偏微分方程式論における幾何学的方法 II

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ボルツマン・ポアソン方程式

渦度場方程式

2D Euler Equation
(simply connected domain)

$$v_t + (v \cdot \nabla) v = -\nabla p$$
$$\nabla \cdot v = 0$$
$$\nu \cdot v|_{\partial\Omega} = 0$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}$$ gradient

vorticity

$$\omega = \nabla^\perp v$$
$$\nabla^\perp = \begin{pmatrix} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{pmatrix}$$
$$x = (x_1, x_2)$$

vortex equation

$$\omega_t + v \cdot \nabla \omega = 0$$
$$v = \nabla^\perp \psi$$
$$\Delta \psi = -\omega$$ stream function
$$\nabla^\perp \cdot \nabla^\perp = \Delta \qquad \psi|_{\partial\Omega} = 0$$

point vortices

$$\omega(dx, t) = \sum_{i=1}^{N} \alpha_i \delta_{x_i(t)}(dx)$$

Kirchhoff equation

$$\alpha_i \frac{dx_i}{dt} = \nabla_i^\perp H, \ 1 \le i \le N$$

Hamiltonian

$$H = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

Green's function

$$-\Delta G(x, x') = \delta_{x'}(dx)$$
$$G(x, x')|_{\partial\Omega} = 0$$
$$(x, x') \in \overline{\Omega} \times \Omega$$

Robin function

$$R(x) = \left[G(x, x') + \frac{1}{2\pi} \log |x - x'| \right]_{x'=x}$$

オンサーガーの予言

Hamiltonian $$H = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

$H = H_N(x_1, \cdots, x_N)$ $N \gg 1$ total energy

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad 1 \leq i \leq N$$

$$p_i = p_i(t), \ q_i = q_i(t) \ \in \mathbf{R}^2$$

micro-canonical ensemble $$\mathbf{R}^{4N}/\{H = E\}$$

$$x = (q_1, \ldots, q_N, p_1, \ldots, p_N)$$

co-area formula $$dx = dE \cdot \frac{d\Sigma(E)}{|\nabla H|}$$

$$d\Sigma(E) \ \leftrightarrow \ \{x \in \mathbf{R}^{4N} \mid H(x) = E\}$$

→ canonical ensemble

thermal equilibrium

Gibbs measure $$\mu^{E,N} = \frac{1}{W(E)} \cdot \frac{d\Sigma(E)}{|\nabla H|}$$

weight factor $$W(E) = \int_{H=E} \frac{d\Sigma(E)}{|\nabla H|}$$

inverse temperature $$\beta = \frac{\partial}{\partial E} \log W(E) = \frac{\Theta''(E)}{\Theta'(E)}$$

$$\Theta(E) = \int_{H<E} dx = \int_{-\infty}^{E} W(E')dE'$$

↑ bounded monotone

<u>Onsager 49</u>

$E \gg 1 \Rightarrow \beta < 0$ ordered structure in <u>negative temperature</u>

平均場方程式導出の原理

micro-canonical statistics

$$\mathbf{R}^{4N}/\{H=E\}$$

$$x=(q_1,\ldots,q_N,p_1,\ldots,p_N)$$

$$dx=dE\cdot\frac{d\Sigma(E)}{|\nabla H|}$$

$$d\Sigma(E)\ \leftrightarrow\ \{x\in\mathbf{R}^{4N}\mid H(x)=E\}$$

micro-canonical measure

$$d\mu^{E,N}=\frac{1}{W(E)}\cdot\frac{d\Sigma(E)}{|\nabla H|}$$

weight factor

$$W(E)=\int_{\{H=E\}}\frac{d\Sigma(E)}{|\nabla H|}$$

canonical statistics

$$\mathbf{R}^{4N}/\{T\}$$

inverse temperature

ボルツマン定数

$$\beta=1/(kT)$$

canonical measure

$$d\mu^{\beta,N}=\frac{e^{-\beta H}dx}{Z(\beta,N)}$$

weight factor

$$Z(\beta,N)=\int_{\mathbf{R}^{4N}}e^{-\beta H}dx$$

thermo-dynamical relation

$$\beta=\frac{\partial}{\partial E}\log W(E)$$

micro-canonical probability measure

$$\mu^N=\mu^N(dx_1,\ldots,dx_N)$$

one point pdf

$$\rho_1^N(x_i)dx_i=\int_{\Omega^{N-1}}\mu^N(dx_1\ldots dx_{i-1}dx_{i+1}dx_N)$$

↑

equal a priori probability (independent of i)

k-point reduced pdf

$$\rho_k^N(x_1,\ldots,x_k)dx_1\ldots dx_k=\int_{\Omega^{N-k}}\mu^N(dx_{k+1},\ldots,dx_N)$$

stationary point vortices

$$\omega_N(x)dx=\sum_{i=1}^N\alpha\delta_{x_i}(dx)$$

⟹

$$\langle\omega_N(x)\rangle=\sum_{i=1}^N\int_{\Omega^N}\alpha\delta(x_i-x)\mu^N(dx_1\ldots dx_N)=N\alpha\rho_1^N(x)$$

phase mean

カオスの伝播と一意性定理

high energy limit
(single intensity)

$$\alpha_i = \hat{\alpha},\ N \uparrow +\infty,\ \hat{\alpha} N = 1$$
$$\hat{H}_N = H,\ \hat{\alpha}^2 N \hat{\beta} = \beta$$

$$\hat{H}_N(x_1, \ldots, x_N) = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

energy $\tilde{E} = H$

inverse temperature $\tilde{\beta} = \frac{\partial}{\partial \tilde{E}} \log W(\tilde{E})$

weight factor $W(\tilde{E}) = \int_{H=\tilde{E}} \frac{d\Sigma_{\tilde{E}}}{|\nabla H|}$

mean field limit $\lim_{N\to\infty} \langle \omega_N(x) \rangle = \rho(x) = \lim_{N\to\infty} N\alpha\rho_1^N(x)$

propagation of chaos
(factorization property) $\rho_k^N \rightharpoonup \rho^{\otimes k} = \Pi_{i=1}^k \rho(x_i)$

(uniqueness theorem)

→ two point pdf compatibility

Boltzmann $\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}$

duality ↕

Poisson $\psi = \int_\Omega G(\cdot, x')\rho(x')dx'$

Joyce-Montgomery 73

rigorous derivation

Caglioti-Lions-Marchioro-Pulvirenti 92, 95. Kiessling 93

1. Bounded Boltzmann weight factors {z}
2. Uniqueness of the solution to the limit equation

→

1. convergence to the limit
2. canonical-micro canonical equivalence in the limit
3. propagation of chaos

OK if $\beta > -8\pi$

楕円型理論への衝撃

循環するハミルトニアン

定理 A（S. 92）

$$0 < \lambda < 8\pi \Rightarrow \exists 1 \text{ solution}$$

Boltzmann Poisson Equation

$\Omega \subset \mathbf{R}^2$ bounded domain $\partial\Omega$ smooth

$\lambda > 0$ constant

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega, \ v = 0 \text{ on } \partial\Omega$$

$\longleftrightarrow$

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}, \ \lambda = -\beta$$

$$\psi = \int_\Omega G(\cdot, x')\rho(x')dx'$$

mean field equation in stream function

定理 B（Nagasaki-S. 90）

$\{(\lambda_k, v_k)\}$ solution sequence s.t.

$\lambda_k \to \lambda_0 \in [0, \infty), \ \|v_k\|_\infty \to \infty$

quantized blowup mechanism

$\Rightarrow \lambda_0 = 8\pi\ell, \ \ell \in \mathbf{N}$

$\exists$sub-sequence, $\exists \mathcal{S} \subset \Omega, \ \sharp\mathcal{S} = \ell$, s.t.

$v_k \to v_0$ loc. unif. in $\overline{\Omega} \setminus \mathcal{S}$

$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0)$$

recursive hierarchy

$$\nabla_{x_i} H_\ell(x_1^*, \cdots, x_\ell^*) = 0, \ 1 \leq i \leq \ell$$

$G = G(x, x')$ the Green's function

$\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$

$$H_\ell(x_1, \ldots, x_\ell) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$$

$$R(x) = \left[G(x, x') + \frac{1}{2\pi}\log|x - x'|\right]_{x=x'}$$

リュービル積分

$$-\Delta u = e^u, \quad u = v + \log\lambda - \log\int_\Omega e^v \qquad z = x_1 + \sqrt{-1}x_2,\ x = (x_1, x_2) \in \Omega$$

$$u_{z\overline{z}} = -\frac{1}{4}e^u,\ \overline{z} = x_1 - \sqrt{-1}x_2 \qquad s = u_{zz} - \frac{1}{2}u_z^2$$ リッカチ方程式

$$\longrightarrow \quad s_{\overline{z}} = u_{zz\overline{z}} - u_z u_{z\overline{z}} = -\frac{1}{4}e^u u_z + \frac{1}{4}e^u u_z = 0 \qquad s = s(z)$$

$$\phi = e^{-u/2} \longrightarrow \phi_{zz} + \frac{1}{2}s\phi = 0 \qquad \{\phi_1, \phi_2\}$$ fundamental system of solutions

$$(\phi_1, \phi_{1z})|_{z=z^*} = (1,0),\ (\phi_2, \phi_{2z})|_{z=z^*} = (0,1)$$

$$\phi = e^{-u/2} = \overline{f}_1(\overline{z})\phi_1(z) + \overline{f}_2(\overline{z})\phi_2(z),\ \exists\overline{f}_1(\overline{z}), \exists\overline{f}_2(\overline{z})$$

$$z^* \leftrightarrow x^*,\ \nabla u(x^*) = 0 \longrightarrow e^{-u/2} = C_1|\phi_1|^2 + C_2|\phi_2|^2$$

$$F = \psi_2/\psi_1,\ \psi_1 = C_1^{1/2}8^{-1/4}\phi_1,\ \psi_2 = C_2^{-1/2}8^{1/4}\phi_2 \longrightarrow \left(\frac{1}{8}\right)^{1/2} e^{u/2} = \frac{|F'|}{1+|F|^2} \equiv \rho(F)$$ spherical derivative

球面導関数と被覆の量子化～8πについて

$$-\Delta v = \sigma e^v, \ \sigma = \frac{\lambda}{\int_\Omega e^v dx}$$

$$\Leftrightarrow \ \exists F = F(z), \ z \in \Omega \subset \mathbf{R}^2 \cong \mathbf{C}$$ 有理型関数

$$\hat{F} = \sqrt{8} \circ F : \Omega \to S^2$$ conformal

$$\left. \frac{d\Sigma}{ds} \right|_{\partial\Omega} = \sigma^{1/2}$$

$(S^2, d\Sigma)$ round sphere

$|S^2| = 8\pi$

$$\rho(F) = \left(\frac{\sigma}{8}\right)^{1/2} e^{v/2} = \frac{|F'|}{1+|F|^2}$$ 球面導関数

$$-\Delta v = \sigma e^v, \ v|_{\partial\Omega} = 0 \ \Leftrightarrow \ \rho(F)|_{\partial\Omega} = \left(\frac{\sigma}{8}\right)^{1/2}$$

$$\int_{\partial\Omega} \frac{d\Sigma}{ds} ds = |\partial\Omega| \, \sigma^{1/2}$$

immersed length of $\hat{F}(\partial\Omega)$

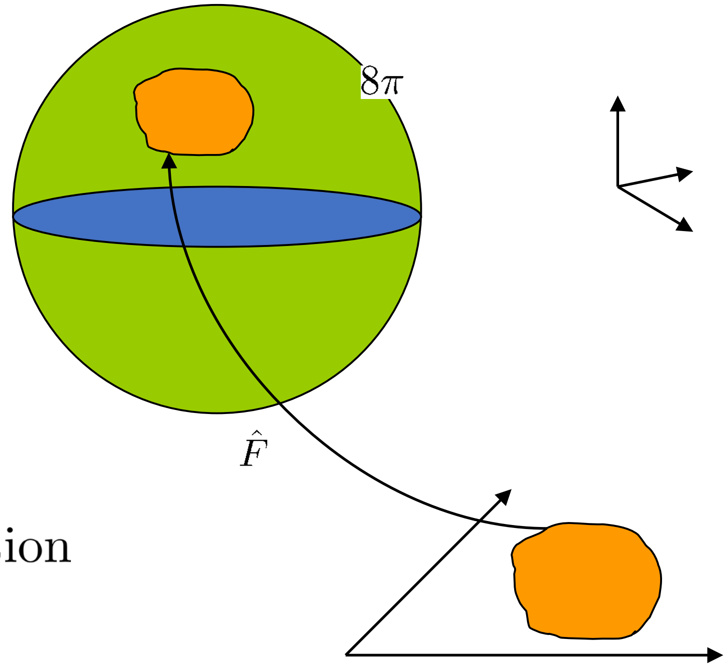
定理 B の証明

1. Liouville integral
2. boundary reflection
3. elliptic regularity

4. complex function theory
4-1. maximum principle
4-2. Montel's theorem
4-3. theorem of coincidence
4-4. residue analysis

$$\int_\Omega \left(\frac{d\Sigma}{ds}\right)^2 dx = 8 \int_\Omega \rho(F)^2 dx = \int_\Omega \sigma e^v$$

immersed area of $\hat{F}(\Omega)$



$$\lambda = \int_\Omega \sigma e^v \to 8\pi\ell$$

$\Leftrightarrow$ total mass quantization

due to ℓ-covering

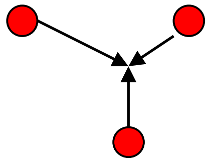
爆発解析の導入

$\Omega \subset \mathbf{R}^2$: open set, $V \in C(\overline{\Omega})$

$-\Delta v = V(x)e^v, \quad 0 \leq V(x) \leq b \quad \text{in } \Omega$

$\int_\Omega e^v \leq C$

定理 C （Li-Shafrir 94）



$\{(V_k, v_k)\}$ solution sequence

$V_k \to V$ loc. unif. in Ω

$\Rightarrow {}^\exists$ sub-sequence with the alternatives;

1. $\{v_k\}$: loc. unif. bdd in Ω

2. $\exists \mathcal{S} \subset \Omega$, $\sharp\mathcal{S} < +\infty$
$v_k \to -\infty$ loc. unif. in $\Omega \setminus \mathcal{S}$
$\mathcal{S} = \{x_0 \in \Omega \mid \exists x_k \to x_0, \ v_k(x_k) \to +\infty$
$V_k(x)e^{v_k}dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx)$ in $\mathcal{M}(\Omega)$
$m(x_0) \in 8\pi\mathbf{N}$

3. $v_k \to -\infty$ loc. unif. in Ω

Comments

1. mass quantization for variable coefficients without boundary condition
2. possible collapse collision
3. many applications together with the proof

prescaled analysis ...Brezis-Merle 91

linear theory $\Rightarrow$

1, 2 with $m(x_0) \geq 4\pi$ (rough estimate), 3

2... localized to $B = B(0, R)$

$-\Delta v_k = V_k(x)e^{v_k}$, $V_k(x) \geq 0$ in B

$V_k \to V$ unif. in $\overline{B}$, $\max_{\overline{B}} v_k \to +\infty$

$\max_{\overline{B} \setminus B_r} v_k \to -\infty$, ${}^\forall r \in (0, R)$

$\lim_k \int_B V_k e^{v_k} = \alpha$, $\int_B e^{v_k} \leq C$

$\Rightarrow \alpha \in 8\pi\mathbf{N}$

リュービル性 〜 再び8πについて

$$v_k(x_k)=\|v_k\|_\infty,\ x_k\to 0$$

$$\tilde v_k(x)=v_k(\delta_k x+x_k)+2\log\delta_k,\ \delta_k=e^{-v_k(x_k)/2}\to 0$$

⟶

$$-\Delta\tilde v_k=V_k(\delta_k x+x_k)e^{\tilde v_k},\ \tilde v_k\le\tilde v_k(0)=0\ \text{in}\ B(0,R/2\delta_k)$$

$$\int_{B(0,R/2\delta_k)}e^{\tilde v_k}\le C$$

Theorem E (Li-Shafrir 94) スケール変数における遠方での挙動の制御

$$-\Delta v=V(x)e^v,\ 0<a\le V(x)\le b\ \text{in}\ \Omega$$

$K\subset\Omega$ コンパクト ⟶

$$\sup_K v+c_1\inf_\Omega v\le\exists c_2,\ \exists c_1\ge 1$$

⟶ **residual vanishing**

Theorem D (Chen-Li 1991) リュービル性

$$-\Delta v=e^v\ \text{in}\ \mathbf{R}^2,\ \int_{\mathbf{R}^2}e^v<+\infty$$

⟶

$$v(x)=\log\left\{\frac{8\mu^2}{(1+\mu^2|x-x_0|^2)^2}\right\},\ x_0\in\mathbf{R}^2,\ \mu>0$$

$$\int_{\mathbf{R}^2}e^v=8\pi$$

Theorem F (Y.Y. Li 99)

$$\max_{\partial B}v_k-\min_{\partial B}v_k\le C,\ \|\nabla V_k\|_\infty\le C$$

⟶ $\alpha=8\pi$

$$\left|v_k(x)-\log\frac{e^{v_k(0)}}{\left(1+\frac{V_k(0)}{8}e^{v_k(0)}|x|^2\right)^2}\right|\le C,\ x\in B$$

1. non-radial bifurcation on annulus (S.S. Lin 89 Nagasaki-S. 90b)

2. effective bound of blowup points for simply-connected domain (S.-Nagasaki 89 Grossi-F.Takahashi 10)

3. classification of singular limits (Nagasaki-S. 90a)

4. spherical mean value theorem (S. 90)

5. localization (Brezis-Merle 91)

6. entire solution (W. Chen-C. Li 91)

7. sup + inf inequality (Shafrir 92)

8. uniqueness (S. 92)

9. field-particle duality (S. 92 Wolansky 92)

10. singular perturbation (Weston 78 Moseley 83 S. 93 Baraket-Pacard 98 Esposito-Grossi-Pistoia 05 del Pino-Kowarzyk-Musso 05)

11. blowup analysis (Li-Shafrir 94)

12. Chern-Simons theory (Tarantello 96)

13. global bifurcation (S.-Nagasaki 89 Mizoguchi-S. 97 Chang-Chen-Lin 03)

14. min-max solution (Ding-Jost-Li Wang 99)

15. local uniform esitmate (Y.Y. Li 99)

16. variable coefficient (Ma-Wei 01)

17. refined asymptotics (Chen-Lin 02)

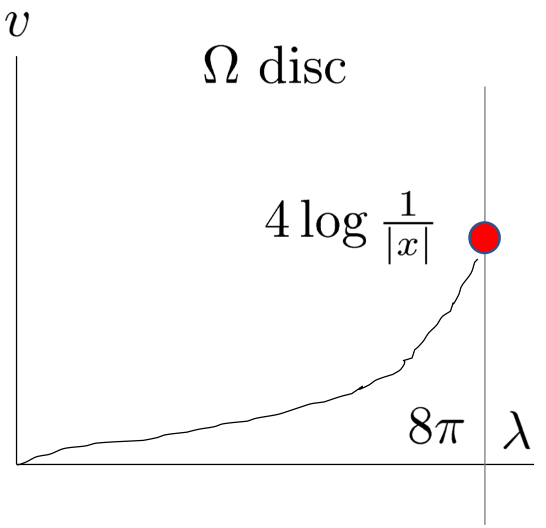
18. topological degree (Li 99 C.C. Chen-C.S. Lin 03 Malchiodi 08)

19. asymptotic non-degeneracy (Gladiali-Grossi 04 Grossi-Ohtsuka-S. 11)

20. isoperimetric profile (Lin-Lucia 06)

21. deformation lemma (Lucia 07)

22. Morse index (Gladiali-Grossi 09)



$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega \subset \mathbf{R}^2$$
$$v|_{\partial\Omega} = 0$$

モース指数の対応

Boltzmann-Poisson-Gel’fand equation

$$-\Delta v = \sigma e^{v}, \ \ v|_{\partial\Omega} = 0$$

$$\{(\sigma_k, v_k)\}, \ \sigma_k \to 0 \ \Rightarrow \ \text{(sub-sequence)}$$

$$\sigma_k \int_\Omega e^{v_k} dx \to 8\pi\ell, \ \ell = 0, 1, 2, \cdots, +\infty$$

$$0 < \ell < +\infty \ \Rightarrow \ \exists \mathcal{S} \subset \Omega, \ \sharp\mathcal{S} = \ell$$

$$v_k \to v_0 \ \text{loc. unif. in } \overline{\Omega} \setminus \mathcal{S} \qquad \mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$$

$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0) \qquad x_* = (x_1^*, \cdots, x_\ell^*)$$

$$\nabla H_\ell(x_*) = 0, \ H_\ell(x_1, \ldots, x_\ell) = \frac{1}{2} \sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$$

定理1 (Gladiali-Grossi-Ohtsuka-S. 14) $k \gg 1$

(augmented) Morse indices

$$\ell + \mathrm{ind}_M\{-H_\ell(x_*)\} \le \mathrm{ind}_M(v_k)$$

$$\mathrm{ind}_M^*(v_k) \le \ell + \mathrm{ind}_M^*\{-H_\ell(x_*)\}$$

Corollary (Gladiali-Grossi 09) x_* non-degenerate

➡ $v_k, \ k \gg 1$ non-degenerate

Theorem 2 (Baraket-Pacard 98)

$(x_1^*, \ldots, x_\ell^*) \in \Omega \times \ldots \times \Omega$

non-degenerate critical point of $H_\ell(x_1, \ldots, x_\ell)$

∃ sequence of ℓ point blow up solutions

Remark

1. only one point blowup and ∃1 blowup spot for convex domain
2. effective bound of the number of blowup points for simply connected domain
3. domain homology and Hamiltonian (Cao 10)
4. inhomogeneous coefficients, equations on manifold, etc. (Ohtsuka-Sato-S.)
5. one-point blowup case
6. refined asymptotics with Morse index correspondence
7. asymptotic non-degeneracy in multi-blowup

非退化性の証明

$$-\Delta v = \lambda e^v \text{ in } \Omega,\ \ v|_{\partial\Omega} = 0$$

$$\lambda_k \to 0,\ \ \lambda_k \int_\Omega e^{v_k} \to 8\pi$$

$$v_k(x) \to 8\pi G(x, x_0),\ \ x \in \overline{\Omega} \setminus \{x_0\}$$ locally uniformly

$$\nabla R(x_0) = 0$$

定理 2（定理 1 の系）

$x_0 \in \Omega$ non-degenerate critical point of $R(x)$

⟹ $-\Delta_D - \lambda_k e^{v_k},\ 0 < \sigma_k \ll 1$ non-degenerate

Proof. otherwise

$$\exists \lambda_k \downarrow 0,\ v_k,\ w_k,\ \ -\Delta v_k = \lambda_k e^{v_k} \text{ in } \Omega,\ \ v_k|_{\partial\Omega} = 0$$

$$-\Delta w_k = \lambda_k e^{v_k} w_k \text{ in } \Omega,\ \ w_k|_{\partial\Omega} = 0,\ \ \|w_k\|_\infty = 1$$

$$v_k(x_k) = \|v_k\|_\infty,\ x_k \to x_0$$

drop k

Green $$\int_{\partial\Omega} w\frac{\partial v_i}{\partial \nu} - v_i \frac{\partial w}{\partial \nu} ds = 0,\ \ v_i = \frac{\partial v}{\partial x_i}$$

scaling $$\delta_k^2 \lambda_k e^{v_k(x_k)} = 1$$

sub-sequence ~ locally uniformly in $\mathbf{R}^2$

$$\tilde{v}_k(x) = v_k(\delta_k x + x_k) - v(x_k) \to v_0(x)$$

$$\tilde{w}_k(x) = w_k(\delta_k x + x_k) \to w_0(x)$$

$$-\Delta v_0 = e^{v_0} \text{ in } \mathbf{R}^2,\ \ \int_{\mathbf{R}^2} e^{v_0} < +\infty$$

$$-\Delta w_0 = e^{v_0} w_0 \text{ in } \mathbf{R}^2,\ \ \|w_0\|_\infty \le 1$$

Liouville property – Baraket-Pacard 98

$$w_0(x) = a \cdot \frac{x}{1+|x|^2} + b\frac{8-|x|^2}{8+|x|^2},\ \ a \in \mathbf{R}^2,\ \ b \in \mathbf{R}$$

補題1 (Nagasaki-S.)

$$v_{ki} \to 8\pi \frac{\partial G}{\partial x_i}(\cdot, x_0)$$ locally uniformly $(\text{except for } x_0)$

補題2 (Gladiali-Grossi 09)

$$\delta_k^{-1} w_k \to 2\pi a \cdot \nabla_{x'} G(\cdot, x_0)$$ locally uniformly

Step 1

$$w_k = \gamma_k \{G(\cdot, x_0) + o(1)\} + 2\pi \delta_k a \cdot \nabla_{x'} G(\cdot, x_0) + o(\delta_k)$$

$$\gamma_k = \int_{\Omega \cap B(x_0, R)} \lambda_k e^{v_k} w_k dx'$$

1. removable singularity theory

$$w_k \to 0$$ locally uniformly

2. Green's formula

$$w_k(x) = \int_\Omega G(x, x') \lambda_k e^{v_k(x')} w_k(x') dx'$$

3. localization around $x' = x_0$

4. Y.Y. Li's estimate $|x - x_0| \geq \delta^k, \ 0 < k < 1/4$

5. Taylor's expansion $G(x, x'), \ x' = x_0, \ |x' - x_0| < \delta^k$

Step 2

$$\overline{w_k}(x = (x - x_0) \cdot \nabla v_k + 2, \ -\Delta \overline{w_k} = \lambda_k e^{v_k} \overline{w_k}$$

$$\int_{\partial B_R(x_0)} \frac{\partial \overline{w_k}}{\partial \nu} w_k - \overline{w_k} \frac{\partial w_k}{\partial \nu} d\sigma = 0$$ ➡ $\gamma_k = o(\delta_k)$

completion of the proof

non-degeneracy + Green+ $$\int_{\partial \Omega} \frac{\partial G}{\partial x_i}(x, y) \frac{\partial}{\partial \nu_x} \frac{\partial}{\partial y_j} G(x, y) ds_x = -\frac{1}{2} \frac{\partial^2 R}{\partial y_i \partial y_j}(y)$$

➡ a=0, b=0 ➡ $|\exists \tilde{x}_k| \to +\infty, \ w_k(\tilde{x}_k) = 1$

exclude by

1. Kelvin transformation
2. Y.Y. Li's estimate
3. maximum principle

Open questions

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \quad v|_{\partial\Omega} = 0$$

$$\{(\lambda_k, v_k)\}, \ \lambda_k \to 8\pi, \ \|v_k\|_\infty \to +\infty$$

$$v_k \to v_0 \ \text{loc. unif. in} \ \overline{\Omega} \setminus \mathcal{S}$$

$$v_0(x) = 8\pi G(x, x_0), \ \nabla R(x_0) = 0$$

$$g : B = B(0,1) \to \Omega \qquad \text{conformal}$$

$$g(z) = x_0 + \sum_{k=1}^{\infty} a_k z^k \qquad \nabla R(x_0) = 0 \ \Leftrightarrow \ a_2 = 0$$

$$\exists \nabla^2 R(x_0)^{-1} \ \Leftrightarrow \ |a_3/a_1| \neq 1/3$$

$$\lambda = 8\pi + C\sigma_k + o(\sigma_k), \ \sigma_k = \frac{\lambda_k}{\int_\Omega e^{v_k}} \to 0$$

$$\frac{C}{\pi} = -|a_1|^2 + \sum_{k=3}^{\infty} \frac{k^2}{k-2}|a_k|^2$$

$|a_3/a_1| \neq 1/3, \ C \neq 0$ **Conjecture**

→ $v_k, \ k \gg 1$ $\quad \mathcal{L}$ non-degenerate

Variation functional $\quad J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v, v \in H_0^1(\Omega)$

Quadratic form $\quad Q(\varphi, \varphi) = \left.\frac{d^2}{ds^2} J_\lambda(v + s\varphi)\right|_{s=0}$

$$\varphi \in H_0^1(\Omega)$$

$$p = \frac{\lambda e^v}{\int_\Omega e^v}]$$

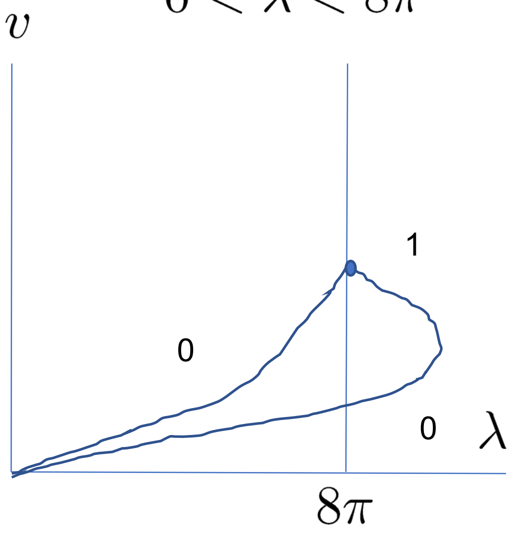
$$= (\nabla\varphi, \nabla\varphi) - \int_\Omega p\varphi^2 + \frac{1}{\lambda}\left(\int_\Omega p\varphi\right)^2$$

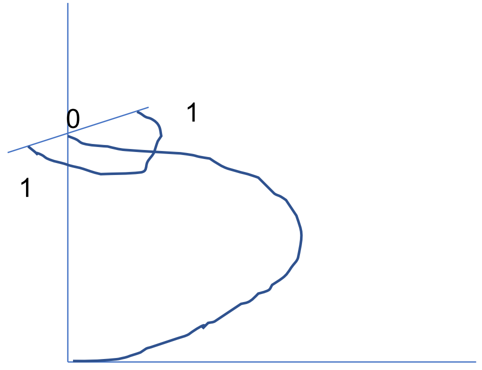
Linearized operator $\quad \mathcal{L}\psi = -\Delta\psi - p\psi + \frac{1}{\lambda}\left(\int_\Omega p\psi\right)p$

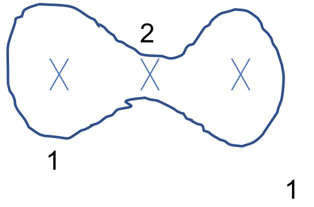
$$D(\mathcal{L}) = H_0^1(\Omega) \cap H^2(\Omega)$$

定理3 (S. 92, Bartoulucci-Lin 15)

$0 < \lambda < 8\pi$ → non-degenerate







Gladial-Grossi 04
Sato-S. 07
Grossi-Ohtsuka-S. 11
Ohtsuka-Sato-S. 13

多強度渦渡場モデル

1. Stochastic Case (Neri 04)

1-species

relative intensity $\alpha \in [-1,1]$ is a random variable subject to the distribution function $P(d\alpha)$

$$-\Delta v = \lambda \frac{\int_{[-1,1]} \alpha e^{\alpha v} P(d\alpha)}{\int_{]-1,1]} \int_\Omega e^{\alpha v} P(d\alpha)}, \ v|_{\partial\Omega} = 0$$

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_{[-1,1]} [\int_\Omega e^{\alpha v}] P(d\alpha)$$

$v \in H_0^1(\Omega)$

$$-\Delta v = \frac{\lambda(e^v - e^{-v})}{\int_\Omega e^v + e^{-v} \ dx}$$

2. Deterministic Case (Onsager's note, Sawada-S. 08)

ℓ-species

$n^i N$-particles take the intensity $\alpha^i \hat{\alpha}$

$0 < n^i < 1, \ -1 \leq \alpha^i \leq 1, \ \sum_i n_i = 1$

$1 \leq i \leq \ell, \ N\hat{\alpha} = 1, \ N \uparrow +\infty$

$$-\Delta v = \lambda \int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}} P(d\alpha), \ v|_{\partial\Omega} = 0$$

$$P(d\alpha) = \sum_{i=1}^{\ell} n^i \delta_{\alpha_i}, \text{ may } \ell = \infty$$

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \int_{[-1,1]} [\log \int_\Omega e^{\alpha v}] P(d\alpha)$$

$v \in H_0^1(\Omega)$

$$-\Delta v = \frac{\lambda}{2} \left(\frac{e^v}{\int_\Omega e^v dx} - \frac{e^{-v}}{\int_\Omega e^{-v} dx} \right)$$

決定論モデルのTM不等式

Sawada-S. functional $E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$

$$J_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \lambda \int_{[-1,1]} [\log \int_\Omega e^{\alpha v}]\, P(d\alpha)$$

$$\overline{\lambda} = \sup\{\lambda \mid \inf_E J_\lambda > -\infty\}$$

approach by blowup analysis (ORS 10)

$$\overline{\lambda} \geq \lambda_* = \inf \frac{8\pi}{\int_{I_\pm} \alpha^2 P(d\alpha)} \qquad \inf_E J_{\lambda_*} > -\infty$$

approach by duality (RS14) $\overline{\lambda} = \lambda^* \geq \lambda_*,$

$$\lambda^* = \inf \left\{ \frac{8\pi P(K_\pm)}{\left[\int_{K_\pm} \alpha P(d\alpha)\right]^2} \middle| K_\pm \subset I_\pm \cap \mathrm{supp}\, P \right\}$$

Theorem 1 (Ohtsuka-Ricciardi-S. 10) $\{v_k\} \subset E$

$$-\Delta v_k = \lambda_k \int_{[-1,1]} \alpha\left(\frac{e^{\alpha v_k}}{\int_\Omega e^{\alpha v_k}} - \frac{1}{|\Omega|}\right) P(d\alpha)$$

non-compact ⟹ sub-sequence

$$\frac{\lambda_k e^{\alpha v_k}}{\int_\Omega e^{\alpha v_k}} dx P(d\alpha) \rightharpoonup [\sum_{x_0 \in \mathcal{S}} m(x_0, \alpha)\delta_{x_0} + r(x,\alpha)] dx P(d\alpha)$$

$$m(x_0,\alpha) \geq 0, \ \sharp\mathcal{S} < +\infty$$

$$0 \leq r = r(x,\alpha) \in L^1(\Omega \times [-1,1], dxdP)$$

$$8\pi \int_{[-1,1]} m(x_0,\alpha) P(d\alpha) = \left\{ \int_{[-1,1]} \alpha m(x_0,\alpha) P(d\alpha) \right\}^2, \ \forall x_0 \in \mathcal{S}$$

$$I_- = [-1,0], \ I_+ = [0,1]$$

双対変数への変換

X Banach space/$\mathbf{R}$

$F : X \to (-\infty, +\infty]$ prop. c'x l.s.c.

$\Rightarrow$

Legendre transformation

$F^* : X^* \to (-\infty, +\infty]$ prop. c'x l.s.c.

$$F^*(p) = \sup_{x \in X} \{\langle x, p \rangle - F(x)\}$$

Fenchel-Moreau duality

$$F^{**} = F$$

$$F^{**}(x) = \sup_{p \in X^*} \{\langle x, p \rangle - F^*(p)\}$$

Toland duality 78, 79

$F, G : X \to (-\infty, +\infty]$ prop. c'x l.s.c.

$$J(x) = G(x) - F(x)$$

$$J^*(p) = F^*(p) - G^*(p)$$

$$L(x,p) = F^*(p) + G(x) - \langle x, p \rangle$$

... Lagrange function

$$\inf_{X \times X^*} L = \inf_X J = \inf_{X^*} J^*$$

Smoluchowski-Poisson equation

point vortex mean field equation (single intensity)

free energy ⟷ field functional

duality

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle (-\Delta)^{-1} u, u \rangle \qquad J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v + \lambda(\log \lambda - 1)$$

$u \geq 0, \ \|u\|_1 = \lambda$

particle distribution

potential density

定理 3 (Ricciardi-S. 14)

$\overline{\lambda} = \lambda^*$

$$\lambda^* = \inf \left\{ \frac{8\pi P(K_\pm)}{\left[\int_{K_\pm} \alpha P(d\alpha)\right]^2} \middle| K_\pm \subset I_\pm \cap \operatorname{supp} P \right\}$$

$I_- = [-1, 0]$

$I_+ = [0, 1]$

$$J_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \lambda \int_{[-1,1]} (\log \int_\Omega e^{\alpha v})\, P(d\alpha)$$

$$E = \{v \in H^1 \mid \int_\Omega v = 0\}$$

field functional

$$L(\oplus\rho_\alpha, v) = \int_{[-1,1]} \left[\int_\Omega \rho_\alpha(\log\rho_\alpha - 1)\right] P(d\alpha) + \frac{1}{2}\|\nabla v\|_2^2 - \int_{[-1,1]} \left[\int_\Omega \alpha\rho_\alpha v\right] P(d\alpha)$$

Lagrangian

$$\inf_{\oplus\Gamma_\lambda \times E} L = \inf_E J_\lambda + \lambda(\log\lambda - 1) = \inf_{\oplus\Gamma_\lambda} \mathcal{F}$$

unfolding-minimality

$$\mathcal{F}(\oplus\rho_\alpha) = \int_{[-1,1]} \left[\int_\Omega \rho_\alpha(\log\rho_\alpha - 1)\right] P(d\alpha) - \frac{1}{2}\int_{[-1,1]^2} \alpha\beta \left\langle \rho_\alpha, (-\Delta)^{-1}\rho_\beta \right\rangle P\otimes P(d\alpha d\beta)$$

free energy

$$\oplus\rho_\alpha \in \oplus\Gamma_\lambda,\ \Gamma_\lambda = \{\rho \geq 0 \mid \int_\Omega \rho = \lambda\} \quad \oplus\Gamma_\lambda = \{\oplus\rho_\alpha \mid \rho_\alpha \in \Gamma_\lambda, P\text{-a.e. } \alpha\}$$

一般化確率論モデルでの階層循環

$$-\Delta v = \lambda \int_I \alpha e^{\alpha v} P(\alpha) \text{ in } \Omega$$

$$v = 0 \text{ on } \partial\Omega$$

$$\lambda \int\int_{I\times\Omega} \alpha e^{\alpha v} P(d\alpha) dx \leq C$$

$\Omega \subset \mathbf{R}^2$ bounded domain

$\partial\Omega$ smooth boundary

$P(d\alpha)$ Borel measure on $I = [0,1]$

$1 \in \text{supp } P(d\alpha)$

Ricciardi-Zecca 16a,b
deMarchis-Ricchiardi 17

$P(d\alpha) = \delta_1(d\alpha)$ single intensity

1. blowup analysis
2. asymptotic non-degeneracy, Morse index calculation
3. deformation theory, topological degree calculation

$P(\{1\}) = \tau > 0$ non-degenerate case $\lambda\tau \mapsto \lambda$

$$-\Delta v = \lambda f(v), \ v|_{\partial\Omega} = 0, \quad 0 \leq f(v) = e^v + o(e^v), \ v \uparrow +\infty$$

Ye 97

otherwise

$$f(v) \equiv \int_I \alpha e^{\alpha v} P(d\alpha) = o(e^{\beta v}), \ v \uparrow +\infty, \forall \beta > 1$$

$$\lim_{v\uparrow+\infty} e^{-\beta v} f(v) = +\infty, \ \forall \beta < 1$$

S. 2022