偏微分方程式論における幾何学的方法 III

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非平衡統計力学のモデル

ボルツマン・ポアソン方程式

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}$$

Euler's equation of motion

$$v_t + (v \cdot \nabla) v = -\nabla p, \ \nabla \cdot v = 0, \ \nu \cdot v|_{\partial\Omega} = 0$$

2D $\omega = \nabla^\perp \cdot v$ ➡ $\omega_t + \nabla \cdot (v\omega) = 0, \ \nabla \cdot v = 0$

$v = \nabla^\perp \psi$ 流れの関数 $\nabla^\perp = \begin{pmatrix} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{pmatrix}$

$$\omega_t + \nabla \cdot (\omega \nabla^\perp \psi) = 0, \ -\Delta \psi = \omega$$ 渦度場方程式

境界条件 ➡ $\psi|_{\partial\Omega} = 0$

グリーン関数

$$-\Delta_x G(x, x') = \delta_{x'}(dx), \ G(x, x')|_{x \in \partial\Omega} = 0$$ ➡

$$\omega_t + \nabla \cdot (\omega \nabla^\perp \psi) = 0, \ \psi(\cdot, t) = \int_\Omega G(\cdot, x') \omega(x', t) dx'$$

$G(x, x') = G(x', x)$ 作用反作用の法則

$\varphi \in C^1(\overline{\Omega}), \ \varphi|_{\partial\Omega} = 0$ ➡ $\omega \otimes \omega = \omega(x,t)\omega(x',t)$

$$\frac{d}{dt} \int_\Omega \varphi\omega = \int_\Omega \omega \nabla^\perp \psi \cdot \nabla\varphi$$

$$= \iint_{\Omega\times\Omega} \nabla_x^\perp G(x, x') \nabla\varphi(x) \ \omega(x,t)\omega(x',t) \ dx dx'$$

$$= \frac{1}{2} \iint_{\Omega\times\Omega} \rho_\varphi(x, x') \omega \otimes \omega \ dx dx'$$

$$\rho_\varphi(x, x') = \nabla\varphi(x) \cdot \nabla_x^\perp G(x, x') + \nabla\varphi(x') \cdot \nabla_{x'}^\perp G(x, x') \in L^\infty(\Omega \times \Omega)$$

点渦系 $\omega(dx, t) = \sum_{i=1}^{\ell} \alpha_i \delta_{x_i(t)}(dx)$

局所2次モーメント p.v. ➡ キルヒホッフ方程式 $\frac{dx_i}{dt} = \nabla_{x_i}^\perp H_N$

点渦ハミルトニアン

$$H_N(x_1, \ldots, x_N) = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

ロバン関数 $R(x) = \left[G(x, x') + \frac{1}{2\pi} \log|x - x'| \right]_{x'=x}$

熱平衡の統計力学

小正準統計

全エネルギー

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \ 1 \leq i \leq N$$

$$\mathbf{R}^{6N}/\{H = E\}$$

co-area formula

$$x = (q_1, \ldots, q_N, p_1, \ldots, p_N), \ dx = dE \cdot \frac{d\Sigma(E)}{|\nabla H|}$$

$$d\Sigma(E) \ \leftrightarrow \ \{x \in \mathbf{R}^{6N} \mid H(x) = E\}$$

micro-canonical measure　　weight factor

$$d\mu^{E,N} = \frac{1}{W(E)} \cdot \frac{d\Sigma(E)}{|\nabla H|}, \ W(E) = \int_{\{H=E\}} \frac{d\Sigma(E)}{|\nabla H|}$$

$$\alpha_i = \hat{\alpha}, \ \hat{\alpha} N = 1, \ \hat{H}_N = H, \ \hat{\alpha}^2 N \hat{\beta} = \beta$$

等重率の仮定 $\xrightarrow{N \uparrow +\infty}$ 高エネルギー極限

正準統計

ボルツマン定数

$$\mathbf{R}^{6N}/\{T\}, \ \beta = 1/(kT)$$

inverse temperature

$$d\mu^{\beta,N} = \frac{e^{-\beta H} dx}{Z(\beta, N)}, \ Z(\beta, N) = \int_{\mathbf{R}^{6N}} e^{-\beta H} dx$$

canonical measure　　weight factor

熱力学的関係式

$$\beta = \frac{\partial}{\partial E} \log W(E)$$

負の逆温度での秩序形成

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}$$

$$\psi = \int_\Omega G(\cdot, x')\rho(x')dx'$$

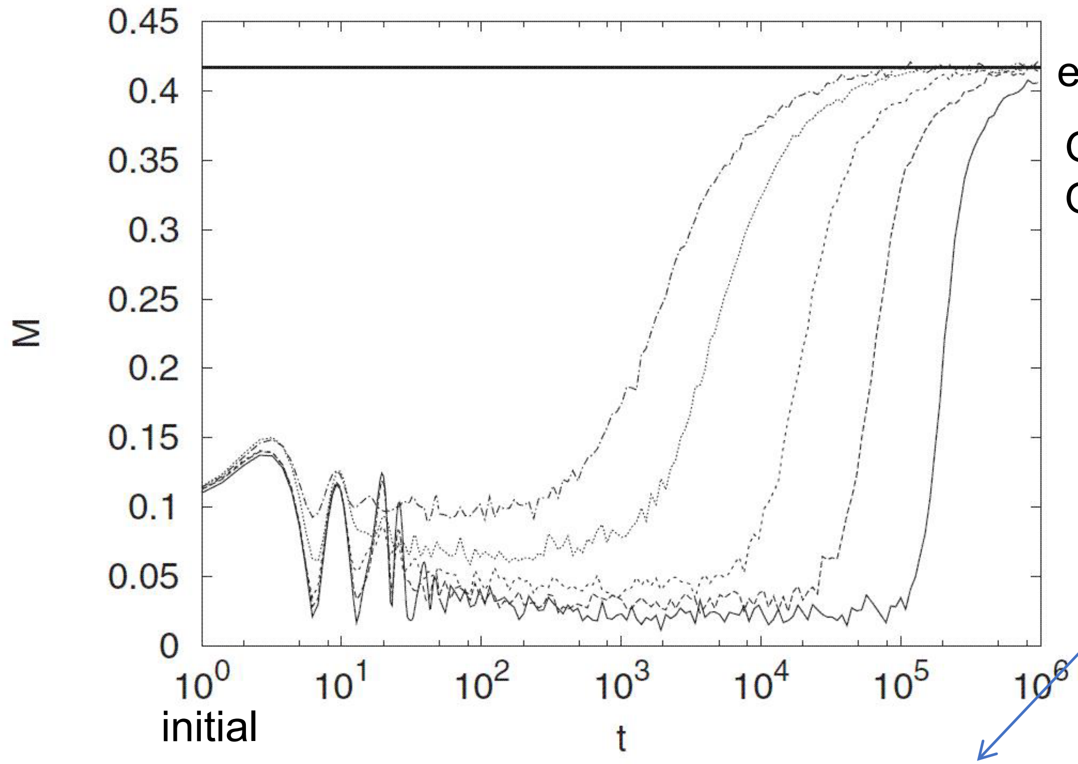
↕ duality

ボルツマン・ポアソン方程式

準平衡から平衡へ

Hamilton System of Many Particles with Long Range Interaction

state of the system



equilibrium

Gibbs
Onsager

kinetic

time

initial

quasi-equilibrium

relaxation

relaxation

static

Staniscia-Chavanis-Ninno-Fanelli 09

Lynden-Bell
Miller-Robert-Sommeria

点渦動的平均場モデル

Chavanis 08 Langevin equation

$\mu > 0$ mobility

摩擦項

$$\frac{dx_i}{dt} = \alpha \nabla_i^{\perp} \hat{H}_N - \mu\alpha^2 \nabla_i \hat{H}_N + \sqrt{2\nu} R_i(t), \ 1 \leq i \leq N$$

$\nu > 0$ viscosity of particles

$R_i(t)$ white noise

$$\langle R_i(t) \rangle = 0, \ \langle R_i^{\alpha}(t) R_j^{\beta}(t') \rangle = \delta_{ij}\delta_{\alpha\beta}\delta(t-t')$$

$P_N(x_1, \cdots, x_N, t)$ N-pdf

$$\frac{\partial P_N}{\partial t} + \alpha \nabla^{\perp} \cdot \hat{H}_N \nabla P_N = \nabla \cdot (\nu \nabla P_N + \mu\alpha^2 P_N \nabla \hat{H}_N)$$

Sire-Chavanis 2002 Smoluchowski-Poisson 方程式

ニュートン粒子

BBGKY hierarchy $\{P_i\}_{i=1,2,\cdots,N}$

factorization (propagation of chaos)

独立同分布

$$P_N(x_1, x_2, \cdots, x_N, t) = \prod_{i=1}^{N} P_1(x_i, t)$$

high-energy limit $\mu\hat{\beta}N\alpha^2 = \nu\beta, \ \alpha N = 1, \ \omega = P_1$

Euler-Smoluchowski-Poisson 方程式

$$\frac{\partial \omega}{\partial t} + \nabla^{\perp}\psi \cdot \nabla\omega = \nu\nabla \cdot (\nabla\omega + \beta\alpha\omega\nabla\psi)$$

$$-\Delta\psi = \omega, \ \psi|_{\partial\Omega} = 0$$

スモルコフスキー・ポアソン方程式

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$\left. \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0, \;\; u|_{t=0} = u_0(x) > 0$$

transport

closed system ↕

potential

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

Sire-Chavanis 02
motion of the mean field of many self-gravitating Brownian particles
kinetic equation + maximum entropy production

Chavanis 08
relaxation to the equilibrium in the point vortices BBGKY hierarchy + factorization

other Poisson parts

a) Debye system (DD model)

$$\Delta v = u, \quad v|_{\partial\Omega} = 0$$

global-in-time existence with compact orbit
Biler-Hebisch-Nadzieja 94

$$\|u\nabla u \cdot \nabla v\|_2 \leq C\|u\|_2 \|\nabla u\|_2 \|\nabla v\|_6$$

b) Childress-Percus-Jager-Luckhaus model (chemotaxis)

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$

$$\left. \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0, \;\; \int_\Omega v = 0$$

blowup threshold
a. Biler 98, Gajewski-Zacharias 98, Nagai-Senba-Yoshida 97
b. Nagai 01, Senba-S. 01b

統計集団と非平衡熱力学

S. Mean Field Theories and Dual Variation, 2nd edition, Atlantis Press, 2015

system	consistency	dynamics	ensemble
isolated	energy	entropy	micro-canonical
closed	temperature	Helmholtz free energy	canonical
open	pressure	Gibbs free energy	grand-canonical

場と粒子の双対性

particle density Smoluchowski ⟷ **duality** ⟷ **field potential** Poisson

$$v=(-\Delta)^{-1}u=\int_\Omega G(\cdot,x')u(x')dx'$$

symmetry

$$u_t=\nabla\cdot(\nabla u-u\nabla v)$$

$$\left.\frac{\partial u}{\partial\nu}-u\frac{\partial v}{\partial\nu}\right|_{\partial\Omega}=0$$

$$-\Delta v=u \quad v|_{\partial\Omega}=0$$

Helmholtz fee energy

$$\mathcal{F}(u)=\int_\Omega u(\log u-1)-\frac{1}{2}\langle(-\Delta)^{-1}u,u\rangle$$

$$\delta\mathcal{F}(u)=\log u-(-\Delta)^{-1}u$$

Model (B) equation

$$u_t=\nabla u\cdot\nabla\delta\mathcal{F}(u),\ \left.\frac{\partial}{\partial\nu}\delta\mathcal{F}(u)\right|_{\partial\Omega}=0$$

total mass conservation, free energy decreaseing

$$\Rightarrow\ \frac{d}{dt}\int_\Omega u=0,\ \frac{d\mathcal{F}}{dt}=-\int_\Omega u|\nabla\delta\mathcal{F}(u)|^2\leq 0$$

トーランド双対

$X/\mathbf{R}$ 実バナッハ空間

適切, 凸, 下半連続

$$F: X \to (-\infty, +\infty], \ \text{prop. c'x, l.s.c.}$$

ルジャンドル変換

$$F^*(p) = \sup_{x \in X} \{\langle x, p\rangle - F(x)\}$$

Fenchel-Moreau 双対

$$F^{**} = F, \ F^{**}(x) = \sup_{p \in X^*} \{\langle x, p\rangle - F^*(p)\}$$

Toland 双対

$$F, G: X \to (-\infty, +\infty], \ \text{prop. c'x l.s.c.}$$

$$J(x) = G(x) - F(x), \ J^*(p) = F^*(p) - G^*(p)$$

$$L(x,p) = F^*(p) + G(x) - \langle x, p\rangle$$ ラグランジュ関数

⟶

$$\inf_{X \times X^*} L = \inf_X J = \inf_{X^*} J^*$$

$$n = 2$$

$$\inf\{\mathcal{F}(u) \mid u \geq 0, \ \|u\|_1 = 8\pi\} > -\infty$$

$$n = 2$$

$$\inf\left\{J_{8\pi}(v) \mid v \in H_0^1(\Omega)\right\} > -\infty$$

自由エネルギー ⟵ 双対 ⟶ 場の汎関数

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle$$

$u \geq 0, \ \|u\|_1 = \lambda$ 粒子密度

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v + \lambda(\log\lambda - 1)$$

$v \in H_0^1(\Omega)$ ポテンシャル分布

基本構造

$$u_t = \nabla\cdot(\nabla u - u\nabla v),\ \ -\Delta v = u$$

$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, v\right)\Bigg|_{\partial\Omega} = 0$$

$$u = u(x,t) \geq 0$$ 粒子密度

$$j = -\nabla u + u\nabla v$$ 流束（拡散+走化性）

$$u_t + \nabla\cdot j = 0$$ 保存則

$$v = (-\Delta)^{-1}u$$ 場のポテンシャル

自己集合（走化性, 重力）
遠隔作用（長距離ポテンシャル）
対称性（作用反作用の法則）

1. 質量保存

モデルB方程式

$$\frac{d}{dt}\|u(t)\|_1 = 0$$

$$G(x,x') = G(x',x)$$ グリーン関数

2. 自由エネルギー減少

内部エネルギー　エントロピー

$$A = U - TS$$

温度一定＝閉じた系　正準集団

ヘルムホルツの自由エネルギー　温度

- (エントロピー)　内部エネルギー

$$\frac{d}{dt}\left\{\int_\Omega u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u\otimes u\right\} = -\int_\Omega u|\nabla(\log u - v)|^2 \leq 0$$

$$u\otimes u = u(x,t)u(x',t)\ dxdx'$$

スケーリング ～ 3たび 8π について

熱力学的汎関数とスケール不変性 → 臨界次元と臨界質量

Scaling and Variation

$$u_t = \nabla \cdot (\nabla u - u \nabla \Gamma * u)$$

$$(x,t) \in \mathbf{R}^n \times (0,T)$$

$$-\Delta \Gamma = \delta$$

self-similar transformation

$$u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t)$$

$$\mu > 0$$

critical dimension

$$u_\mu(x) = \mu^2 u(\mu x), \ \mu > 0$$

$$\|u\|_1 = \|u_\mu\|_1 \equiv \lambda \ \Leftrightarrow \ n = 2$$

Critical mass

$$\mathcal{F}(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2}\langle \Gamma * u, u \rangle$$

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

→

$$\mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi} \right) \log \mu + \mathcal{F}(u)$$

$$\lambda = 8\pi$$

dual Trudinger-Moser inequality

$$\inf \{ \mathcal{F}(u) \mid u \geq 0, \ \|u\|_1 = 8\pi \} > -\infty$$

$$u_t = \nabla \cdot (\nabla u - u \nabla v), \ -\Delta v = u$$

$$\left(\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}, v \right) \Big|_{\partial \Omega} = 0$$

blowup threshold

$$\lambda = \|u_0\|_1 < 8\pi \quad \Rightarrow \quad T = +\infty, \ \|u(t)\|_\infty \leq C$$

$$\|\exists u_0\|_1 > 8\pi, \ T < +\infty$$

グリーン関数の対称性と弱形式

$$\varphi \in C^2(\overline{\Omega}),\ \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0 \quad \Longrightarrow \quad \frac{d}{dt}\int_\Omega \varphi u(\cdot,t) = \int_\Omega \Delta\varphi \cdot u(\cdot,t) + \frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\varphi(x,x')u\otimes u$$

$$\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x') \in L^\infty(\Omega\times\Omega)$$

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$G(x,x') = \Gamma(x-x') + K(x,x'),\ K = K(x,x') \in C^{1+\theta,\theta}(\Omega\times\overline{\Omega})\cap C^{\theta,1+\theta}(\overline{\Omega}\times\Omega)$$

$$x_0 \in \partial\Omega \Longrightarrow G(x,x') = E(X,X') + K(x,x'),\ K = K(x,x') \in (C^{1+\theta,\theta}\cap C^{\theta,1+\theta})(\overline{\Omega\cap B(x_0,R)}\times\overline{\Omega\cap B(x_0,R)})$$

conformal diffeomorphism

$$X : \overline{\Omega\cap B(x_0,2R)} \to \overline{\mathbf{R}_+}^2 = \{(X_1,X_2) \mid X_2 \geq 0\} \qquad X(\partial\Omega\cap B(x_0,2R)) \subset \partial\mathbf{R}_+^2$$

$$E(X,X') = \Gamma(X-X') - \Gamma(X-X'_*) \qquad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mapsto X_* = \begin{pmatrix} X_1 \\ -X_2 \end{pmatrix}$$

主定理 (有限時間爆発) $T < +\infty$

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \text{ in } \mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})'$$

$$m(x_0) \in 8\pi\mathbf{N}$$

Collapse 質量の量子化と sub-collapse の衝突

境界爆発の排除

blowup set

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0,\ t_k \uparrow T,\ u(x_k,t_k) \to +\infty\} \subset \Omega,\ \sharp\mathcal{S} < +\infty$$

$$0 < f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$$

測度論的正則部分

ポアソン方程式 $$v = \int_\Omega G(\cdot,x')u(x')dx' \ \Leftrightarrow\ -\Delta v = u,\ \ v|_{\partial\Omega} = 0$$

対角部分 $$D = \{(x_i) \in \Omega^\ell \mid \exists i \neq j,\ x_i = x_j\}$$

ロバン関数 $$R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x-x'|\right]_{x'=x}$$

ε 正則性

$$\frac{1}{p+1}\frac{d}{dt}\|u\|_{p+1} = -\frac{4p}{(p+1)^2}\|\nabla u^{\frac{p+1}{2}}\|_2^2 + \frac{p}{p+1}\|u\|_{p+3}^{p+3}$$

$$\|z\|_{p+1}^{p+1} \leq C_p \|z\|_1 \|z\|_{H^1}^p$$

$n = 2$

Gagliardo-Nirenberg不等式

楕円型正則性

semi-group 評価

Moserの反復列

maximal regularity

$$\exists \varepsilon_0 > 0,\ \|u_0\|_1 < \varepsilon_0 \ \Rightarrow\ T = +\infty,\ \|u(\cdot,t)\|_\infty \leq C$$

$$\lambda < 8\pi \ \Rightarrow\ \|u(\cdot,t)\|_{L\log L} \leq C$$

$$\|u(\cdot,t)\|_{L\log L} \leq C \ \Rightarrow\ \exists s \gg 1,\ \|u(\cdot,t) \vee s\|_1 < \varepsilon_0$$

定理 1 $\|u_0\|_1 < 8\pi \ \Rightarrow\ T = +\infty,\ \|u(\cdot,t)\|_\infty \leq C$

局所化

$$\lim_{R\downarrow 0} \limsup_{t\uparrow T} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists \varepsilon_0 \Rightarrow x_0 \notin \mathcal{S}$$

nice cut-off function

$$x_0 \in \overline{\Omega},\ 0 < R \ll 1, \varphi = \varphi_{x_0,R} \in \mathcal{Y}$$

$$0 \leq \varphi \leq 1,\ \varphi = \begin{cases} 1, & x \in B(x_0, \frac{R}{2}) \\ 0, & x \in \mathbf{R}^2 \setminus B(x_0, R) \end{cases}$$

$$|\nabla \varphi| \leq CR^{-1}\varphi^{\frac{5}{6}}$$

$$|\nabla^2 \varphi| \leq CR^{-2}\varphi^{\frac{2}{3}}$$

$$\mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0\}$$

コラプスの形成

symmetry of the Green function → weak form (symmetrization)

$$\varphi \in C^2(\overline{\Omega}), \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\frac{d}{dt}\int_\Omega \varphi u(\cdot,t) = \int_\Omega \Delta\varphi \cdot u(\cdot,t) + \frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\varphi(x,x')u\otimes u$$

$$\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x')$$

$$\|\rho_\varphi\|_\infty \leq C\|\nabla\varphi\|_{C^1}$$

boundary behavior of the Green function
singularity cancellation by the symmetry

monotonicity formula $\lambda = \|u(\cdot,t)\|_1$

$$\left|\frac{d}{dt}\int_\Omega u\varphi\right| \leq C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1}$$

→ **weak continuation**

$$0 \leq \exists\mu(dx,t) \in C_*([0,T],\mathcal{M}(\overline{\Omega}))$$

$$u(x,t)dx = \mu(dx,t), \ 0 \leq t < T$$

ε -regularity

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists\varepsilon_0 \Rightarrow x_0 \notin \mathcal{S}$$

→

$$x_0 \in \mathcal{S} \Rightarrow \lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0$$

⟷ monotonicity formula

$$\lim_{R\downarrow 0}\liminf_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0$$

→ $\sharp\mathcal{S} < +\infty$ → **formation of collapse**

$$\mu(dx,T) = \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0} + f(x)dx, \ m(x_0) \geq \varepsilon_0, \ 0 \leq f = f(x) \in L^1(\Omega)$$

弱解の生成

$0 \le \mu = \mu(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$ weak solution

$\overset{\text{def}}{\Longleftrightarrow}$ $0 \le \exists \mathcal{N} = \mathcal{N}(\cdot,t) \in L^\infty_*([0,T], \mathcal{X}')$

1. $t \in [0,T] \mapsto \langle \varphi, \mu(dx,t) \rangle, \ \varphi \in \mathcal{Y}$ a.c.
2. $\frac{d}{dt}\langle \varphi, \mu \rangle = \langle \Delta\varphi, \mu \rangle + \frac{1}{2}\langle \rho_\varphi, \mathcal{N}(\cdot,t) \rangle$ a.e. $t \in [0,T]$
3. $\mathcal{N}|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu \otimes \mu$

$$\mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0\} \quad \overset{\text{可分}}{\mathcal{X}} = [\mathcal{X}_0]^{L^\infty(\Omega\times\Omega)}$$

$$\mathcal{X}_0 = \{\rho_\varphi + \psi \mid \varphi \in \mathcal{Y}, \ \psi \in C(\overline{\Omega}\times\overline{\Omega})\}$$

$\Longrightarrow$

$$\mu(\overline{\Omega},t) = \mu(\overline{\Omega},0) \equiv \lambda, \ 0 \le t \le T$$

$$\left|\frac{d}{dt}\langle \varphi, \mu(dx,t) \rangle\right| \le C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1}$$

$u = u(x,t)$ classical solution

$\Longrightarrow$ $\mathcal{N}(\cdot,t) = u(x,t) \otimes u(x',t) \ dxdx'$

$$\|\mathcal{N}(\cdot,t)\|_{\mathcal{X}'} = \lambda^2, \ \lambda = \|u_0\|_1$$

定理 2 $\mu_k(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$

$\mathcal{N}_k \in L^\infty_*([0,T], \mathcal{X}')$ weak solutions

$0 \le \mu_k(\overline{\Omega},t) \le C$

$\|\mathcal{N}_k(\cdot,t)\|_{\mathcal{X}'} \le C$ $\Longrightarrow$ sub-sequence

$\mu_k(dx,t) \rightharpoonup \mu(dx,t) \quad \text{in } C_*([0,T], \mathcal{M}(\overline{\Omega}))$

$\mathcal{N}_k(\cdot,t) \rightharpoonup \mathcal{N}(\cdot,t) \quad \text{in } L^\infty_*([0,T], \mathcal{X}')$ weal solution

後方自己相似変換 $x_0 \in \mathcal{S}$

$$y = (x - x_0)/(T-t)^{1/2}, \ s = -\log(T-t)$$
$$z(y,s) = (T-t)u(x,t)$$

weak limit $s_k \uparrow +\infty$ subsequence

$$z(y, s+s_k)dy \rightharpoonup \exists \zeta(dy,s) \text{ in } C_*(-\infty,+\infty; \mathcal{M}(\mathbf{R}^2))$$

Put $z(y,s) = 0$ where it is not defined.

$$\mathcal{M}(\mathbf{R}^2) = C_\infty(\mathbf{R}^2)'$$
$$C_\infty(\mathbf{R}^2) = \{z \in C(\mathbf{R}^2 \cup \{\infty\}), \ z(\infty) = 0\}$$

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

第1放物包 parabolic envelope

$$\left| \frac{d}{dt} \int_\Omega u(\cdot,t)\varphi_{x_0,R} \right| \leq C_\lambda R^{-2}, \ 0 < R \ll 1$$

$$|\langle \varphi_{x_0,R}, u(\cdot,t)dx\rangle - \langle \varphi_{x_0,R}, \mu(dx,T)\rangle| \leq C_\lambda (T-t)/R^2$$
$$s_k + s = -\log(T-t), \ R = b(T-t)^{1/2}$$

$\longrightarrow$

$$|\langle \varphi_{0,b}, z(\cdot, s+s_k)dy\rangle - \langle \varphi_{x_0, be^{-(s+s_k)/2}}, \mu(dx,T)\rangle|$$
$$\leq C_\lambda / b^2$$

$$\mu(dx,T) = \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$$k \to \infty, \ b \uparrow +\infty \quad \longrightarrow \quad m(x_0) = \zeta(\mathbf{R}^2, s)$$

第2放物包 $\langle |y|^2, \zeta(dy,s)\rangle \leq C$

弱スケール極限　$x_0 \in \mathcal{S}$

$$y = (x - x_0)/(T-t)^{1/2}, \ s = -\log(T-t)$$
$$z(y,s) = (T-t)u(x,t)$$

$$y \in (T-t)^{-1/2}(\Omega - \{x_0\}) = \Omega_s$$

$$-\log T \leq s < +\infty, \ \|z(\cdot,s)\|_1 = \lambda$$

$$z_s = \nabla \cdot (\nabla z - z\nabla(w + |y|^2/4))$$
$$\frac{\partial z}{\partial \nu} - z\frac{\partial}{\partial \nu}(w + |y|^2/4)\Big|_{\partial\Omega_s} = 0$$

$$w(\cdot,s) = \int_{\Omega_s} G_s(\cdot,y')z(y',s)dy'$$
$$G_s(y,y') = G(x,x')$$

定理 3　$x_0 \in \Omega$ ⟹ $s_k \uparrow +\infty$ subsequence

$$z(y,s+s_k)dy \rightharpoonup \exists \zeta(dy,s) \text{ in } C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$
$$\zeta_s = \nabla \cdot (\nabla \zeta - \zeta \nabla(\Gamma * \zeta + |y|^2/4)) \ \text{ in } \mathbf{R}^2 \times (-\infty,+\infty)$$

証明　$\varphi \in C_0^2(\mathbf{R}^2), \ s \gg 1$

$$\frac{d}{ds}\int_{\mathcal{O}_s} z\varphi = \int_{\mathcal{O}_s}(\partial_s\varphi + y\cdot\nabla\varphi + \Delta\varphi)z$$
$$+\frac{1}{2}\int_{\mathcal{O}_s\times\mathcal{O}_s}\rho_\varphi^s(y,y')z\otimes z$$

$$\mathcal{O}_s = \Omega_s \times \{s\}$$
$$\rho_\varphi^s(y,y') = \nabla\varphi(y)\cdot\nabla_y G_s(y,y')$$
$$+\nabla\varphi(y')\cdot\nabla_{y'}G_s(y,y')$$

$$G(x,x') = \Gamma(x-x') + K(x,x')$$
$$(x,x') \in (\overline{\Omega}\times\Omega)\cup(\Omega\times\overline{\Omega})$$
$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$G_s(y,y') = \Gamma(y-y') - \frac{s}{4\pi}$$
$$+K(e^{-s}y + x_0, e^{-s}y' + x_0)$$

□

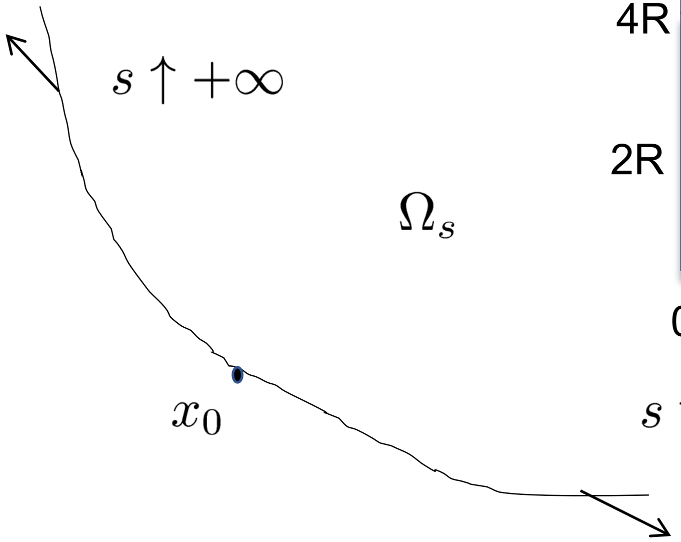
境界爆発点の消滅 $x_0 \in \partial\Omega$

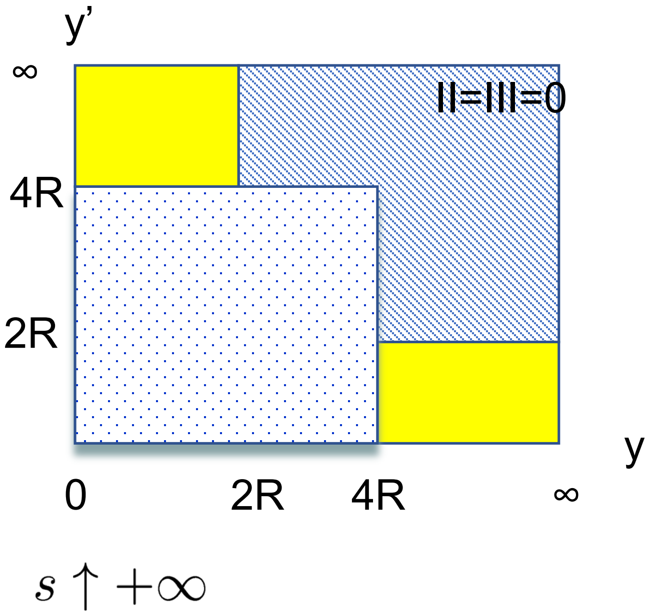
$$0 \le \zeta(dy,s),\ \zeta(\mathbf{R}^2,s) \le \lambda \equiv \|u_0\|_1,\ \mathrm{supp}\ \zeta(dy,s) \subset \overline{\mathbf{R}^2_+}$$

$$\zeta_s = \nabla\cdot(\nabla\zeta - \zeta\nabla(E*\zeta + |y|^2/4))\ \text{in}\ \mathbf{R}^2_+ \times (-\infty,+\infty)$$

$$\frac{\partial z}{\partial\nu} - z\frac{\partial}{\partial\nu}(E*\zeta + \frac{|y|^2}{4})\Big|_{\partial\mathbf{R}^2_+} = 0$$

$$E(y,y') = \Gamma(y-y') - \Gamma(y-y'_*)$$





$$\varphi = |y|^2\psi_R,\ \psi_R(y) = \psi(y/R)$$
$$\psi = \varphi_{0,2}(|y|)$$

$$\Delta\varphi = 4\psi_R + 4\frac{y}{R}\cdot\nabla\psi(\frac{y}{R}) + \frac{|y|^2}{R^2}\Delta\psi(\frac{y}{R})$$

$$y\cdot\nabla\varphi = 2|y|^2\psi_R + |y|^2\frac{y}{R}\cdot\nabla\psi(\frac{y}{R})$$

$$\langle 1+|y|^2, \zeta(dy,s)\rangle \le C$$

$\Rightarrow$ (dominated convergence theorem)

$$\lim_{R\uparrow+\infty}\int_{s_1}^{s_2}\langle\Delta\varphi + \frac{y}{2}\cdot\nabla\varphi, \zeta(dy,s)\rangle$$
$$= 4(s_2-s_1)m(x_0) + \int_{s_1}^{s_2} I(s)ds$$
$$I(s) = \langle |y|^2, \zeta(dy,s)\rangle$$

$$\rho^0_\varphi(y,y') = \nabla\varphi(y)\cdot\nabla_y E(y,y')$$
$$+\nabla\varphi(y')\cdot\nabla_{y'}F(y,y')$$
$$= I + II + III,\ \varphi = |y|^2\psi_R$$

$$I = \psi_R(y)\nabla|y|^2 \cdot \nabla_y F(y,y')$$
$$+\psi_R(y')\nabla|y'|^2 \cdot \nabla_{y'} F(y,y')$$

$$II = (|y|^2 - |y'|^2)\nabla\psi_R(y) \cdot \nabla_y F(y,y')$$

$$III =$$
$$|y'|^2(\nabla\psi_R(y) - \nabla\psi_R(y')) \cdot \nabla_{y'} F(y,y')$$

$$\rho^0_{|y|^2}(y,y') = 0 \ \Rightarrow$$
$$I = (\psi_R(y) - \psi_R(y'))\nabla|y|^2 \cdot \nabla_y F(y,y')$$

$$|y| < 2R \Rightarrow$$
$$|I| \le \|\nabla\psi\|_\infty \cdot \frac{|y-y'|}{R} \cdot 2|y| \cdot \frac{1}{\pi|y-y'|}$$
$$\le \frac{2}{\pi}\|\nabla\psi\|_\infty \varphi_{0,4R}(y) \cdot \frac{|y|}{R}$$

$$|I| \le C\left(\varphi_{0,4R}(y)\frac{|y|}{R} + \varphi_{0,4R}(y')\frac{|y'|}{R}\right)$$
$$y,y' \in \mathbf{R}^2$$

$$0 \le \varphi_{0,4R}(y)\frac{|y|}{R} + \varphi_{0,4R}(y')\frac{|y'|}{R} \le C$$
$$\to 0$$
$$\forall y,y' \text{ as } R\uparrow +\infty$$

$$|II| + |III| \le CH_R(y,y'), \quad y,y' \in \mathbf{R}^2$$

$$0 \le H_R(y,y') =$$
$$(\varphi_{0,8R}(y)(1+|y|) + \varphi_{0,8R}(y')(1+|y'|))$$
$$\cdot \left[\frac{|y|}{R} + \frac{|y|^2}{R^2} + \frac{|y'|}{R} + \frac{|y'|^2}{R^2}\right]$$
$$\le C(1+|y|^2)(1+|y'|^2)$$

$$\lim_{R\uparrow+\infty} H_R(y,y') = 0, \quad \forall y,y' \in \mathbf{R}^2$$

$$\langle 1+|y|^2, \zeta(dy,s)\rangle \leq C$$

$$\lim_{R\uparrow+\infty}\int_{s_1}^{s_2} \langle \rho_\varphi^0(y,y'), \mathcal{K}(\cdot,s)\rangle_{\mathcal{E},\mathcal{E}'}\, ds = 0$$

$$I(s_2)-I(s_1)=\int_{s_1}^{s_2} 4m(x_0)+I(s)\ ds$$

$$I(s)=\langle |y|^2, \zeta(dy,s)\rangle \leq C$$

$$\frac{dI}{ds}=4m(x_0)+I(s)\ \text{a.e.}\ \ s$$

$\Rightarrow$

$$\lim_{R\uparrow+\infty} I(s)=+\infty, \text{ a contradiction} \qquad \square$$

スケーリングバック

$$\zeta(dy,s)=e^{-s}A(dy',s'),\ \ y'=e^{-s/2}y,\ \ s'=-e^{-s}$$

$$A_s=\nabla\cdot(\nabla A-A\nabla\Gamma * A) \text{ in } \mathbf{R}^2\times(-\infty,0)$$
$$A=A(dy,s)\geq 0,\ \ A(\mathbf{R}^2,s)=m(x_0)$$

弱リュービル性～黒木場・小川の定理

弱解

$$a_s=\nabla\cdot(\nabla a-a\nabla\Gamma * a) \text{ in } \mathbf{R}^2\times(-\infty,+\infty)$$
$$\Rightarrow\ \ a(\mathbf{R}^2,s)=0 \text{ or } 8\pi$$

局所2次モーメント＋スケーリング