Optimal Consumption and Portfolio Decisions with Partially Observable Real Prices.

Alain Bensoussan, Jussi Keppo, Suresh P. Sethi
Introduction

We consider the optimal consumption and portfolio investment problem of an investor who is interested in maximizing his utilities from consumption and terminal wealth subject to a random inflation in the consumption basket prices over time. The measurement of the consumption basket price level is difficult because the goods in the basket change over time and the price data is scarce and/or noisy. These lead to a situation where the consumers do not exactly know the current price of their consumption basket. In other words, inflation in the consumption basket price is not fully observed and, therefore, the current real asset prices are also incompletely observed.
We first consider fully observed basket prices. In this case, the real asset market is complete, and the optimal policy can be solved using a Hamilton-Jacobi-Bellman equation. The real optimal consumption process equals the optimal policy studied in the classical case in Merton (1971) and Karatzas et al. (1986). However, since the consumption basket price is also stochastic, its presence affects the optimal portfolio selection. We show that the optimal portfolio can be characterized as a combination of three funds: a growth optimal fund and a risk free fund as in the classical case, and a fund that hedges the consumption basket price as well as possible.
Every investor uses the first two funds, whereas the composition of the last fund depends on his consumption basket price dynamics. We then study the situation where the consumption basket prices are not directly observed. Instead, the investor receives noisy observations on these prices called *signals*. These signals, for instance, could be prices of certain items in the basket, consumer price index (CPI), or monthly credit card bills. With the help of these signals, the investor can obtain the conditional probability distribution of the current basket price and thereby, the distribution of the current real asset prices.
Naturally, the new risk due to the partial observability of the basket prices affects the optimal policy. The higher the variance of the current consumption basket price, the lower is the nominal consumption. Furthermore, since the investor can update the distributions of the consumption basket prices from the signals, it is better for him to postpone some current consumption in favor of future consumption when the basket price is better known. Thus, our risk-averse investor hedges the additional uncertainty by postponing consumption. A similar hedging occurs in Liu and Zhou (2006), where the consumption-saving puzzle is explained by a wealth shock and the lack of insurance against such a shock.
Interestingly enough, the components of the optimal portfolio in the partially observed case are the same as in the fully observed case. Thus, in both cases, the optimal portfolio is a linear combination of the growth optimal fund, the fund that replicates the inflation uncertainty as well as possible, and the risk free fund. In general, the relative wealth allocation in these funds are different because the investor’s beliefs about the consumption basket price are different in the two cases.
In the particular case of the constant relative risk aversion (CRRA) utilities, we show that the optimal portfolios are the same in both cases. More specifically, since the consumption is lower due to the partial observability risk, the investor saves more. However, since the proportional wealth in the financial assets is the same as in the fully observed case, our model predicts that the additional uncertainty in the current consumption basket price and the current real asset prices decreases consumption and increases savings, but without affecting the proportional nature of the optimal portfolio.
Information Sets

Let \((\Omega, \mathcal{F}, P)\) be a probability space hosting Wiener processes \(w_I(\cdot), w_Z(\cdot),\) and \(w(\cdot)\). The first two processes are scalar valued, whereas the last one is \(n\) dimensional. The process \(w_I(\cdot)\) models the random nature of inflation, \(w_Z(\cdot)\) is the signal observed when the consumption basket is not directly observable, and \(w(\cdot)\) models the uncertainties in the risky financial assets.
There is a correlation between the inflation and the market, i.e., the process \( w_I(\cdot) \) and \( w(\cdot) \) are correlated. The process \( w_Z(\cdot) \) is independent of the others. The signal algebra

\[
\mathcal{F}_t = \sigma\{w_I(s), w_Z(s), w(s), s \leq t\}
\]

denotes the basic filtration on which all processes are adapted, except possibly the initial conditions.
Stochastic Processes

We shall introduce several stochastic processes: $Y_0(t)$ represents the riskless asset; $Y(t)$ is an $n$ dimensional stochastic process representing the prices of the risky assets; $X(t)$ denotes the investor’s wealth; $L(t)$ is a process measuring the consumption basket price; and $Z(t)$ is the signal when the process $L(t)$ is not directly observable. In this case, the pair $(w(t), Z(t))$ is observable and we let

$$\mathcal{G}_t = \sigma \{ w(s), Z(s), s \leq t \},$$

which denotes the filtration of the observations.
Evolution of the Market

The evolution of the riskless asset is described by

$$dY_0(t) = r(t)Y_0(t)dt, \quad y_0(0) = y_0,$$

where $y_0 > 0$ is a given constant and $r(t)$ is the deterministic nominal risk free interest rate. The evolution of the prices of the risky assets is given by

$$dY_i(t) = Y_i(t)(\alpha_i(t))dt + \sum_{j=1}^{n} \sigma_{ij}(t)dw_j(t))$$

$$Y_i(0) = y_i, \quad i = 1, \ldots n$$
We assume that $y_i > 0$ are given constants, and $\alpha_i(t)$ and $\sigma_{ij}(t)$ are deterministic and bounded expected returns and volatility functions, respectively. We assume also that the volatility matrix 

$$
\sigma(t) = (\sigma_{ij}(t))_{1 \leq i \leq n, 1 \leq j \leq n}
$$

is invertible for each $t$. This implies that our market is complete.
Basket Price

We denote the consumption basket price, or simply the basket price, of the investor by $B(t)$. The basket price acts as a numeraire in order to get the real consumption process and terminal wealth from the nominal ones. The dynamics of the basket price is given by

$$dB(t) = B(t) \left( Idt + \zeta dw_I(t) \right), \quad B(0) = b_0,$$

in which $b_0$ represents the initial condition.
\(I\) and \(\zeta\) are constants, and the correlation between \(w(t)\) and \(w_I(t)\) is defined by a vector \(\rho = (\rho_1, \ldots, \rho_n)^T\), where

\[
E [dw_j(t)dw_I(t)] = \rho_j dt, \ i = 1, \ldots n,
\]

and \(y^T\) is the transpose of \(y\). The number \(I\) denotes the expected instantaneous inflation and \(\zeta > 0\) is the inflation volatility. Since the basket prices usually rise, it is natural to assume \(I > 0\). The initial basket price \(b_0\) is known when there is full observation. In the case of partial information for the process \(B(t)\), the initial condition can be a random variable independent of \(\mathcal{F}_t\).
Wealth Process

Let \( L(t) = \log B(t) \)

\[
dL(t) = (I - \frac{1}{2} \zeta^2) dt + \zeta dW_I(t), \quad L(0) = \log b_0.\]

The nominal wealth at time \( t \) is defined by

\[
X(t) = \varpi_f(t) Y_0(t) + \varpi(t) Y(t),
\]

where \( \varpi_f(t) \) and \( \varpi(t) \) denote the amount of riskless and risky assets possessed by the investor. We assume that the self-financing condition holds

\[
dX(t) = \varpi_f(t) dY_0(t) + \varpi(t) dY(t) - C(t) dt,
\]

where \( C(t) \) is the instantaneous consumption rate.
Market Price of Risk

We define the *market price of risk* $\theta$ by

$$
\alpha_i(t) - r = \sum_{j=1}^{n} \sigma_{ij}(t)\theta_j(t), \ i = 1, \ldots n.
$$

The wealth evolution can be written as

$$
dX(t) = r(t)X(t)dt + X(t)\pi(t)\sigma(t)(dw(t) + \theta(t)dt) - C'(t)dt
$$

(1)

where $\pi = (\pi_1, \ldots, \pi_n)$ is the vector of the proportional wealth in the risky assets, i.e.,

$$
\varpi_i(t)Y_i(t) = \pi_i(t)X(t), \ i = 1, \ldots n.
$$
In the full information case, the investor observes the process $L(t)$ and his information set is $\mathcal{F}_t$. We shall consider a problem starting at time $t$ with known initial conditions $X(t) = x$ and $L(t) = L$. The role of the consumption basket is to discount nominal consumption and wealth. Thus, the real consumption and the real wealth are $Ce^{-L}$ and $Xe^{-L}$, respectively, where $C$ and $X$ are the amount of money spent on consumption and the nominal wealth. The agent gets utility from the real consumption and wealth. Therefore, we introduce the respective utility functions $U_1(\cdot)$ and $U_2(\cdot)$, which are twice differentiable, increasing, and concave.
Utility Functions

Moreover,

\[ U'_i(0) = \infty, \quad U'_i(\infty) = 0, \quad i = 1, 2, \quad (2) \]

\[ \frac{\partial U_1(Ce^{-L})}{\partial C} = U'_1(Ce^{-L})e^{-L} > 0 \]

\[ \frac{\partial U_1(Ce^{-L})}{\partial L} = -U'_1(Ce^{-L})Ce^{-L} < 0 \]

\[ \frac{\partial^2 U_1(Ce^{-L})}{\partial C^2} = U'''_1(Ce^{-L})e^{-2L} < 0 \]

\[ \frac{\partial^2 U_1(Ce^{-L})}{\partial L^2} = Ce^{-L}[U'_1(Ce^{-L}) + U''_1(Ce^{-L})Ce^{-L}] \quad (3) \]
The second derivative with respect to $L$ is nonnegative if, and only if,

$$-\frac{U''_1(Ce^{-L})Ce^{-L}}{U'_1(Ce^{-L})} \leq 1$$

i.e., iff the relative risk aversion is less than one. Next

$$\frac{\partial^2 U_1(Ce^{-L})}{\partial L \partial C} = -e^{-L}[U'_1(Ce^{-L}) + U''_1(Ce^{-L})Ce^{-L}] =$$

$$= -\frac{1}{C} \frac{\partial^2 U_1(Ce^{-L})}{\partial L^2}$$
By the second derivative above, this is negative if the relative risk aversion is less than one. Thus, in this case, the marginal utility from the nominal consumption goes down with respect to the consumption basket price. The results crucially depend on whether the value of the relative risk aversion is greater or less than unity.
Objective Function

We define the objective function

\[ J(C(\cdot), \pi(\cdot); x, L, t) = E \left[ \int_t^T e^{-\beta(s-t)} U_1(C(s)e^{-L(s)}) ds \right. \]

\[ \left. + e^{-\beta(T-t)} U_2(X(T)e^{-L(T)}) | L(t) = L, X(t) = x \right] \]

(4)

where wealth \( X(t) \) follows (1), terminal time \( T > 0 \), \( \beta \) is the utility discount rate and it can be different from the risk-free rate. The value function is defined by

\[ V(x, L, t) = \sup_{C(\cdot), \pi(\cdot)} J(C(\cdot), \pi(\cdot); x, L, t) \].
Optimal Policy

The HJB equation is

\[
\frac{\partial V}{\partial t} - \beta V + rx \frac{\partial V}{\partial x} + (I - \frac{1}{2}\zeta^2) \frac{\partial V}{\partial L} + \frac{1}{2}\zeta^2 \frac{\partial^2 V}{\partial L^2} \\
+ \max_C \left\{ U_1(Ce^{-L}) - C \frac{\partial V}{\partial x} \right\} \\
+ \max_\pi \left\{ x\pi\sigma \left( \frac{\partial V}{\partial x} \theta + \zeta \frac{\partial^2 V}{\partial L\partial x} \rho \right) + \frac{1}{2}x^2 \frac{\partial^2 V}{\partial x^2} \pi a\pi \right\} = 0
\]

with the terminal condition \( V(x, L, T) = U_2(xe^{-L}) \),

where \( a = \sigma\sigma^\top \).
To solve the two maximization problems appearing in (5), we define the function $\ell_1(\lambda)$ as the inverse of $U'(c)$, i.e.,

$$U'_1(c) = \lambda \iff c = \ell_1(\lambda).$$

The inverse exists and is unique by (2) and the concavity of $U_1$. The problem of solving for consumption in (5) is a concave maximization problem. Therefore, the first-order necessary condition is also sufficient, and it is

$$e^{-L}U'_1(Ce^{-L}) - V_x(x, L, t) = 0,$$
Optimal Consumption

From this we get the optimal feedback nominal consumption

\[ C^*(x, L, t) = e^L \ell_1 \left( e^L V_x(x, L, t) \right), \quad (6) \]

which transforms the real consumption in the classical case into nominal consumption. Once we know the value function derivative, we get the optimal consumption.
Optimal Portfolio

We also get

\[ \pi^*(x, L, t)^T = \]

\[ -\frac{(\sigma^T)^{-1}(t)}{xV_{xx}(x, L, t)} [V_x(x, L, t)\theta(t) + \zeta V_{Lx}(x, L, t)\rho]. \]

(7)

The first term is the classical solution and the second term is the effect of the uncertainty in inflation. The inflation effect depends also on the inflation volatility \( \zeta > 0 \) and on \( V_{Lx} \), i.e., on the sensitivity of the marginal value \( V_x \) with respect to the log basket price.
Mutual Fund

Theorem 1. With fully observed inflation, the following hold: (i) The optimal portfolio involves an allocation between the risk free fund $F^0$ and two risky funds that consist only of risky assets:

$$F^1(t) = (\sigma^T)^{-1}(t)\theta(t) \quad \text{and} \quad F^2(t) = (\sigma^T)^{-1}(t)\rho,$$

where the vector $F^k(t)$ represents the $k$th portfolio’s weights of the risky assets at time $t$, $k = 1, 2$. (ii) The optimal relative allocation of wealth in the funds $F^1(t)$, $F^2(t)$, and $F^0(t)$ at time $t$ is given by

$$\mu^1(t) = -\frac{V_{xx}(x,L,t)}{xV_{xx}(x,L,t)}, \quad \mu^2(t) = -\frac{\xi V_{Lx}(x,L,t)}{xV_{xx}(x,L,t)}, \quad \text{and}$$

$$\mu^0(t) = 1 - \mu_1(t) - \mu_2(t),$$

respectively.
Interpretation

The first fund is the growth optimum portfolio fund in the classical problem. The second fund arises from the correlation between the inflation uncertainty and the market risk, and the third fund is the risk free asset also as in the classical problem. Three fund theorems are not new. In the three fund theorem obtained by Brennan and Xia (2002), one fund is the classical growth optimal fund, another one replicates real interest rate uncertainty, and the last one replicates the fully observed inflation uncertainty. They do not consider partially observed inflation.
HJB Equation

By (7), we have

\[ x^2 V_{xx} \pi^* a(\pi^*)^T = - \left| \frac{V_x \theta(t) + \zeta V_{Lx} \rho}{V_{xx}} \right|^2. \]

The HJB equation (5) becomes

\[ V_t - \beta V + r(t) x V_x + (I - \frac{1}{2} \zeta^2) V_L + \frac{1}{2} \zeta^2 V_{LL} + \]

\[ + U_1(\ell_1(e^L V_x)) - e^L V_x \ell_1(e^L V_x) - \frac{1}{2} \left| \frac{V_x \theta(t) + \zeta V_{Lx} \rho}{V_{xx}} \right|^2 = 0 \]

with the terminal condition \( V(x, L, T) = U_2(x e^{-L}) \).
Example

Let us consider the constant relative risk aversion (CRRA) utility:

$$U_i(y) = \frac{y^{1-\phi}}{1 - \phi}, \ i = 1, 2,$$

where the relative risk aversion coefficient $\phi > 0$, $\phi \neq 1$. Note that $y^{1-\phi}$ is increasing in $y$ if $\phi \in (0, 1)$ but decreasing if $\phi > 1$. Therefore, dividing by $1 - \phi$ ensures that the marginal utility is positive for all values of $\phi$. 
Further, note that the third derivative is positive, implying a positive motive for precautionary saving. From the CRRA utility, we get $U'_1(c) = c^{-\phi}$ and $\ell_1(\lambda) = \lambda^{-1/\phi}$. Therefore, we can obtain the convex dual

$$\max_{0 < c < \infty} [U_1(c) - \lambda c] = U_1(\ell_1(\lambda)) - \lambda \ell_1(\lambda) = \frac{\phi}{1-\phi} \lambda^{\frac{\phi-1}{\phi}}$$

for all $\lambda \in (0, \infty)$. 
Then (8) can be written as

\[ V_t - \beta V + r(t)xV_x + (I - \frac{1}{2}\xi^2)V_L + \frac{1}{2}\xi^2V_{LL} + \]

\[ + \frac{\phi}{1-\phi} \left( e^L V_x \right)^{\frac{\phi-1}{\phi}} - \frac{|V_x \theta(t) + \xi V_{Lx} \rho|^2}{2V_{xx}} = 0 \]

(9)

where \( V(x, L, T) = \frac{1}{1-\phi} \left( xe^{-L} \right)^{1-\phi} \).
The solution of this equation is given by

\[ V(x, L, t) = \frac{1}{1-\phi} \left( xe^{-L} \right)^{1-\phi} g(t), \quad (10) \]

where \( g(t) \) solves

\[ g' + g[(1 - \phi)(r(t) - I + \frac{1}{2}\zeta^2) - \beta + \frac{1}{2}\zeta^2(1 - \phi)^2 + \]

\[ + \frac{1}{2} \frac{1-\phi}{\phi} \theta(t) - \zeta(1 - \phi)\rho|^2] + \phi g \frac{\phi-1}{\phi} = 0 \]

with the terminal condition \( g(T) = 1. \)
By (6) and (7), the optimal nominal policy is given by:

\[ C^*(x, L, t) = \frac{x}{g(t)^{\frac{1}{\phi}}} \]

\[ \pi^*(x, L, t)^T = \frac{1}{\phi} (\sigma^T)^{-1}(t)[\theta(t) - (1 - \phi)\xi\rho] \]
Analysis

Naturally, the consumption rate is higher when the investor is wealthier. Also he invests more in the risky assets when the market price of risk is higher and the risks of the assets lower. If $\rho > 0$ and $\phi > 1$, then the investor invests more in the financial market than a corresponding investor who does not consider inflation. Note that $\phi > 1$ implies $\frac{\partial^2 U_i(ye^{-L})}{\partial L^2} < 0$ for all $y > 0$ and $i = 1, 2$, i.e., $\frac{\partial U_i(ye^{-L})}{\partial L}$ falls when $L$ rises. Thus, if $\phi > 1$ and $I > 0$, then the agent invests more in the risky assets and tries to hedge the decrease in the future utility from the rising consumption basket price. Conversely, if $\frac{\partial^2 U_i(ye^{-L})}{\partial L^2} > 0$, then the agent invests less in the risky assets.
Partially Observed Basket Prices

We now assume that the process $L(t)$ is not observable, and the investor has information $\mathcal{G}_t = \sigma\{w(s), Z(s), s \leq t\}$. The investor observes noisy signal $Z(t)$ on the consumption basket price. Some examples of this signal are monthly credit card bills and the consumer price index (CPI). The signal process evolves as follows:

$$dZ(t) = L(t)dt + mdw_{Z}(t), \quad Z(0) = 0, \quad (13)$$

where $m > 0$ is a constant signal volatility. Recall that the Wiener process $w_{Z}(t)$ is independent of the other uncertainties implied by $w(t)$ and $w_{I}(t)$. 
The investor also observes the process $w(t)$ from the asset prices in the market, i.e., by the invertibility of the matrix $\sigma(t)$, he recovers the process $w(t)$ from the asset price $Y(t)$. Because the agent’s information is described by the filtration $\mathcal{G}_t$, the decisions $C(\cdot)$ and $\pi(\cdot)$ must be adapted to $\mathcal{G}_t$. Moreover, since the objective function depends on $L(t)$, we need to compute the conditional probability of $L(t)$ given $\mathcal{G}_t$. Therefore, we have a non-linear filtering problem to solve.
Transformation

We perform the following transform

\[
d\tilde{w}_I(t) = \frac{dw_I(t) - \rho^T dw(t)}{\sqrt{1 - |\rho|^2}}
\]

\[
d\tilde{w}(t) = dw(t) + \theta(t) dt
\]

\[
d\tilde{Z}(t) = \frac{dZ(t)}{m}
\]

with initial condition 0. The first one is obviously a Wiener process under \( P \) and it is independent of the market. The last two are diffusion processes with drifts under \( P \).
Define the process $M(t)$ as follows:

$$dM(t) = -M(t)(\theta(t)^T dw(t) + \frac{L(t)}{m} dw_Z(t)), \quad M(0) = 1,$$

$$M(t) = \exp \left( - \int_0^t (\theta(s)^T dw(s) + \frac{L(s)}{m} dw_Z(s)) \right)$$

$$- \frac{1}{2} \int_0^t (|\theta(s)|^2 + \frac{L^2(s)}{m^2}) ds \right).$$

Hence, $M(t)$ is a $(P, \mathcal{F}_t)$ martingale.
Therefore, we can define the probability $\tilde{P}$ on $(\Omega, \mathcal{F})$ by the Radon-Nikodym derivative

$$\frac{d\tilde{P}}{dP} = M(t)$$

on $\mathcal{F}_t$. By the Girsanov Theorem we get

**Lemma 1.** The processes $\tilde{w}_I(t)$, $\tilde{w}(t)$, and $\tilde{Z}(t)$ are independent standard Wiener processes for $\tilde{P}$ and $\mathcal{F}_t$. 
Conditional Density of Log Basket Price

Given a smooth test function \( \psi_t(L) = \psi(L, t) \), we want to derive the operator

\[
\Pi(t)(\psi_t) = E[\psi(L(t), t)|\mathcal{G}_t],
\]

(15)

where the notation means that the operator \( \Pi(t) \) for each fixed \( t \) is a linear operator on functions of \( L \). This operator is the solution of a functional equation. We postulate a conditional probability density \( p(L, t) \):

\[
E[\psi(L(t), t)|\mathcal{G}_t] = \int p(L, t)\psi(L, t) dL
\]

for any test function \( \psi(L, t) \).
Zakai Equation

The density $p(L, t)$ is the solution of a Kushner equation, which can be obtained from an un-normalized probability density $q(L, t)$ derived from a Zakai equation (see Zakai (1969)).

**Lemma 2.** The process of un-normalized probability density $q(L, t)$ is as follows

$$dq = \left[-q_L(I - \frac{1}{2}\zeta^2) + \frac{1}{2}\zeta^2 q_{LL}\right]dt + (q\theta - q_L\zeta\rho)^T d\tilde{w}(t) + q\frac{L}{m}d\tilde{Z}(t)$$

(16)

where $q(L, 0) = p_0(L)$
For any test function $\psi$ we have

$$\int p(L, t)\psi(L, t)dL = \frac{\int q(L, t)\psi(L, t)dL}{\int q(L, t)dL}$$

**Theorem 2.** If

$$p_0(L) = \frac{1}{\sqrt{2\pi S_0}}e^{-\frac{1}{2}(L-L_0)^2/S_0}.$$  

Then

$$q(L, t) = K(t)e^{-\frac{1}{2}(L-\hat{L}(t))^2/S(t)}$$  \hspace{1cm} (17)$$

where the variance $S(t) = E[(L(t) - \hat{L}(t))^2|\mathcal{G}_t]$ is deterministic.
The variance $S(t)$ is given by the formula

$$S(t) = \begin{cases} 
  m\Lambda_1 \frac{\Lambda_2 \exp(2\Lambda_1 t/m) - 1}{\Lambda_2 \exp(2\Lambda_1 t) + 1} & \text{if } S_0 < m\Lambda_1 \\
  m\Lambda_1 & \text{if } S_0 = m\Lambda_1 \\
  m\Lambda_1 \frac{\Lambda_2 \exp(2\Lambda_1 t/m) + 1}{\Lambda_2 \exp(2\Lambda_1 t) - 1} & \text{if } S_0 > m\Lambda_1, 
\end{cases}$$

(18)

where $\Lambda_1 = \zeta \sqrt{1 - |\rho|^2}$ and $\Lambda_2 = \left| \frac{m\Lambda_1 + S_0}{m\Lambda_1 - S_0} \right|$. 
Kalman Filter

The Belief \( \hat{L}(t) = E[L(t)|\mathcal{G}_t] \) is the solution to the Kalman filter:

\[
d\hat{L}(t) = (I - \frac{1}{2}\zeta^2 - \zeta \rho^T \theta(t))dt + \\
+ \zeta \rho^T d\tilde{w}(t) + \frac{S(t)}{m} (d\tilde{Z}(t) - \frac{\hat{L}(t)}{m} dt)
\]

where \( \hat{L}(0) = L_0 \).

(19)
The variable $K(t)$ in (17) is given by

$$K(t) = \exp\left(-\frac{1}{2} \int_0^t \left( \frac{\hat{L}^2(s)}{m^2} + |\theta(s)|^2 \right) ds + \right.$$  

$$+ \int_0^t \frac{\hat{L}(s)}{m} d\tilde{Z}(s) + \int_0^t \theta(s)^T d\tilde{w}(s) \right)$$

(20)

If $S_0 = 0$ then

$$S(t) = m\Lambda_1 \tanh\left( \frac{\Lambda_1}{m} t \right)$$

with

$$\lim_{t \to \infty} S(t) = m\Lambda_1$$
By (19) and the definition of \( \tilde{Z}(t) \), the dynamics of \( \hat{L} \) in terms of the original processes are as follows:

\[
d\hat{L}(t) = (I - \frac{1}{2}\zeta^2)\,dt + \zeta\rho^\top\,dw(t) + \frac{S(t)}{m^2} (dZ(t) - \hat{L}(t)\,dt)
\]  

(21)
The innovation process defined by

\[ d\tilde{w}_Z(t) = \frac{1}{m} \left( dZ(t) - \hat{L}(t)dt \right), \quad \tilde{w}_Z(0) = 0. \]

is a $P$-Wiener process. Then we can write the belief process:

\[ d\hat{L}(t) = (I - \frac{1}{2} \zeta^2)dt + \zeta \rho^T dw(t) + \frac{S(t)}{m} d\tilde{w}_Z(t) \]  

(22)
It is driven by two Wiener processes: one from the signals (13) and one from the wealth process (1). By definition, the signals give information about the consumption basket price and, due to correlation $\rho$, also the asset prices give some information about the inflation. (22) combines the two Wiener processes. Let us consider an optimal consumption and investment problem starting at time $t$ with the initial conditions $\hat{L}(t) = \hat{L}$ and $X(t) = x$. The objective function of the investor is given similarly as in the full observation case, but now with respect to $\mathcal{G}_t$. 
It is given by

\[ \tilde{J}(C(\cdot), \pi(\cdot); x, \hat{L}, t) = \]

\[ E \left[ \int_t^T e^{-\beta(s-t)} U_1(C(s)e^{-L(s)}) ds + e^{-\beta(T-t)} U_2(X(T)e^{-L(T)}) \bigg| \mathcal{G}_t \right] \]

\[ = E \left[ \int_t^T e^{-\beta(s-t)} E[U_1(C(s)e^{-L(s)}) \bigg| \mathcal{G}_s] ds + e^{-\beta(T-t)} E[U_2(X(T)e^{-L(T)}) \bigg| \mathcal{G}_T] \bigg| \mathcal{G}_t \right]. \]

(23)
Hence, we integrate over the $L$-distribution and calculate first the expected utilities:

\[
\tilde{U}_1(C, \hat{L}, s) = \frac{1}{\sqrt{2\pi}} \int U_1(C e^{-\hat{L} - y\sqrt{S(s)}}) e^{-\frac{1}{2}y^2} dy
\]

\[
\tilde{U}_2(X, \hat{L}, s) = \frac{1}{\sqrt{2\pi}} \int U_2(X e^{-\hat{L} - y\sqrt{S(s)}}) e^{-\frac{1}{2}y^2} dy
\]

which are, thus, the expected utilities over the $L$-distribution.
Objective Function

The objective function can be written as

$$\tilde{J}(C(\cdot), \pi(\cdot); x, \hat{L}, t) =$$

$$E \left[ \int_t^T e^{-\beta(s-t)} \tilde{U}_1(C(s), \hat{L}(s), s) ds \right]$$

$$+ e^{-\beta(T-t)} \tilde{U}_2(X(T), \hat{L}(T), T) | \hat{L}(t) = \hat{L}, X(t) = x \right]$$

(25)

and the value function as

$$\tilde{V}(x, \hat{L}, t) = \sup_{C(\cdot), \pi(\cdot)} \tilde{J}(C(\cdot), \pi(\cdot); x, \hat{L}, t).$$
Solution

We first write the HJB which now is given as follows:

\[
\tilde{V}_t - \beta \tilde{V} + r x \tilde{V}_x + (I - \frac{1}{2} \zeta^2) \tilde{V}_{\hat{L}} + \\
+ \frac{1}{2} (\zeta^2 |\rho|^2 + \frac{S^2(t)}{m^2}) \tilde{V}_{\hat{L}\hat{L}} + \sup_C \{ \tilde{U}_1(C, \hat{L}, t) - C \tilde{V}_x \} + \\
+ \sup_{\pi} \{ x \pi \sigma (\tilde{V}_x \theta + \zeta \tilde{V}_{\hat{L}x} \rho) + \frac{1}{2} x^2 \tilde{V}_{xx} \pi \pi^T \} = 0
\]

(26)

with \( \tilde{V}(x, \hat{L}, T) = \tilde{U}_2(x, \hat{L}, T) \). Note that we get (5), if we assume zero initial uncertainty due to partial observation (\( S'(0) = 0 \)) and perfect observation (\( \rho = 1 \)).
By the first order conditions, the optimal consumption and portfolio strategy \((\hat{C}, \hat{\pi})\) satisfies

\[
\left. \frac{\partial \tilde{U}_1(C, \hat{L}, t)}{\partial C} \right|_{C = \hat{C}(x, \hat{L}, t)} = \tilde{V}_x(x, \hat{L}, t)
\]

\[
\hat{\pi}(x, \hat{L}, t)^T = -\frac{(\sigma^T)^{-1}(t)}{x \tilde{V}_{xx}(x, \hat{L}, t)} \left[ \tilde{V}_x(x, \hat{L}, t)\theta(t) + \zeta \tilde{V}_{Lx}(x, \hat{L}, t)\rho \right]
\]

\[(27)\]
Three Fund Theorem

**Theorem 3.** Under the partially observable basket process, Theorem 1 holds with a modified relative allocations of wealth between the funds:

\[ \hat{\mu}_1(t) = - \frac{\tilde{V}_x(x, \hat{L}, t)}{x \tilde{V}_{xx}(x, \hat{L}, t)} \]

\[ \hat{\mu}_2(t) = - \frac{\zeta \tilde{V}_{\hat{L}x}(x, \hat{L}, t)}{x \tilde{V}_{xx}(x, \hat{L}, t)} \]

and \( \hat{\mu}_0(t) = 1 - \hat{\mu}_1(t) - \hat{\mu}_2(t) \), where \( \hat{\mu}_i(t) \) is the relative wealth invested in the \( i \)th fund at time \( t \).
Further Analysis

Theorems 1 and 3 imply that the components of the funds are the same under the fully observed and partially observed basket prices, only the relative allocations of the wealth invested in these funds are different. Thus, in both the cases the optimal portfolio is a linear combination of the growth optimum fund, the fund that replicates the inflation uncertainty as well as possible, and the risk free fund. The proportions of the wealth invested in these funds are different because the investor’s beliefs on the consumption basket prices are not the same under different information sets, i.e., because \( \hat{L} \neq L \). Thus, the noisy signals affect the optimal solution through the value function derivatives.
Brennan (1998) and Xia (2001) consider the effects of uncertainty about the stock return predictability on optimal dynamic portfolio choice, i.e., in their models, the expected returns are unknown and are learned from market variables. Our model is quite different from theirs since in the present paper we have uncertainty on the deflator, i.e., on the consumption basket price. Our investor is only interested in the real prices \( \{Y_i/B\} \), but he does not observe the values of \( B \), which implies that the current real prices are random. Thus, in our model there is no uncertainty on the expected returns, but on their current real values.
From (26) and (27) we get

\[ \tilde{V}_t - \beta \tilde{V} + rx \tilde{V}_x + \]

\[ + (I - \frac{1}{2}\zeta^2) \tilde{V}_L + \frac{1}{2}(\zeta^2|\rho|^2 + \frac{S^2(t)}{m^2}) \tilde{V}_{L \hat{L}} \]

\[ + \tilde{U}_1(\hat{C}, \hat{L}, t) - \hat{C} \tilde{V}_x - \frac{|\tilde{V}_x \theta(t) + \zeta \tilde{V}_x \hat{L} \rho|^2}{2\tilde{V}_{xx}} = 0 \]

where \( \tilde{V}(x, \hat{L}, T) = \tilde{U}_2(x, \hat{L}, T) \).
Example

We use the CRRA utility function introduced in the fully observable case. Therefore,

\[
\tilde{U}_1(C, \hat{L}, t) = \frac{1}{1-\phi} C^{1-\phi} e^{-(1-\phi)\hat{L}} + \frac{1}{2} (1-\phi)^2 S(t)
\]

\[
\tilde{U}_2(x, \hat{L}, T) = \frac{1}{1-\phi} x^{1-\phi} e^{-(1-\phi)\hat{L}} + \frac{1}{2} (1-\phi)^2 S(T)
\]

where the relative risk aversion \( \phi > 0, \phi \neq 1 \). From (27) we get

\[
\hat{C}(x, \hat{L}, t) = \tilde{V}_x^{-1/\phi} e^{\frac{\phi-1}{\phi} \hat{L}} + \frac{1}{2} \frac{(1-\phi)^2}{\phi} S(t)
\]
The HJB equation reduces to

\[
\tilde{V}_t - \beta \tilde{V} + r x \tilde{V}_x + \\
(I - \frac{1}{2} \zeta^2) \tilde{V}_{\hat{L}} + \frac{1}{2} (\zeta^2 |\rho|^2 + \frac{S^2(t)}{m^2}) \tilde{V}_{\hat{L}} \hat{L} \\
+ \frac{\phi}{\phi - 1} \left( \tilde{V}_x e^{\hat{L} - \frac{1}{2} (1-\phi) S(t)} \right)^{\frac{\phi - 1}{\phi}} \\
- \frac{2 \tilde{V}_{xx}}{2 \tilde{V}_{xx}} = 0
\]

(29)
We have

\[ \tilde{V}(x, \hat{L}, T) = \frac{1}{1-\phi} x^{1-\phi} e^{-(1-\phi)\hat{L}} + \frac{1}{2} (1-\phi)^2 S(T) \]

Its solution is given by

\[ \tilde{V}(x, \hat{L}, t) = \frac{1}{1-\phi} x^{1-\phi} e^{-(1-\phi)\hat{L}} h(t) \quad (30) \]
$h(t)$ is the solution of

$$
h' + h'[(1 - \phi)(r - I + \frac{1}{2}\zeta^2) - \beta + \\
+ \frac{1}{2}(1 - \phi)^2(\zeta^2|\rho|^2 + \frac{S^2(t)}{m^2}) + \frac{1}{2} \frac{1 - \phi}{\phi} |\theta(t) - \zeta(1 - \phi)\rho|^2] \\
+ \phi h \phi \frac{1 - \phi}{\phi} e^{\frac{1}{2}} \frac{(1 - \phi)^2}{\phi} S(t) = 0$$

(31)

with $h(T) = e^{\frac{1}{2}}(1 - \phi)^2 S(T)$. 
Now (27) gives the optimal feedback consumption and portfolio policies:

\[
\hat{C}(x, \hat{L}, t) = \frac{x}{(h(t)e^{-\frac{1}{2}(1-\phi)^2 S(t)\frac{1}{\phi}}} \\
\hat{\pi}(x, L, t)^T = \frac{1}{\phi}(\sigma^T)^{-1}(t)[\theta(t) - (1 - \phi)\zeta\rho].
\]
By comparing with (12), we get the following result.

**Proposition 1.** *Let the investor have the CRRA utilities.*

(i) The optimal portfolio process is not affected by the uncertainty in the current consumption basket price.

(ii) The higher the consumption basket price uncertainty, the lower is the optimal nominal consumption.

The nominal consumption process hedges the consumption basket price uncertainty and, therefore, the portfolio in (32) is the same as in (12).
Conclusions

We have derived a new optimal portfolio and consumption decision model under partially observed real asset prices. The investor observes noisy signals on the consumption basket price. Based on these, he updates his real asset and consumption basket price beliefs, and then decides on his portfolio and his consumption rate. We show that a modified Mutual Fund Theorem consisting of three funds holds. The funds are a growth optimum fund, a fund that replicates the inflation uncertainty as well as possible, and a risk-free fund.
In general, the wealth invested in these funds depends on the investor’s utility function and on his beliefs about the consumption basket price. However, the funds are robust over different information sets on the consumption basket price. That is, the investor uses the same three funds regardless of the noise in the consumption basket price signals.
We solve our model explicitly for the CRRA utility functions. In this case, the noise in the consumption basket price signals lowers the nominal consumption but does not change the relative portfolio. The investor hedges the consumption basket price uncertainty due to its partial observability by postponing a part of his consumption. Thus, our model predicts that the additional uncertainty in the current consumption basket price decreases consumption and increases savings, but without affecting the proportional nature of the optimal portfolio.
References


