Mean-Variance Hedging in Discrete Time with Execution Uncertainty

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Introduction

• An illiquid asset is not sufficiently traded in the market.
• The execution ratio (percentage of successful trade) is in \([0, 1]\).

Number of Units Invested in Illiquid Asset

An investor cannot perfectly control his portfolio.
▷ Partial Execution Risk
An investor hedges a contingent claim, using one illiquid risky asset and the saving account. The saving account is the numeraire. All prices are discounted prices.

\[ T \in \mathbb{N} \] : Fixed Maturity
\[(\Omega, \mathcal{F}, P, \{\mathcal{F}_k; k = 0, 1, \ldots, T\}) \] : Filtered probability space
\[ X_k \ (k = 0, 1, \ldots, T) \] : Price Process of the Illiquid Risky Asset
\[ H \] : Payoff of Contingent Claim at \( T \)
\[ \nu_k \in [0, 1], \ k = 0, 1, \ldots, T - 1 \] : Execution Ratio (Realization/Plan)

**Assumption**
- \( X \) is adapted and square-integrable. \( X_0 \) is constant.
- \( H \) is \( \mathcal{F}_T \)-measurable and square-integrable.
- \( \nu \) is adapted.

Note that \( \nu_0 \) can be random.
Trading Strategy

\[ \pi_k (k = 0, 1, \ldots, T - 1) : \text{Planned number of } X, \]
\[ \theta^\pi_k (k = 0, 1, \ldots, T - 1) : \text{Realized number of } X \text{ with } \pi \]
\[ \theta^\pi_{-1} = \theta_0 : \text{Initial number of units invested in } X \]

\[ \theta^\pi_k = \theta^\pi_{k-1} + (\pi_k - \theta^\pi_{k-1}) \nu_k, \quad k \geq 0. \]

Planned Trade

**Assumption**  Let \( G_k \) be the investor’s information immediately before the trade. \( G_k \) satisfies

\[ \sigma\{X_l; l \leq k\} \vee \sigma\{\nu_l; l < k\} \subset G_k \subset F_k, \quad G_0 = \{\emptyset, \Omega\}. \]

\( \pi_k \) must be adapted to \( G_k \).

Note that \( \nu_k \) is **not** assumed to be \( G_k \)-measurable.
Problem

For a trading strategy $\pi$, the value of the hedging portfolio at $k$ is given by

$$W^\pi_k = w_0 + \sum_{0 \leq j \leq k-1} \theta^\pi_j \triangle X_{j+1}, \quad k = 0, 1, \ldots, T$$

where $w_0$ is the initial cost and $\triangle X_{j+1} = X_{j+1} - X_j$.

Let $A = \{\pi : \mathcal{G}_k\text{-adapted}, \ W^\pi_T \in L^2(P)\}$ be the set of admissible trading strategies.

Mean-Variance Hedging Problem

\[ \inf_{(\pi, \theta_0, w_0) \in A \times \mathbb{R}^2} E[(H - W^\pi_T)^2] \]

\[ \text{ESHE} = \text{Expected Square Hedging Error} \]

We want to find an optimal strategy $\pi^*$ and an optimal initial condition $y^* = t(\theta^*_0, w^*_0)$. 
Auxiliary Process

We use the mathematical convention $0/0 = 0$. We write $E_k[.] = E[·|G_k]$. We introduce three auxiliary processes

$$A_k = \begin{pmatrix} a_{k,11} & a_{k,12} \\ a_{k,21} & a_{k,22} \end{pmatrix}, \quad b_k = \begin{pmatrix} b_{k,1} \\ b_{k,2} \end{pmatrix}, \quad c_k, \quad k = 0, 1, \ldots, T.$$ 

Set

$$A_T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad b_T = \begin{pmatrix} 0 \\ H \end{pmatrix}, \quad c_T = H^2.$$

For $k = 0, 1, \ldots, T - 1$ we define

$$a_{k,11} = E_k[ t z_{k+1} A_{k+1} z_{k+1} (1 - \nu_k)^2 ] - \frac{E_k[ t z_{k+1} A_{k+1} z_{k+1} \nu_k (1 - \nu_k) ]^2}{\psi_k},$$

$$a_{k,12} = E_k[ t z_{k+1} A_{k+1} e (1 - \nu_k) ] - \frac{E_k[ t z_{k+1} A_{k+1} z_{k+1} \nu_k (1 - \nu_k) ] E_k[ t z_{k+1} A_{k+1} e \nu_k ]}{\psi_k},$$

$$a_{k,21} = a_{k,12},$$

$$a_{k,22} = E_k[ a_{k+1,22} ] - \frac{E_k[ t z_{k+1} A_{k+1} e \nu_k ]^2}{\psi_k},$$

$$E_k[ t z_{k+1} A_{k+1} z_{k+1} \nu_k (1 - \nu_k) ] = E_k[ t z_{k+1} A_{k+1} e (1 - \nu_k) ] - \frac{E_k[ t z_{k+1} A_{k+1} z_{k+1} \nu_k (1 - \nu_k) ] E_k[ t z_{k+1} A_{k+1} e \nu_k ]}{\psi_k}.$$
**Auxiliary Process**

\[
\begin{align*}
    b_{k,1} &= E_k[^t b_{k+1} z_{k+1} (1 - \nu_k)] - \frac{E_k[^t z_{k+1} A_{k+1} z_{k+1} \nu_k (1 - \nu_k)] E_k[^t b_{k+1} z_{k+1} \nu_k]}{\Psi_k}, \\
    b_{k,2} &= E_k[b_{k+1,2}] - \frac{E_k[^t z_{k+1} A_{k+1} e \nu_k] E_k[^t b_{k+1} z_{k+1} \nu_k]}{\Psi_k}, \\
    c_k &= E_k[c_{k+1}] - \frac{E_k[^t b_{k+1} z_{k+1} \nu_k]^2}{\Psi_k}
\end{align*}
\]

where \( z_{k+1} = \begin{pmatrix} 1 \\ \triangle X_{k+1} \end{pmatrix}, e = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \Psi_k = E_k[^t z_{k+1} A_{k+1} z_{k+1} \nu_k^2]. \)

- \( A_k \) is independent of \( H \). \( b_k \) and \( c_k \) depend on \( H \).
- We can show that these processes are well-defined and satisfy
  \[
  a_{k,11} \in \mathcal{L}^1(P), \quad a_{k,12} = a_{k,21} \in \mathcal{L}^2(P), \quad 0 \leq a_{k,22} \leq 1 \quad P\text{-a.s.},
  \]
  \[
  b_{k,1} \in \mathcal{L}^1(P), \quad b_{k,2} \in \mathcal{L}^2(P), \quad c_k \in \mathcal{L}^1(P).
  \]
Optimal Strategy

For \( y \in \mathbb{R}^2 \), \( k = 0, 1, \ldots, T \), let

\[
V_k(y) = \operatorname{essinf}_{\pi \in \mathcal{A}_k(y)} E_k[(H - W_T^\pi)^2],
\]

where \( \mathcal{A}_k(y) = \{ \pi \in \mathcal{A} : t(\theta_{k-1}^\pi, W_k^\pi) = y \} \).

**Theorem 1** Fix the initial condition \((\theta_0, w_0)\). Let

\[
\pi_k^* = \frac{E_k[tb_{k+1}z_{k+1}\nu_k]}{\psi_k} - \frac{E_k[tz_{k+1}A_{k+1}z_{k+1}\nu_k(1 - \nu_k)]}{\psi_k} \theta_{k-1}^\pi \\
- \frac{E_k[tz_{k+1}A_{k+1}e\nu_k]}{\psi_k} W_k^\pi, \quad k = 0, 1, \ldots, T - 1.
\]

Then \( \pi_k^* \) is well-defined and \( \pi^* = \{ \pi_k^* ; k = 0, 1, \ldots, T - 1 \} \in \mathcal{A} \). \( \pi^* \) is an optimal strategy. For all \( y \in \mathbb{R}^2 \) and \( k = 0, 1, \ldots, T \),

\[
V_k(y) = y A_k y - 2^t b_k y + c_k.
\]

Note that \( V_0(t(\theta_0, w_0)) \) is ESHE with \( \pi^* \).

- \( A_k, b_k, c_k \) can be calculated by the one-step backward equations.
- \( \pi_k^* \) and \( V_k(y) \) can be calculated using the auxiliary processes.
Optimal Initial Condition

For all initial conditions, \((\theta_0, w_0) \in \mathbb{R}^2\), the optimal strategy \(\pi^*\) exists. Though \(\pi^*\) depends on \((\theta_0, w_0)\), I continue to use the shorter notation \(\pi^*\).

**Theorem 2** An optimal initial condition \(y_0^* = t(\theta_0^*, w_0^*)\) is a solution of

\[
A_0 y_0^* = b_0.
\]

Optimal initial condition may not be unique.

Especially when \(A_0\) is invertible, \(y_0^* = A_0^{-1}b_0\).

For this optimal initial condition, we can show \(E[H - W^*_T] = 0\).

The least ESHE is the variance of the hedging error.

**Theorem 3** There exist non-negative constants \(\lambda_1, \lambda_2\) and an orthogonal matrix \(P\) such that \(tPA_0P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\). The least ESHE is given by

\[
c_0 - \frac{\tilde{b}_{0,1}^2}{\lambda_1} - \frac{\tilde{b}_{0,2}^2}{\lambda_2}
\]

where \(\tilde{b}_0 = t(\tilde{b}_{0,1}, \tilde{b}_{0,2}) = tPb_0\). Especially when \(A_0\) is invertible, the least ESHE is \(c_0 - t^*b_0A_0^{-1}b_0\). Note that \(c_0\) is ESHE with \(\pi^*\) and \((\theta_0, w_0) = (0, 0)\).
No Execution Risk \((\nu_k = 1)\)

\[
A_k = \begin{pmatrix} 0 & 0 \\ 0 & E_k[a_{k+1,22}] - \frac{E_k[\Delta X_{k+1} a_{k+1,22}]^2}{E_k[(\Delta X_{k+1})^2 a_{k+1,22}]} \end{pmatrix},
\]

\[
b_k = \begin{pmatrix} 0 \\ E_k[b_{k+1,2}] - \frac{E_k[\Delta X_{k+1} a_{k+1,22}]E_k[b_{k+1,2} \Delta X_{k+1}]}{E_k[(\Delta X_{k+1})^2 a_{k+1,22}]} \end{pmatrix},
\]

\[
c_k = E_k[c_{k+1}] - \frac{E_k[b_{k+1,2} \Delta X_{k+1}]^2}{E_k[(\Delta X_{k+1})^2 a_{k+1,22}]},
\]

\[
\pi_k^* = \frac{E_k[b_{k+1,2} \Delta X_{k+1}]}{E_k[(\Delta X_{k+1})^2 a_{k+1,22}]} - \frac{E_k[\Delta X_{k+1} a_{k+1,22}]}{E_k[(\Delta X_{k+1})^2 a_{k+1,22}]} W_k^{\pi^*} \cdot \theta_k^{\pi^*} \text{ disappears!}
\]

- Since \(A_0\) is not invertible, the optimal initial condition is not unique.

\[
\theta_0^* \in \forall \mathbb{R}, \quad w_0^* = \frac{b_{0,2}}{a_{0,22}} \text{ if } a_{0,22} \neq 0.
\]

- The least ESHE is \(c_0 - \frac{b_{0,2}^2}{a_{0,22}}\) if \(a_{0,22} \neq 0\).

These results correspond to Gugushvili (2003).
Illiquid Market ($\nu_k = 0$)

$$
A_k = \begin{pmatrix}
E_k[(X_T - X_k)^2] & E_k[X_T - X_k] \\
E_k[X_T - X_k] & 1
\end{pmatrix},
$$

$$
b_k = \begin{pmatrix}
E_k[H(X_T - X_k)] \\
E_k[H]
\end{pmatrix},
$$

$$
c_k = E_k[H^2],
$$

$$
\pi_k^* = 0, \quad \text{A strategy does not influence the hedging error.}
$$

- Suppose that $\text{Var}[X_T] = |A_0|$ is positive. \(\Rightarrow\) $A_0$ is invertible.

- The least ESHE is $\text{Var}[H](1 - \rho_{H,X_T}^2)$.

These results correspond to the static hedging.
Multi-Period Binomial Model

\[ \delta t = 1/12, \ u = e^{\sigma \sqrt{\delta t}}, \ d = e^{-\sigma \sqrt{\delta t}}. \]

\[ \{ (\nu_k, X_{k+1}/X_k); \ k = 0, 1, \ldots, T-1 \} \text{ is independent and identically distributed.} \]

\[
P[\left( \nu_k, \frac{X_{k+1}}{X_k} \right) = \left( \frac{i}{M}, j \right) | G_k] = p_{i,j}, \ i = 0, 1, \ldots, M, \ j = u, d,
\]

\[
P[\nu_k = \frac{i}{M} | G_k] = \begin{cases} 
\lambda, & \text{if } i = M, \\
\frac{1-\lambda}{M} \left( \alpha \frac{i}{M-1} + 1 - \frac{\alpha}{2} \right), & \text{otherwise.}
\end{cases}
\]
Basic Condition

Suppose that $H = \max(X_T - 110, 0)$. Set

$X_0 = 100, \quad T = 36, \quad M = 5, \quad \mu = 10\%, \quad \sigma = 25\%, \quad \lambda = 10\%, \quad \alpha = 0,$

$$
p_{i,u} : p_{i,d} = \frac{e^{\mu \delta t} - d}{u - d} : \frac{u - e^{\mu \delta t}}{u - d}, \quad i = 0, 1, \ldots, M.
$$

This assumption implies that $X$ and $\nu$ are independent. Under this condition, the optimal solution is given by

$$
\pi_0^* = \theta_0^* = 0.508, \quad w_0^* = 13.33.
$$

We change the following parameters.

1. Execution Ratio, $\nu_k$: $\lambda, \alpha, M$.
2. Price Process, $X_k$: $\mu, \sigma$.
3. Relation between $\nu_k$ and $X_k$: $p_{M,u}^1 = P[X_1^1 = u|X_0 = 1]$. 
$\pi^*_0, \theta^*_0$ vs. $p^1_{M,u}, \lambda$

- $\pi_0^*$ is more stable than $\theta_0^*$.
- When $p^1_{M,u} = 0.54$, $X$ and $\nu$ are independent and $\pi_0^* = \theta_0^*$.
- Positively correlated case: $\pi_0^* \geq \theta_0^*$. Negatively correlated case: $\pi_0^* \leq \theta_0^*$.

The correlation between $X$ and $\nu$ is an important factor.
\( w^*_0 \) vs. \( \alpha, M \)

- When \( M \) or \( \alpha \) increase, \( w^*_0 \) increases.
- When \( \alpha = 2 \), regardless of \( M \), \( w^*_0 (=13.47) \) is close to the no arbitrage price (=13.52).

\( \alpha \) influences more strongly the optimal solution.
\( \Rightarrow \) It is important whether \( \nu \) is right-skewed or left-skewed.
\( \theta_0^* (= \pi_0^*), \ w_0^* \) vs. \( \mu, \sigma \)

- \( \theta_0^* \) and \( w_0^* \) increase with \( \sigma \).
  Natural for OTM call.
- \( w_0^* \) decreases with \( \mu \).

If \( \mu \) is large,

1. many people want to buy the call option,
2. \( w_0^* \) is small and can be negative.

\( \Rightarrow \) \( w_0^* \) is not an approximate price but an optimal initial hedging cost.
Concluding Remarks

We have solved the mean-variance hedging problem in the discrete time model with execution uncertainty, by the dynamic programming method.

- Three auxiliary processes, $A_k, b_k, c_k$ given by the one-step backward equations play an important role in solving the problem.
- We give a computationally-efficient formula of an optimal strategy.
- An optimal initial condition is a solution of linear simultaneous equation.

Future Problems

1. Fair price of the contingent claim of illiquid asset
2. Continuous time model
References I


