_i,x_j)$$

系 1 $T<+\infty$ if

(1) $\lambda\notin 8\pi\mathbf{N},\ \nexists$ stationary solution or $\mathcal{F}(u_0)\ll -1$

(2) $\lambda\in 8\pi\ell,\ \ell\in\mathbf{N},\ \nexists$ critical point of H_ℓ

系 2 Ω convex $\lambda\neq 8\pi$

$\Rightarrow$ $T<+\infty$ or $T=+\infty$ compact orbit

c.f. Grossi-F. Takahashi $\exists$ stationary solution

弱極限

assume $T = +\infty,\ t_k \uparrow +\infty,\ \lim_{k\to\infty} \|u(\cdot,t_k)\|_\infty = +\infty$

subsequence $u(\cdot, t+t_k)dx \rightharpoonup \mu(dx,t) \in C_*(-\infty,+\infty;\mathcal{M}(\overline{\Omega}))$ weak solution

$$\mu(dx,t) = \sum_{x_0\in\mathcal{S}_t} m(x_0)\delta_{x_0}(dx) + f(x,t)dx$$

improved regularity
formation of collapse in infinite time

blowup set

exclusion of boundary blowup

$$m(x_0) \geq \varepsilon_0,\ 0 \leq f = f(\cdot,t) \in L^1(\Omega)$$

$$\mathcal{S}_t = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0,\ \lim_k u(x_k, t+t_k) = +\infty\} \subset \Omega$$

dilation $x_0 = 0 \in \mathcal{S}_0,\ \beta > 0$

$$\mu_\beta(dx',t') = \beta^2\mu(dx,t),\ x' = \beta x,\ t' = \beta^2 t$$

$\beta_k \downarrow 0$ subsequence

$$\mu_{\beta_k}(dx,t) \rightharpoonup \tilde{\mu}(dx,t) \in C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$ scaling limit

$$m(x_0) = \tilde{\mu}(\mathbf{R}^2,0) = 8\pi \geq \varepsilon_0$$ full orbit of weak solutions on the whole space

Liouville property

collapse mass quantization

local second moment traces the collapse dynamics

$$\sharp\, \mathcal{S}_t \equiv \ell,\ \mu^s(dx,t) = \sum_{i=1}^{\ell} 8\pi\delta_{x_i(t)}(dx) \qquad |\dot{x}_j(t)| \leq C \qquad \{x(t)\} \subset\subset \Omega^\ell \setminus D$$

residual vanishing

$$x_i = x_i(t),\ u_k(x,t) = u(x, t+t_k),\ v_k(x,t) = v(x,t+t_k),\ 0 < r \ll 1$$

$$\frac{d}{dt}\int_{B(x_i,r)} |x-x_i|^2 u_k = \int_{B(x_i,r)} \frac{\partial}{\partial t}(|x-x_i|^2 u_k) + \dot{x}_i \cdot \nabla(|x-x_i|^2 u_k)\ dx$$

リュービルの第1体積公式

$$= \int_{B(x_i,r)} |x-x_i|^2 u_{kt} + \dot{x}_i \cdot |x-x_i|^2 \nabla u_k\ dx$$

$$\int_{B(x_i,r)} |x-x_i|^2 u_{kt}$$

$$= \int_{B(x_i,r)} |x-x_i|^2 \nabla\cdot(\nabla u_k - u_k \nabla v_k)\ dx$$

$$\leq r^2 \int_{\partial B(x_i,r)} \frac{\partial u_k}{\partial \nu} - u_k \frac{\partial v_k}{\partial \nu}\ dS$$

$$+ \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k \nabla v_k\ dx$$

$$= \int_{B(x_i,r)} \cancel{r^2 u_{kt}} + 4u_k + 2(x-x_i)\cdot u_k\nabla v_k\ dx$$

$$\int_{B(x_i,r)} \dot{x}_i \cdot |x-x_i|^2 \nabla u_k$$

$$= \int_{\partial B(x_i,r)} (\dot{x}_i\cdot\nu)|x-x_i|^2 u_k\ dS$$

$$- \int_{B(x_i,r)} 2(x-x_i)\cdot \dot{x}_i u_k$$

$$= \int_{B(x_i,r)} \cancel{r^2 \dot{x}_i \cdot} \nabla u_k$$

$$-2(x-x_i)\cdot\dot{x}_i u_k\ dx$$

$$\frac{d}{dt}\int_{B(x_i,r)} u_k = \int_{B(x_i,r)} \cancel{u_{kt}}$$

$$+\dot{x}_i \cdot \nabla \cancel{u_k}\ dx$$

defect measure

$$\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2)u_k$$

$$\leq \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k\nabla v_k - 2(x-x_i)\cdot\dot{x}_i u_k\ dx$$

$$x_i = x_i(t)$$
$$u_k(x,t) = u(x, t+t_k)$$
$$v_k(x,t) = v(x, t+t_k)$$
$$0 < r \ll 1$$

$\longrightarrow$

$$\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2)u_k$$
$$\leq \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k \nabla v_k$$
$$-2(x-x_i)\cdot \dot{x}_i u_k \; dx$$

$$v_k(x,t) = \sum_{i=0}^{3} v_k^i(x,t)$$

$$v_k^0(x,t) = \int_{B(x_i,r)} \Gamma(x-x')u_k(x',t)dx'$$

$$v_k^1(x,t) = \int_{B(x_i,r)} K(x,x')u_k(x',t)\; dx'$$

$$v_k^2(x,t) = \int_{\Omega\setminus \mathcal{S}_t^{2r}} G(x,x')u_k(x',t)dx'$$

$$v_k^3(x,t) = \int_{\mathcal{S}_t^{2r}\setminus B(x_i,r)} G(x,x')u_k(x',t)dx'$$

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$
$$G(x,x') = \Gamma(x-x') + K(x,x')$$

$$2\int_{B(x_i,r)} (x-x_i)\cdot u_k \nabla v_k^0 \; dx$$
$$= -\frac{1}{2\pi}\left(\int_{B(x_i,r)} u_k \; dx\right)^2$$

$$\|u_k(\cdot,t)\|_1 = \lambda, \;\; K(x,x') \in C^1(\Omega\times\Omega)$$
$$\sup_x \int_\Omega |\nabla_x G(x,x')|\; dx' \leq C$$

$\longrightarrow$

$$\|\nabla v^i(\cdot,t)\|_{L^\infty(B(x_i,r))} \leq C, \;\; 1\leq i\leq 3$$

$$\frac{d}{dt}\int_{B(x_i,r)}(|x-x_i|^2-r^2)u_k$$

$$\leq 4\int_{B(x_i,r)} u_k - \frac{1}{2\pi}\left(\int_{B(x_i,r)} u_k\right)^2$$

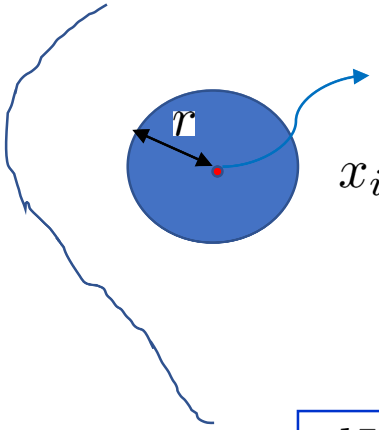
$$+C\int_{B(x_i,r)} |x-x_i|u_k$$

$$r^2\int_{B(x_i,r)} u_k \to 8\pi r^2 + \int_{B(x_i,r)} f$$

$k \to \infty$

as distributions in time

defect measure



$$x_i = x_i(t)$$

$$\frac{d}{dt}\int_{B(x_i,r)}(|x-x_i|^2-r^2)f$$

$$\leq 4\left(8\pi + \int_{B(x_i,r)} f\right)$$

$$-\frac{1}{2\pi}\left(8\pi + \int_{B(x_i,r)} f\right)^2$$

$$+C\int_{B(x_i,r)} |x-x_i|f$$

$$0 < r \ll 1$$

$$\frac{dI}{dt} \leq \int_{B(x_i,r)} -4f + C|x-x_i|f\ dx \leq \frac{2I}{r^2}$$

$$I(t) \equiv \int_{B(x_i,r)}(|x-x_i|^2-r^2)f \leq 0$$

$$I(t) \equiv 0$$

$$f = 0 \text{ in } B(x_i,r)$$

$$f \equiv 0$$

コラプスの運動 〜 3たび循環するハミルトニアン

$$\mu(dx,t) = 8\pi \sum_{i=1}^{\ell} \delta_{x_i(t)}(dx)$$

$$\frac{dx_i}{dt} = 8\pi \nabla_{x_i} H_\ell(x_1, \cdots, x_\ell), \ 1 \leq i \leq \ell$$

a blowup criterion excludes the collapse collision in infinite time

$$x(t) = (x_i(t)) \in \Omega^\ell \setminus D$$ pre-compact

$$D = \{(x_i) \mid \exists i \neq j, \ x_i = x_j\}$$

⟶

$$\exists x^* \in \Omega^\ell \setminus D, \ \nabla_{x_i} H_\ell(x^*) = 0, \ 1 \leq i \leq \ell$$

自由エネルギーの有界性

定理 3 (Senba-S. 02)

$$T = +\infty, \ \lim_{t \uparrow +\infty} \|u(\cdot,t)\|_\infty = +\infty$$

$$\lim_{t \uparrow +\infty} \mathcal{F}(u(\cdot,t)) > -\infty$$

⟶

$$\lambda = \|u_0\|_1 = 8\pi\ell, \ \ell \in \mathcal{N}, \ \exists (x_1^*, \cdots, x_\ell^*) \in \Omega^\ell \setminus D$$

$$\nabla_{x_i} H_\ell(x_1^*, \cdots, x_\ell^*) = 0, \ 1 \leq i \leq \ell$$

$$\mu(dx,t) = 8\pi \sum_{i=1}^{\ell} \delta_{x_i^*}(dx)$$

勾配不等式　　クリニック軌道

$$\exists x_\pm^* \in \Omega^\ell \setminus D, \ \lim_{t \to \pm\infty} x(t) = x_\pm^*, \ \nabla_{x_i} H_\ell(x_\pm^*) = 0, \ 1 \leq i \leq \ell$$

スモルコフスキー方程式（補足）

全空間の力学系

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$u_t = \Delta u - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^2 \times (0,T)$$
$$u|_{t=0} = u_0(x) \in L^\infty \cap L^1(\mathbf{R}^2)$$

$$\int_{\mathbf{R}^2} |x|^2 u_0\, dx < +\infty \longrightarrow \not\exists$$ dichotomy

$$0 < \lambda < 8\pi \longrightarrow T = +\infty,\ \|u(\cdot,t)\|_\infty \le C$$

$$\lambda > 8\pi \longrightarrow T < +\infty$$

$$\lambda = 8\pi \longrightarrow T = +\infty$$

vanishing

$$\lim_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = 0$$

compact (concentration)

$$\lim_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = +\infty$$

再び走化性方程式について

$$\varepsilon u_t = \nabla\cdot(\nabla u - u\nabla v),\ -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$

$$\left(\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu}, \frac{\partial v}{\partial\nu}\right)\Big|_{\partial\Omega} = 0,\ \int_\Omega v = 0$$

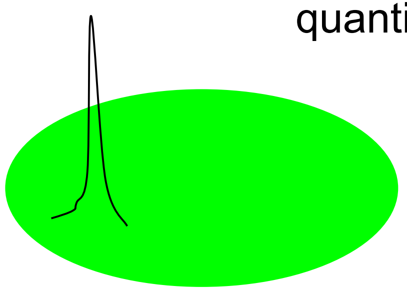
$$n = 2,\ T = T_{\max} < +\infty$$

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$$m(x_0) \in m_*(x_0)\mathbf{N},\ m_*(x_0) = \begin{cases} 8\pi, & x_0 \in \Omega \\ 4\pi, & x_0 \in \partial\Omega \end{cases}$$

$$0 \le f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$$ 境界爆発点の存在

quantized blowup mechanism



also in infinite time

緩和時間のある場合

full system of chemotaxis

$$\varepsilon u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$\tau v_t = \Delta v + u \text{ in } \Omega \times (0,T)$$

$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, v\right)\Big|_{\partial\Omega} = 0$$

$\tau = 0$ Smoluchowski-Poisson quantized blowup mechanism

$$\varepsilon = 0 \qquad v_t = \Delta v + \frac{\lambda e^v}{\int_\Omega e^v}, \quad v|_{\partial\Omega} = 0$$

non-local parabolic equation

$$\Omega = B(0,1) \subset \mathbf{R}^2, \ v = v(|x|,t), \ \lambda \geq 8\pi$$

Wolansky 97

$$\longrightarrow \quad \frac{\lambda e^v}{\int_\Omega e^v} dx \ \rightharpoonup \ \lambda\delta_0, \ t \uparrow T = T_{\max} \in (0,+\infty]$$

Kavallaris-S. 07

$$\lambda > 8\pi \ \Rightarrow \ T = T_{\max} < +\infty$$

場と粒子の双対性

Lagrangian in Toland duality

$$L(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}|\nabla v|^2 - vu \ dx$$

→ unfolding-minimality

$$\inf\{L(u,v) \mid u \geq 0, \|u\|_1 = 8\pi, \ v \in H_0^1(\Omega)\} > -\infty$$

Model C equation

$$\varepsilon u_t = \nabla \cdot (u\nabla L_u(u,v)), \ \tau v_t = -L_v(u,v)$$

非平衡熱力学モデル semi-unfolding-minimality

infinitesimal stability → dynamical stabilitiy

local minimum of the analytic field functional → infinitesimal stable

phase transition, phase separation, shape memory alloys

高次元の量子化

$$n > 2,\ m = \frac{n}{n-2}$$

weak solution

$$u_t = \frac{m-1}{m}\Delta u^m - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^n \times (0,T)$$

$$u|_{t=0} = u_0(x) \geq 0 \in L^\infty \cap L^1(\mathbf{R}^n) \quad \int_{\mathbf{R}^n} |x|^2 u_0 dx < +\infty$$

$$\Gamma(x) = \frac{|x|^{2-n}}{|\partial B|},\ B = B(0,1) \quad \Rightarrow \quad \frac{d}{dt}\int_{\mathbf{R}^n} u\ dx = 0$$

$$\frac{d}{dt}\mathcal{F}(u) = -\int_{\mathbf{R}^n} u|\nabla u^{m-1} - \Gamma * u|^2 dx \leq 0$$

$$\mathcal{F}(u) = \int_{\mathbf{R}^n} \frac{u^m}{m} - \frac{1}{2}\langle \Gamma * u, u\rangle$$

Tsallis entropy

定理 $T < +\infty \quad \Rightarrow \quad \mathcal{S} \subset \mathbf{R}^n \qquad \sharp\mathcal{S}_{II} < +\infty$

$$\mathcal{S} = \{x_0 \in \mathbf{R}^n \cup \{\infty\} \mid \exists x_k \to x_0,\ t_k \uparrow T,\ \lim_{k\to\infty} u(x_k, t_k) = +\infty\}$$

$$S_{II} = \{x_0 \in \mathcal{S} \mid \lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,r_0))} = +\infty,\ \forall r_0 > 0$$

楕円型一様化理論に向けて

$$-\Delta w = w_+^m \text{ in } \Omega,\ w = c \in \mathbf{R} \text{ on } \partial\Omega$$

solution sequence

$$\int_\Omega w_+^m = \lambda \qquad (w_k, c_k, \lambda_k) \qquad \lambda_k \to \lambda_0$$

$\Rightarrow$ Subsequence alternatives

(a) $\|w_k\|_\infty \leq C$

(b) $\sup_\Omega w_k \to -\infty$

(c) $\lambda_0 = m_* \ell,\ \ell \in \mathbf{N}$

$$\mathcal{S} = \{x_1^*, \cdots, x_\ell^*\} \in \Omega$$

$$\nabla_{x_j} H_\ell(x)\big|_{x=x_*} = 0,\ 1 \leq j \leq \ell$$

$$x = (x_1, \cdots, x_\ell),\ x_* = (x_1^*, \cdots, x_\ell^*)$$

$$H_\ell(x_1, \cdots, x_\ell) = \frac{1}{2}\sum_j R(x_j) + \sum_{i<j} G(x_i, x_j)$$