Thou Shalt *Still* Buy and Hold

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This Talk Based on ...

Merton’s model
Classical Continuous-Time Portfolio Selection Models

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- Markowitz’s model
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- Markowitz’s model
- Behavioral model (recently)
Classical Continuous-Time Portfolio Selection Models

- Merton’s model
- Markowitz’s model
- Behavioral model (recently)
- No transaction costs
Classical Continuous-Time Portfolio Selection Models

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- Markowitz’s model
- Behavioral model (recently)
- No transaction costs
- Optimal portfolios are to “trade all the time” so as to keep certain “constant proportions”
“Continuous Trading” clearly incurs infinite costs
Portfolio Selection with Transaction Costs

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- In the presence of transaction costs: sell region, buy region and no trade region
Portfolio Selection with Transaction Costs

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- Trade only necessary
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- Liu and Loewenstein (2002), *Rev Fin Studies*, CRRA investor, transaction costs and finite horizon
Portfolio Selection with Transaction Costs

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- Liu and Loewenstein (2002), *Rev Fin Studies*, CRRA investor, transaction costs and finite horizon
  - An investor might optimally *never* buy stock if the horizon is short
“Continuous Trading” clearly incurs infinite costs

In the presence of transaction costs: sell region, buy region and no trade region

Trade only necessary

Liu and Loewenstein (2002), *Rev Fin Studies*, CRRA investor, transaction costs and finite horizon

- An investor might optimally *never* buy stock if the horizon is short
- An investor should *largely* buy and hold if the horizon is long
Conventional Wisdom: Buy and Hold

- Buy and Hold: Buy a *good* stock and leave it alone for long time
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  - “Never sell a security unless you need money”
    (EfficientMarket.ca)
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  - “Trading is hazardous to your wealth” (Barber and Odean, 2000, *J Fin*)
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- Grounds
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- Grounds
  - Efficient market hypothesis (EMH): every security is fairly valued at all times, so there is no point to trade (you trade because something happens to *you*, not because something happens to the *market*)
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  - Pure costs: bid/offer spread, brokerage, capital gain tax etc. (Warren Buffett is a buy-and-hold advocate rejecting EMH)
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- However, no well-established portfolio models produced *pure* buy-and-hold strategy
If I Were an Innocent Investor ...

- Let’s say I just bought a stock, and *must* sell it in one year
If I Were an Innocent Investor ...

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- Need to decide when to sell.
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  - Mean-Variance? Expected utility?
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- Criterion?
  - Mean-Variance? Expected utility? *I don’t know what you are talking about*
Let’s say I just bought a stock, and \textit{must} sell it in one year.

Need to decide when to sell.

Criterion?

- Mean-Variance? Expected utility? \textit{I don’t know what you are talking about}
- Let’s sell higher ...
Let’s say I just bought a stock, and *must* sell it in one year.

Need to decide when to sell.

Criterion?

- Mean-Variance? Expected utility? *I don’t know what you are talking about*?

- Let’s sell higher ... say, *at the maximum price*?
Let’s say I just bought a stock, and *must* sell it in one year

Need to decide when to sell

Criterion?

- Mean-Variance? Expected utility? *I don’t know what you are talking about*
- Let’s sell higher ... say, *at the maximum price*?

- Selling at the maximum price is a “mission impossible”
If I Were an Innocent Investor ...

- Let’s say I just bought a stock, and *must* sell it in one year
- Need to decide when to sell
- Criterion?
  - Mean-Variance? Expected utility? *I don’t know what you are talking about*
  - Let’s sell higher ... say, *at the maximum price?*
- Selling at the maximum price is a “mission impossible”
- How about selling at the price closest to the maximum?
Let’s say I just bought a stock, and must sell it in one year:

- Need to decide when to sell
- Criterion?
  - Mean-Variance? Expected utility? *I don’t know what you are talking about*
  - Let’s sell higher ... say, at the maximum price?

Selling at the maximum price is a “mission impossible”

- How about selling at the price closest to the maximum?
- ...or sell at the time when the expected relative error between the current price and the maximum price is minimised
The Model

- A Black–Scholes market with a stock and a saving account
The Model

- A Black–Scholes market with a stock and a saving account
- The discounted stock price follows, on \((\Omega, F, P)\):

\[
dP_t = (a - r)P_t dt + \sigma P_t dB_t, \quad \text{or} \quad P_t = e^{\mu t + \sigma B_t}
\]

where \(\mu = a - r - \frac{1}{2} \sigma^2\)
A Black–Scholes market with a stock and a saving account

The *discounted* stock price follows, on $(\Omega, \mathcal{F}, P)$:

$$dP_t = (a - r)P_t dt + \sigma P_t dB_t,$$

or

$$P_t = e^{\mu t + \sigma B_t},$$

where $\mu = a - r - \frac{1}{2} \sigma^2$

Let $M_t = \max_{0 \leq s \leq t} P_s$, $0 \leq t \leq T$
The Model

- A Black–Scholes market with a stock and a saving account
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- Let $M_t = \max_{0 \leq s \leq t} P_s, \ 0 \leq t \leq T$
- Consider the following optimal stopping problem

$$\min_{0 \leq \tau \leq T} E \left[ \frac{M_T - P_{\tau}}{M_T} \right]$$

where $\tau \in [0, T]$ is a $\{ B_t \}$-stopping time
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- Let $M_t = \max_{0 \leq s \leq t} P_s, \ 0 \leq t \leq T$
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$$\min_{0 \leq \tau \leq T} \mathbb{E} \left[ \frac{M_T - P_\tau}{M_T} \right]$$

where $\tau \in [0, T]$ is a $\{B_t\}$-stopping time

- ... or equivalently

$$\max_{0 \leq \tau \leq T} \mathbb{E} \left[ \frac{P_\tau}{M_T} \right]$$

\[
\min_{0 \leq \tau \leq T} E(B^0_\tau - S^0_T)^2
\]

where \( S^0_t := \max_{0 \leq s \leq t} B^0_s \), and obtained optimal

\[
\tau^* = \inf \left\{ 0 \leq t \leq T \left| \frac{S^0_t - B^0_t}{\sqrt{T - t}} \geq z^* \right. \right\}
\]

where \( z^* = 1.12... \)
Related (Probabilistic) Literature


\[
\min_{0 \leq \tau \leq T} E(B^0_{\tau} - S^0_T)^2
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\]

where \( z^* = 1.12 \ldots \)

- Pedersen (2003), *Stoch Stoch Rep*, considered

\[
\min_{0 \leq \tau \leq T} E|B^0_{\tau} - S^0_T|^p, \quad 0 \leq p < +\infty
\]
Assume $\sigma = 1$ (otherwise rescale the time)
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Rewrite $B_t^\mu = \mu t + B_t$, $S_t^\mu = \max_{0 \leq s \leq t} B_s^\mu$, $0 \leq t \leq T$
Assume $\sigma = 1$ (otherwise rescale the time)

Rewrite $B_t^\mu = \mu t + B_t$, $S_t^\mu = \max_{0 \leq s \leq t} B_s^\mu$, $0 \leq t \leq T$

The problem is

$$\max_{0 \leq \tau \leq T} E \left[ \frac{e^{B_\tau^\mu}}{e^{S_T^\mu}} \right]$$
The Model Rewritten

- Assume $\sigma = 1$ (otherwise rescale the time)
- Rewrite $B^\mu_t = \mu t + B_t$, $S^\mu_t = \max_{0 \leq s \leq t} B^\mu_s$, $0 \leq t \leq T$
- The problem is
  \[
  \max_{0 \leq \tau \leq T} E \left[ \frac{e^{B^\mu_\tau}}{e^{S^\mu_T}} \right]
  \]
- Not a standard optimal stopping problem: $S^\mu_T$ is not $B_t$-adapted!
For any $\{B_t\}$-stopping time $0 \leq \tau \leq T$, we have

$$E \left[ \frac{e^{B_\tau^\mu}}{e^{S_T^\mu}} \right] = E \left[ \frac{e^{B_\tau^\mu}}{\max\{e^{S_\tau^\mu}, e^{\max_{\tau \leq t \leq T} B_t^\mu}\}} \right]$$
For any \( \{B_t\} \)-stopping time \( 0 \leq \tau \leq T \), we have

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\]

\[
= E \left[ \min \left\{ e^{-(S^\mu_\tau - B^\mu_\tau)}, e^{-\max_{\tau \leq t \leq T} (B^\mu_t - B^\mu_\tau)} \right\} \right]
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\]
For any \( \{ B_t \} \)-stopping time \( 0 \leq \tau \leq T \), we have

\[
E \left[ \frac{e^{B_{\tau}}}{e^{S_{T}}} \right] = E \left[ \frac{e^{B_{\tau}}}{\max\{e^{S_{\tau}}, e^{\max_{\tau \leq t \leq T} B_{t}}\}} \right]
\]

\[
= E \left[ \min \left\{ e^{-(S_{\tau} - B_{\tau})}, e^{-\max_{\tau \leq t \leq T} (B_{t} - B_{\tau})} \right\} \right]
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\]

\[
= E \left[ E \left[ \min \left\{ e^{-x}, e^{-S_{T-\tau}} \right\} \bigg| x = S_{\tau} - B_{\tau} \right] \right]
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For any \( \{B_t\} \)-stopping time \( 0 \leq \tau \leq T \), we have

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\]

\[
= E \left[ E \left[ \min \left\{ e^{-x}, e^{-S^\mu_T - \tau} \right\} \mid x = S^\mu_\tau - B^\mu_\tau \right] \right]
\]

\[
= E \left[ G(\tau, X_\tau) \right]
\]

where \( G(t, x) = E \left[ \min \left\{ e^{-x}, e^{-S^\mu_T - t} \right\} \right], X_t = S^\mu_t - B^\mu_t \).
For any \( \{B_t\} \)-stopping time \( 0 \leq \tau \leq T \), we have

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E \left[ \frac{e^{B_\tau^\mu}}{e^{S_{T^\mu}}} \right] = E \left[ \frac{e^{B_\tau^\mu}}{\max \{e^{S_{\tau}^\mu}, e^{\max_{\tau \leq t \leq T} B_t^\mu} \}} \right]
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= E \left[ G(\tau, X_\tau) \right]
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where \( G(t, x) = E \left[ \min \left\{ e^{-x}, e^{-S_{T-t}^\mu} \right\} \right] \), \( X_t = S_t^\mu - B_t^\mu \)

\( X_t \) – drawdown process – is \( B_t \)-adapted!
Function $G$

Assuming $\mu \neq \frac{1}{2}$, then

$$G(t, x) = \frac{2(\mu - 1)}{2\mu - 1} e^{-(\mu - \frac{1}{2})(T-t)} \Phi \left( \frac{-x + (\mu - 1)(T-t)}{\sqrt{T-t}} \right)$$

$$+ \frac{1}{2\mu - 1} e^{-(1-2\mu)x} \Phi \left( \frac{-x - \mu(T-t)}{\sqrt{T-t}} \right)$$

$$+ e^{-x} \Phi \left( \frac{x - \mu(T-t)}{\sqrt{T-t}} \right),$$

where $\Phi$ is the CDF of standard normal distribution.
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where $\Phi$ is the CDF of standard normal distribution

- If $\mu = \frac{1}{2}$, then $G$ has a different (explicit) expression.
Define \( X_{t+s}^x = x \vee S_s^\mu - B_s^\mu, \ s \geq 0 \)
Define $X^x_{t+s} = x \lor S^\mu_s - B^\mu_s, \ s \geq 0$

$X^x_{t+s}$ is Markovian under $P, \ \forall (t, x)$
Dynamic Programming

- Define \( X_{t+s}^x = x \lor S_s^\mu - B_s^\mu, \ s \geq 0 \)
- \( X_t^x \) is Markovian under \( P, \ \forall (t, x) \)
- Value function

\[
V(t, x) = \sup_{0 \leq \tau \leq T-t} E \left[ G(t + \tau, X_{t+\tau}^x) \right]
\]
Dynamic Programming

- Define $X_{t+s}^x = x \lor S_s^\mu - B_s^\mu$, $s \geq 0$
- $X_t^x$ is Markovian under $P$, $\forall(t, x)$
- Value function
  \[ V(t, x) = \sup_{0 \leq \tau \leq T-t} E \left[ G(t + \tau, X_{t+\tau}^x) \right] \]
- In particular
  \[ V(0, 0) = \sup_{0 \leq \tau \leq T} E \left[ \frac{e^{B_\tau^\mu}}{e^{S_{T}^{\mu}}} \right] \]
Variational Inequalities

Dynamic programming equation (Variational Inequalities)

\[
\min\{-\mathcal{L}V, V - G\} = 0, \quad (t, x) \in [0, T) \times (0, \infty) \tag{1}
\]

\[
V(T, x) = G(T, x), \quad x \in (0, \infty) \tag{2}
\]

\[
V_x(t, 0+) = 0, \quad t \in [0, T) \text{ (normal reflection)} \tag{3}
\]

where

\[
\mathcal{L}V = V_t - \mu V_x + \frac{1}{2} V_{xx} \tag{4}
\]
Holding and Selling Region

Holding region

\[ C = \{(t, x) \in [0, T] \times [0, \infty) : V(t, x) > G(t, x)\} \]
Holding and Selling Region

- Holding region

\[ C = \{(t, x) \in [0, T] \times [0, \infty) : V(t, x) > G(t, x)\} \]

- Selling region

\[ D = \{(t, x) \in [0, T] \times [0, \infty) : V(t, x) = G(t, x)\} \]
Holding and Selling Region

- **Holding region**

\[ C = \{(t, x) \in [0, T] \times [0, \infty) : V(t, x) > G(t, x)\} \]

- **Selling region**

\[ D = \{(t, x) \in [0, T] \times [0, \infty) : V(t, x) = G(t, x)\} \]

- An optimal selling time is

\[ \tau^* = \inf\{t \in [0, T] : (t, S^\mu_t - B^\mu_t) \in D\} \]
Holding and Selling Region

- Holding region
  \[ C = \{(t, x) \in [0, T] \times [0, \infty) : V(t, x) > G(t, x)\} \]

- Selling region
  \[ D = \{(t, x) \in [0, T] \times [0, \infty) : V(t, x) = G(t, x)\} \]

- An optimal selling time is
  \[ \tau^* = \inf \{t \in [0, T] : (t, S_t^\mu - B_t^\mu) \in D\} \]

- So it boils down to finding \( V \) - which is hard
To get around: turn *terminal* payoff into *running* payoff
To get around: turn *terminal* payoff into *running* payoff

\[ X_{t+}^x = |Y_t| \text{ in law, where} \]

\[ dY_t = -\mu \text{sign}(Y_t) dt + dB_t, \quad Y_0 = x \]
Optimal Stopping with Running Payoff

- To get around: turn \textit{terminal} payoff into \textit{running} payoff
- $X_{t^+}^x = |Y|$ in law, where

$$dY_t = -\mu \text{sign}(Y_t)dt + dB_t, \quad Y_0 = x$$

- By Tanaka’s formula

$$|Y_s| = x - \mu \int_0^s I(Y_u \neq 0)du + \int_0^s \text{sign}(Y_u)I(Y_u \neq 0)dB_u + \ell^0_s(Y)$$
Itô’s formula then implies

\[ G(t + s, X^x_{t+s}) = G(t, x) + \int_0^s H(t + u, X^x_{t+u})du + M_s \]

provided that \( G_x(t, 0+) = 0 \) (so as to kill the local time term), where \( M \) is a martingale and

\[ H(t, x) = \mathcal{L}G(t, x) = G_t(t, x) - \mu G_x(t, x) + \frac{1}{2} G_{xx}(t, x) \]
Itô’s formula then implies

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\[ H(t, x) = \mathcal{L}G(t, x) = G_t(t, x) - \mu G_x(t, x) + \frac{1}{2} G_{xx}(t, x) \]

Hence

\[ V(t, x) = G(t, x) + \sup_{0 \leq \tau \leq T-t} E \left[ \int_0^\tau H(t + u, X_{t+u}^x) du \right] \]
Optimal Stopping with Running Payoff (Cont’d)

- Itô’s formula then implies

\[ G(t + s, X_{t+s}^x) = G(t, x) + \int_0^s H(t + u, X_{t+u}^x)\,du + M_s \]

provided that \( G_x(t, 0+) = 0 \) (so as to kill the local time term), where \( M \) is a martingale and

\[ H(t, x) = \mathcal{L}G(t, x) = G_t(t, x) - \mu G_x(t, x) + \frac{1}{2} G_{xx}(t, x) \]

- Hence

\[ V(t, x) = G(t, x) + \sup_{0 \leq \tau \leq T-t} E \left[ \int_0^\tau H(t + u, X_{t+u}^x)\,du \right] \]

- Need to calculate \( H \) and show \( G_x(t, 0+) = 0 \)
Function \( H \)

Write \( G(t, x) = \frac{2(\mu-1)}{2\mu-1} A(t, x) + \frac{1}{2\mu-1} B(t, x) + C(t, x) \) where

\[
A = e^{-(\mu - \frac{1}{2})(T-t)} \Phi \left( \frac{-x + (\mu - 1)(T - t)}{\sqrt{T - t}} \right)
\]

\[
B = e^{-(1-2\mu)x} \Phi \left( \frac{-x - \mu(T - t)}{\sqrt{T - t}} \right)
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C = e^{-x} \Phi \left( \frac{x - \mu(T - t)}{\sqrt{T - t}} \right)
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Function $H$

- Write $G(t, x) = \frac{2(\mu - 1)}{2\mu - 1} A(t, x) + \frac{1}{2\mu - 1} B(t, x) + C(t, x)$ where

  \[
  A = e^{-(\mu - \frac{1}{2})(T-t)} \Phi \left( \frac{-x + (\mu - 1)(T-t)}{\sqrt{T-t}} \right)
  
  B = e^{-(1-2\mu)x} \Phi \left( \frac{-x - \mu(T-t)}{\sqrt{T-t}} \right)
  
  C = e^{-x} \Phi \left( \frac{x - \mu(T-t)}{\sqrt{T-t}} \right)
  \]

- Lengthy calculations show for $\varphi = A, B, C$,

  \[
  \mathcal{L}\varphi = (\mu - \frac{1}{2})\varphi - \varphi x
  \]
Function $H$

- Write $G(t, x) = \frac{2(\mu - 1)}{2\mu - 1} A(t, x) + \frac{1}{2\mu - 1} B(t, x) + C(t, x)$ where

  
  
  $$
  A = e^{-(\mu - \frac{1}{2})(T-t)} \Phi \left( \frac{-x + (\mu - 1)(T - t)}{\sqrt{T - t}} \right)
  $$

  $$
  B = e^{-(1 - 2\mu)x} \Phi \left( \frac{-x - \mu(T - t)}{\sqrt{T - t}} \right)
  $$

  $$
  C = e^{-x} \Phi \left( \frac{x - \mu(T - t)}{\sqrt{T - t}} \right)
  $$

- Lengthy calculations show for $\varphi = A, B, C,$

  $$
  \mathcal{L} \varphi = (\mu - \frac{1}{2})\varphi - \varphi x
  $$

- Hence $H = \mathcal{L} G = (\mu - \frac{1}{2})G - G_x$
Function $H$

Write $G(t, x) = \frac{2(\mu - 1)}{2\mu - 1} A(t, x) + \frac{1}{2\mu - 1} B(t, x) + C(t, x)$ where

$$A = e^{-(\mu - \frac{1}{2})(T-t)} \Phi \left( \frac{-x + (\mu - 1)(T-t)}{\sqrt{T-t}} \right)$$

$$B = e^{- (1 - 2\mu)x} \Phi \left( \frac{-x - \mu(T-t)}{\sqrt{T-t}} \right)$$

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- Moreover $G_x = B - C$

- So $G_x(t, 0+) = B(t, 0+) - C(t, 0+) = 0$
The Good…

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... and The Bad

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Finding More Good Guys

- If $\mu = 0$:

$$V(t, x) \geq E_{t,x}[G(T, X^x_T)] > G(t, x), \quad \forall t \in [0, T), \quad x > 0$$
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\frac{\partial}{\partial \mu} \left\{ e^{\frac{1}{2} \mu^2 (T-t)} \left( E_{t,x}[G(T, X_T^x)] - G(t, x) \right) \right\} > 0
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Define a *goodness index* of the stock

\[ \alpha = \frac{a - r}{\sigma^2} \]
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**Theorem**

(Shiryaev, Xu and Zhou, *Quantitative Finance* 2008) *Optimal selling time* \( \tau^* = T \) if \( \alpha \geq 0.5 \), and \( \tau^* = 0 \) if \( \alpha \leq 0 \)
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**Theorem**

(Dai, Jin, and Zhou 2008) *Optimal selling time* \( \tau^* = T \) if \( \alpha \geq 0.5 \), and \( \tau^* = 0 \) if \( \alpha < 0.5 \).
Let $P_t = e^{\mu t + \sigma B_t}$, $M_t = \max_{0 \leq s \leq t} P_s$

Assume $\alpha \geq 0.5$
Optimal Relative Error - Good Stock

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Optimal selling time $\tau^* = T$

Need to determine $E \left[ \frac{M_T - P_T}{M_T} \right]$: the optimal relative error

The joint density function of $(P_T, M_T)$ is

$$f_T(p, m) = \frac{2}{\sigma^3 \sqrt{2\pi T^3}} \frac{\ln(m^2/p) e^{-\frac{\ln^2(m^2/p)}{2\sigma^2 T}} + \frac{\beta}{\sigma} \ln(p) - \frac{1}{2} \beta^2 T}{pm}$$

where $0 < p \leq m$, $m \geq 1$, $\beta = \frac{\mu}{\sigma} \equiv (\alpha - \frac{1}{2}) \sigma$
We have

\[ E \left[ \frac{P_T}{M_T} \right] = \int_0^1 P \left( \frac{P_T}{M_T} > y \right) dy \]

\[ = \int_0^1 \int_y^\infty \int_{p \lor 1}^{p/y} f_T(p, m) dm dp dy \]

\[ = \ldots \]

\[ = \left( 1 - \frac{1}{2\alpha} \right) \Phi \left( (\alpha - \frac{1}{2})\sigma \sqrt{T} \right) + \left( 1 + \frac{1}{2\alpha} \right) e^{\alpha \sigma^2 T} \Phi \left( \left( -\alpha + \frac{1}{2} \right) \sigma \sqrt{T} \right) \]
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Optimal relative error \( r^*(\alpha, \sigma) \) decreases in \( \alpha \) and increases in \( \sigma \).
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$$= \left(1 - \frac{1}{2\alpha}\right) \Phi \left((\alpha - \frac{1}{2})\sigma\sqrt{T}\right) + \left(1 + \frac{1}{2\alpha}\right) e^{\alpha\sigma^2T} \Phi \left(-\left(\alpha + \frac{1}{2}\right)\sigma\sqrt{T}\right)$$

Optimal relative error $r^*(\alpha, \sigma)$ decreases in $\alpha$ and increases in $\sigma$.

$$0 \leq r^*(\alpha, \sigma) < \frac{1}{2\alpha}$$
Assume $\alpha < 0.5$
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- Optimal selling time $\tau^* = 0$
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- Optimal selling time $\tau^* = 0$
- Optimal relative error

$$r^*(\alpha, \sigma) = 1 - \frac{2\alpha - 1}{2(\alpha - 1)} \Phi \left( \frac{1}{2} - \alpha \right) \sigma \sqrt{T} - \frac{2\alpha - 3}{2(\alpha - 1)} e^{(1-\alpha)\sigma^2 T} \Phi \left( \alpha - \frac{3}{2} \right) \sigma \sqrt{T}$$
The Messages

If one accepts the investment criterion in our model, then

- The stock is said to be “good” if $\alpha \geq 0.5$
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Thou Shalt Still Buy and Hold.
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  - The better the stock the smaller relative error; the latter diminishes to zero as $\alpha$ goes to infinity

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  - In this case one should not buy in the first place, or should short if possible
Mehra & Prescott (1985): $\alpha - r = 6.18\%$, $\sigma = 16.67\%$ based on S&P 500 (1889-1978)

$\alpha = 2.2239 > 0.5$ (by large margin)!
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Taking $T = 1$, $r^*(\alpha, \sigma) = 10.15\%$

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Taking $T = 1$, $r^*(\alpha, \sigma) = 10.15\%$

Buy and hold an S&P 500 index fund for one year: Statistically expected to achieve almost 90% of the maximum possible return!
Market Timing

- Buy-and-hold rule believed to be the antithesis of market timing
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  - You can’t really enter on lows and sell on highs so you might as well just buy and hold
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  - So the market timing is consistent with buy-and-hold
Our optimal solutions *insensitive* to the market parameters
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Definition of good/bad stock involves a *range* of parameters, instead of specific values
Insensitivity to Market Parameters

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On the other hand: an interesting problem to test hypothesis $\alpha \geq 0.5$
How Essential is Bang–Bang Policy?

Consider

\[ \max_{0 \leq \tau \leq T} E \left[ U \left( \frac{P_\tau}{M_T} \right) \right] \quad (5) \]

- We have shown that if \( U \) is linear then optimal stopping is bang–bang (either \( \tau^* = T \) or \( \tau^* = 0 \))

Xunyu Zhou/Oxford

Thou Shalt Still Buy and Hold
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- If \( U \) is logarithm or power then optimal stopping times are exactly the same.
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- We have shown that if \( U \) is linear then optimal stopping is bang–bang (either \( \tau^* = T \) or \( \tau^* = 0 \))
- If \( U \) is logarithm or power then optimal stopping times are exactly the same
- What about a general \( U \)?
Let \( v(x) := U(e^x) \). The problem is equivalent to

\[
\max_{0 \leq \tau \leq T} E \left[ v(B^\mu_\tau - S^\mu_T) \right]
\]

**Theorem**

(Dai, Jin and Zhou 2008)

- If \( v \) is increasing and convex, then optimal stopping is bang–bang (\( \tau^* = T \) if \( \alpha \geq 0.5 \), and \( \tau^* = 0 \) if \( \alpha < 0.5 \)).
Let $v(x) := U(e^x)$. The problem is equivalent to

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**Theorem**

(Dai, Jin and Zhou 2008)

- *If* $v$ *is increasing and convex, then optimal stopping is bang–bang* ($\tau^* = T$ *if* $\alpha \geq 0.5$, *and* $\tau^* = 0$ *if* $\alpha < 0.5$).

- Assume $v$ is increasing, $C^2$, and $|v''(x)| \leq e^{k(x^2+1)}$. *If* optimal stopping is bang–bang, *then* $v$ *must be convex.*
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Convexity and Behavioural Finance

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Yet the two experiments have exactly the same final wealth positions!
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- Risk-averse on gains, and **risk-seeking on losses**
In this model the maximum $S_T^\mu$ implicitly taken as reference point.

$$\max_{0 \leq \tau \leq T} E \left[ v(B_\tau^\mu - S_T^\mu) \right]$$
Maximum as Reference Point

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\max_{0 \leq \tau \leq T} E \left[ \nu(B_{\tau}^{\mu} - S_{T}^{\mu}) \right]
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- In this model the maximum \( S^\mu_T \) implicitly taken as reference point
- Always in a “loss” situation
- Hence risk-seeking or having convex \( \nu \)
- Bang-bang (or buy-and-hold) behaviour consistent with behavioural theory
It is only natural that an investor wishes to sell a stock closest to the maximum price over a given planning horizon.
Conclusions

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- The problem is formulated as an optimal stopping problem.
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- The problem is formulated as an optimal stopping problem.
- We have shown that, even in the absence of transaction costs, one should *buy and hold*, if the stock is good, that is...
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- This, at least from one angle, reinforces the conventional maxim of *buy and hold*.
- That our model produces buy-and-hold rule suggests the criterion proposed sensible and warrants further investigations.
A Serenity Prayer: Illusion of Control

Thou Shalt Still Buy and Hold
A Serenity Prayer: Illusion of Control

God, grant me the serenity to accept the things I cannot control, courage to control the things I can, and the wisdom to know the difference.

—— Meir Statman