The Birth of Financial Bubbles

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Famous bubbles of history

• Tulipomania; Amsterdam, 17th century (circa 1630s)
• John Law and the Banque Royale (Paris, 1716 – 1720)
• The South Sea Company (London, 1711 – 1720)
• In the United States:
  • After the War of 1812, real estate speculation, fostered by the Second Bank of the United States, created in 1816;
  • Runaway speculation tied to advances in infrastructure through the building of canals and turnpikes, ending in the crash of 1837;
  • Speculation due to the creation of the railroads led to the panic of 1873.
• The Wall Street panic of 1907; (Banking crisis due to speculation; stock market fell 50%, led to development of Federal Reserve in 1913 [Glass-Owen bill]); role of J.P. Morgan.
• Florida land speculation in the first half of the 1920s, followed by stock market speculation in the second half of the 1920s created in part by margin loans, led to the Great Crash of 1929, leading to many bank failures and the worldwide depression of the 1930s.
US Stock Prices 1929 (Donaldson & Kamstra [1996])

Figure 1
S&P 500 stock price index.
More recent bubbles

- Minor crashes in the 1960s and 1980s
- Junk bond financing led to the major crash of 1987
- Japanese housing bubble circa 1970 to 1989
- The “dot com” crash, from March 11th, 2000 to October 9th, 2002. Led by speculation due to the promise of the internet; The Nasdaq Composite lost 78% of its value as it fell from 5046.86 to 1114.11.
- Current US housing bubble and subprime mortgages
Figure 1. Returns on equally weighted Internet index, S&P 500 and Nasdaq composite. Comparison of index levels of the equally weighted Internet index, the S&P 500 index, and the Nasdaq composite index for the period 1/1/1998–12/31/2000. All three indexes are scaled to be 100 on 12/31/1997.
Current US Housing Price Trend (Center for Responsible Lending)
Oil Futures (WTRG Economics)
Oil Futures (WTRG Economics)
The Basic Framework

- We assume the **No Free Lunch With Vanishing Risk** framework of F. Delbaen and W. Schachermayer. In words, there are no arbitrage opportunities and there are no trading strategies which approximate arbitrarily closely arbitrage opportunities.

- A risky asset with maturity $\tau$ and a money market account with constant value 1 are traded.

- $D = (D_t)_{0 \leq t \leq \tau} \geq 0$ is the **cumulative dividend process**.

- $X_\tau \geq 0$ is the payoff at time $\tau$;

- The **market price** is $S = (S_t)_{0 \leq t \leq \tau} \geq 0$.

- The **wealth process** is

$$W_t = S_t + \int_0^{t \land \tau} dD_u + X_\tau 1\{t \geq \tau\}.$$
• A **trading strategy** is a pair of adapted processes \((\pi, \eta)\) representing the number of units of the risky asset and money market account held at time \(t\).

• The **wealth process** \(V\) of the trading strategy \((\pi, \eta)\) is given by

\[
V_{t}^{\pi,\eta} = \pi_{t}S_{t} + \eta_{t}.
\]

(1)

• A **self-financing trading strategy** is a trading strategy \((\pi, \eta)\) with \(\pi\) predictable and \(\eta\) optional such that \(V_{0}^{\pi} = 0\) and

\[
V_{t}^{\pi,\eta} = \int_{0}^{t} \pi_{u}dW_{u} = (\pi \cdot W)_{t}
\]

(2)
• We say that the trading strategy $\pi$ is a—admissible if it is self-financing and $V_t^\pi \geq -a$ for all $t \geq 0$ almost surely.

• We say a trading strategy is admissible if it is self-financing and there exists an $a \in \mathbb{R}_+$ such that $V_t^\pi \geq -a$ for all $t$ almost surely.

  • Admissibility needed to exclude doubling strategies.
  • Admissibility is the reason for the existence of bubbles.
  • Admissibility is an implicit restriction on shorting the risky asset.
Theorem (D & S, 1998; First Fundamental Theorem)

A process $S$ has No Free Lunch with Vanishing Risk (NFLVR) if and only if there exists an equivalent probability measure $Q$ such that $S$ is a sigma martingale under $Q$.

Definition

A market is complete if every bounded contingent claim can be perfectly hedged.

Theorem (Second Fundamental Theorem)

A market is complete if and only if there is only one and only one risk neutral measure (sigma martingale measure)

- Since $W \geq 0$ always, we can replace sigma martingale with local martingale.
A market is said to satisfy **No Dominance** if, given any two assets with their associated payoff structures (dividends + terminal payoff) and market prices, neither asset’s payoff structure is always (weakly) greater than the other’s, and also has a strictly lower market price.

**Lemma**

*No Dominance implies NFLVR; however the converse is false.*

From now on, we assume No Dominance holds.
The Fundamental Price

In complete markets with a finite horizon $T$, we use the risk neutral measure $Q$, and for $t < T$ the fundamental price of the risky asset is defined to be:

$$S_t^* = E_Q\left\{ \int_t^T dD_u + X_T | \mathcal{F}_t \right\}$$

**Definition (Bubble)**

A bubble in a static market for an asset with price process $S$ is defined to be:

$$\beta = S - S^*$$
Static Markets

Theorem (Three types of bubbles)

1. $\beta$ is a local martingale (which could be a uniformly martingale) if $P(\tau = \infty) > 0$;
2. $\beta$ is a local martingale but not a uniformly integrable martingale, if it is unbounded, but with $P(\tau < \infty) = 1$;
3. $\beta$ is a strict $Q$ local martingale, if $\tau$ is a bounded stopping time.

- Type 1 is akin to fiat money
- Type 2 is tested in the empirical literature
- Type 3 is essentially “new.” Type 3 are the most interesting!
Theorem (Bubble Decomposition)

The risky asset price admits a unique decomposition

\[ S = S^* + (\beta^1 + \beta^2 + \beta^3) \]

where

1. \( \beta^1 \) is a càdlàg nonnegative uniformly integrable martingale with \( \lim_{t \to \infty} \beta^1_t = X_\infty \) a.s.
2. \( \beta^2 \) is a càdlàg nonnegative NON uniformly integrable martingale with \( \lim_{t \to \infty} \beta^2_t = 0 \) a.s.
3. \( \beta^3 \) is a càdlàg non-negative supermartingale (and strict local martingale) such that \( \lim_{t \to \infty} E\{\beta^3_t\} = 0 \) and \( \lim_{t \to \infty} \beta^3_t = 0 \) a.s.
Why doesn’t “no arbitrage” exclude bubbles in an NFLVR economy?

- The obvious candidate strategy: short the risky asset during the bubble, and cover the short after the bubble crashes
  - For type 1 and type 2 bubbles, the trading strategy fails to be an arbitrage because all trading strategies must terminate in finite time, and the bubble may outlast this trading strategy with positive probability
  - For type 3 bubbles this trading strategy fails because of the admissibility requirement. With positive probability a type 3 bubble can increase such that the short position’s losses violate the admissibility condition

- In a complete market, No Dominance excludes these bubbles because there are two ways to create the asset’s payoff (synthetic versus buy and hold)
- In an incomplete market, synthetic replication need not be possible. Hence, bubbles can exist!
A static market with NFLVR only

Corollary

Any asset price bubble has the following properties:

- Bubbles are non-negative
- For assets with possibly unbounded but finite lifetimes, bubbles may burst at the asset’s maturity
- Bubbles cannot be “born” after time 0

Implications

- As a local martingale, a typical pattern (a price increase, then a decrease) may not occur.
- A bubble is a supermartingale (a local martingale which is bounded below)
- Bubbles may be more common (and exist in individual assets as well as in sectors) than is widely believed
Black-Scholes Model (Static Market, Finite Horizon)

- Fix $T$ and let $S$ be the price process of a stock without dividends following

$$S_t = \exp \left\{ (\mu - \frac{\sigma^2}{2})t + \sigma B_t \right\}, \quad 0 \leq t \leq T,$$

where $\mu, \sigma \in \mathbb{R}_+$, and $B$ is a standard Brownian motion.

- The finite horion $\Rightarrow$ only a type 3 bubble can exist.
- Since $S$ is a $Q$ martingale, no type 3 bubbles are possible.
- This holds more generally for complete markets, under NFLVR, and without needing No Dominance.
Black-Scholes Model (Static Market, Infinite Horizon)

- If we extend $S$ to times in $[0, \infty)$ then the situation changes.
- The fundamental value of the stock is $S^*_t = 0$. (There are no dividends.)
- The definition of the bubble $\beta$ is

$$\beta = S_t - S^*_t = S_t,$$

and the entire stock is a bubble!
- Under No Dominance, if the asset does not have a bubble, $S$ must be the zero process, since there are no dividends and the terminal payout is zero.
- Therefore the model is a bubble, and only the finite horizon Black-Scholes model is reasonable.
Incomplete markets

- There are an infinite number of risk neutral measures
- We need to choose one to define the concept of fundamental value.
- We assume that enough derivative securities trade so that a risk neutral measure is uniquely determined by the market. To do this we could use the ideas of Jacod and Protter, or alternatively Schweizer and Wissel.
- We allow regime/structural shifts in the economy to generate changes in the market selected risk neutral measure across time (this might be compared to Ising models for phase change)
  - If there are no regime shifts, we say the market is static
  - If there is at least one regime shift possible, we say that the market is dynamic.
Regime Change

• This idea of regime change is new; previously a risk neutral measure in an incomplete market was chosen in some manner (often ad hoc) and fixed for all $t \geq 0$

• The new approach is that the market has chosen one of the infinitely many risk neutral measures with which to price derivatives; in theory, one can determine this choice if (for example) there are enough put options traded, and they are priced consistently with each other and with the price process (Jacod and Protter, 2007; Schweizer and Wissel, 2007)

• Then, it seems possible that over time the risk neutral measure chosen by the market can change, from one to another member of the infinite collection

• This idea is roughly analogous to the Ising model (and related models) of phase changes in physics
The Fundamental Price

- In complete markets with a finite horizon $T$, we use the risk neutral measure $Q$, and for $t < T$ the fundamental price of the risky asset is defined to be:

$$S^*_t = E_Q\left\{ \int_t^T dD_u + X_T | \mathcal{F}_t \right\}$$

- In incomplete markets, if one $Q$ is chosen by the market for all time (i.e., a static market), the definition is analogous.

- If an incomplete market is dynamic with an infinite horizon, then the fundamental price of the risky asset is defined to be, with end time $\tau$ for the asset, $t < \tau$, and supposing we are in regime $i$ at time $t$:

$$S^*_t = E_{Q^i}\left\{ \int_t^\tau dD_u + X_{\tau} 1_{\{\tau<\infty\}} | \mathcal{F}_t \right\}$$

where $Q^i$ is the risk neutral measure chosen by the market.

- Note that $X_{\tau} 1_{\{\tau=\infty\}}$ is not included.
We can piece all of these measures $Q^i$ together to get one measure $Q^*$, but $Q^*$ need not be risk neutral measure; we call $Q^*$ the evaluation measure, and write it $Q^{t*}$ to denote that it changes with the time $t$.

Written this way, the previous equation becomes:

$$S^*_t = E_{Q^{t*}} \left\{ \int_t^\tau dD_u + X_\tau 1_{\{\tau < \infty\}} \middle| \mathcal{F}_t \right\}$$
Recall the definition of a bubble:

**Definition (Bubble)**

A bubble in a static market for an asset with price process $S$ is defined to be:

$$\beta = S - S^*$$

A bubble in a dynamic market for $t < \tau$ in regime $i$ is:

$$\beta = S - E_{Q^t} \{ \int_t^{\tau} dD_u + X_\tau 1_{\{\tau<\infty\}} |\mathcal{F}_t} \}

Since we are in regime $i$, we have in this case $Q^t = Q^i$.

If there are no bubbles, a change to a new risk neutral measure can create a bubble; we call this **bubble birth**.
Derivative Securities

• Assume $S$ pays no dividends
• A derivative security is written on the market price of $S$
• Let $H$ be such a contingent claim, and denote its market price by $\Lambda^H_t$
• Suppose we are in regime $i$ at time $t$; the fundamental price of $H$ is $E_{Q^t\star}\{H|\mathcal{F}_t\}$
• The derivative security’s price bubble is defined as

$$\delta_t = \Lambda^H_t - E_{Q^t\star}\{H|\mathcal{F}_t\}.$$
European Call and Put Options

We have a risky asset with market price \( S = (S_t)_{t \geq 0} \). We consider contingent claims with a maturity date \( T \) and a strike price \( K \):

- A **forward contract** has payoff \( S_T - K \). Its market price at time \( t \) is denoted \( V_t^f(K) \).
- A **European call option** has payoff \( (S_T - K)_+ \). Its market price at time \( t \) is denoted \( C_t(K) \).
- A **European put option** has payoff \( (K - S_T)_+ \). Its market price at time \( t \) is denoted \( P_t(K) \).
- We let \( V_t^*(K), C_t^*(K) \) and \( P_t^*(K) \) be the **fundamental prices** of the forward, call, and put, respectively.
Theorem (Put-Call parity for Fundamental Prices)

\[ C_t^*(K) - P_t^*(K) = V_t^f(K). \]

Theorem (Put-Call Parity for Market Prices)

\[ C_t(K) - P_t(K) = V_t^f(K) = S_t - K \]

• The Fundamental Price Theorem follows by properties of expectations

• The Market Price Theorem follows by No Dominance using the argument of Merton (1973)
Theorem (Equality of European Put Prices)  
For all $K \geq 0$

$$P_t(K) = P_t^*(K)$$

European puts have no bubbles, due to the payoff being bounded. 

Theorem (European Call Prices)  
For all $K \geq 0$

$$C_t(K) - C_t^*(K) = S_t - EQ^t\star\{S_T|\mathcal{F}_t\} = \beta_t^3 - EQ^t\star\{\beta_T^3|\mathcal{F}_t\}$$

- Only type 3 bubbles are reflected in call prices
- Risk neutral valuation need not hold in an NFLVR and No Dominance market
American Call Options (Static Market)

• We introduce a risk free savings account $D$ given by

$$D_t = \exp\left(\int_0^t r_s ds\right)$$

where $r$ is a non-negative, adapted process representing the default free spot rate of interest.

• The **fundamental value** of an American Call option with strike price $K$ and maturity $T$ is

$$C_t^{A*}(K) = \sup_{\eta \in [t, T]} E_Q\{ (S_\eta - \frac{K}{D_\eta})_+ | F_t \}$$

where $\eta$ is a stopping time and $Q$ is the risk neutral measure.

• We let $C_t^A(K)$ denote the **market price** at time $t$ of this same option.
Theorem

Assume that the jumps of the asset price $S$ satisfy some mild regularity conditions. Then for all $K$,

$$C_t^E(K) = C_t^A(K) = C_t^{A*}(K)$$

- This is an extension of Merton's famous “no early exercise” theorem (1973)
- American call options do not exhibit bubbles
- $C_t^A(K) - C_t^{E*}(K) = \beta_t^3 = S_t - E_{Q^*}[S_T|\mathcal{F}_t]$
- While the market prices of European and American options agree, the fundamental prices need not agree.
Prices of Forwards and Futures

- $S$ denotes the price in dollars of the risky asset, and $\frac{S}{D}$ is the price in units of the numéraire.
- Assume no dividends are paid over the time interval $(0, T]$ and that $\tau > T$ a.s.
- If $Q$ is an equivalent local martingale measure implies $\frac{S}{D}$ is a $Q$ local martingale.
- We denote $p(t, T)$ as the market price at time $t$ of a sure dollar paid at time $T$.
- A forward contract on $S$ with strike price $K$ and maturity $T$ is defined by its time $T$ payoff $(S_T - K)$.
- The forward price, denoted $f_{t,T}$, is defined to be that strike price $K$ that gives the $T$ maturity forward contract zero market value at time $t$. 
Theorem

\[ f_{t,T} \times p(t, T) = S_t \]

Corollary (Forward Price Bubbles)

1. \( f_{t,T} \geq 0 \)
2. \( f_{t,T} \times p(t, T) \) is a Q local martingale for each risk neutral measure Q
3. \( f_{t,T} \times p(t, T) = E_Q\{S_T|\mathcal{F}_t\} + \beta_t \), where \( \beta_t = S_t - S_t^* \)
• A **futures contract** is a financial contract, written on the risky asset $S$, with a fixed maturity $T$, which represents the purchase of the risky asset at time $T$ via a prearranged payment procedure.

• Marking-to-market obligates the purchaser (long position) to accept a continuous cash flow stream equal to the continuous changes in the futures prices for this contract.

• The time $t$ **futures price**, denoted $F_{t,T}$, is set (by market convention) such that newly issued futures contracts (at time $t$) on the same risky asset with the same maturity date $T$, have **zero market value**

• At maturity, the last futures price must equal the asset’s price: $F_{t,T} = S_T$
The futures contract’s accumulated wealth process is

\[ V_t^F = \int_0^t \frac{1}{D_s} dF_{s,T}. \]

Definition (Prices of Futures)

Semimartingales \((F_t,T)_{0 \leq t \leq T}\) are called \textbf{NFLVR futures prices processes} if they satisfy all of:

1. \(V_t^F\) is locally bounded from below
2. There exist a risk neutral measure \(Q\) such that \((V_t^F)_{t \geq 0}\) is a \(Q\) local martingale and such that there exists increasing stopping times \(\nu_n\) such that \((V_{t \wedge \nu_n}^F)_{t \geq 0}\) is bounded from below for each \(n\)
3. \(F_t,T = S_T\)

\textbf{Nota Bene:} We do not require futures prices \((F_t,T)_{t \geq 0}\) to be non-negative
We let $\Phi^F$ denote the class of all NFLVR futures price processes.

**Theorem**

A risk neutral measure $Q$ is chosen and fixed. Define $(F'_{t,T})_{t \geq 0} = (E_Q\{S_T|\mathcal{F}_t\})_{t \geq 0}$. Then $(F'_{t,T})_{t \geq 0} \in \Phi^F$.

This is the usual definition of the futures price in the literature.

**Theorem**

A risk neutral measure $Q$ is chosen and fixed. Let $\beta$ be a local $Q$ martingale, locally bounded from below with $\beta_T = 0$.

Define $F_{t,T} = E_Q\{S_T|\mathcal{F}_t\} + \beta_t$. Then $(F_{t,T})_{t \geq 0} \in \Phi^F$.

- Bubbles can exist in futures prices
- They are unrelated to bubbles in the underlying risky asset
- **Futures price bubbles can be negative**
How do we test to see if we are in a bubble?

Perhaps the easiest way is to try to detect a difference in prices between European and American options. (Perhaps not too realistic.)

With **Soumik Pal**, we have analyzed the behavior of the inverse Bessel process in some detail, as well as other strict local martingales which behave similarly. With these price processes, the prices of European calls decrease as a function to time to expiration: That is, for $S$ the inverse Bessel process, the function

$$T \mapsto E\{(S_T - K)_+\}$$

is monotone decreasing if $K \leq \frac{1}{2}$, and otherwise it is initially increasing and then strictly decreasing for

$$T \geq \left(K \log \frac{2K + 1}{2K - 1}\right)^{-1}.$$