ラプラスシアンの Green 関数の Hadamard 変分

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数学協働プログラム 工学と現代数学の接点を求めて (1)
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The Green Function

\( \Omega \subset \mathbb{R}^n \): a bounded domain with the Lipschitz boundary \( \partial \Omega \) \( (n \geq 2) \).

\( f \in L^2(\Omega) \): given.

Consider the following Poisson problem:

\[
-\Delta w = f \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial \Omega.
\]

The unique solution of the Poisson problem is written by

\[
w(x) = \int_{\Omega} G(x, y) f(y) \, dy,
\]

where \( G(x, y) \) is the Green function of \( \Delta \) on \( \Omega \).

The Green function of the above BVP is defined so that

- \( -\Delta G(x, y) = \delta(x - y), x, y \in \Omega \). \( \delta \) is the Dirac’s delta function.
- \( G(x, y) = 0, x \in \partial \Omega, y \in \Omega \).
Define the fundamental solution $\Gamma(x)$ of $\Delta$ by

$$
\Gamma(x) := \begin{cases} 
-\frac{1}{2\pi} \log |x|, & n = 2 \\
\frac{1}{(n-2)\omega_n} |x|^{2-n}, & n \geq 3.
\end{cases}
$$

Then, we have

$$-\Delta \Gamma(x-y) = \delta(x-y),$$

or, for $w \in C^2(\overline{\Omega})$,

$$w(y) = \int_{\Omega} (-\Delta w(x)) \Gamma(x-y) \, dx$$

$$+ \int_{\partial \Omega} \left[ \frac{\partial w}{\partial \nu}(x) \Gamma(x-y) - w(x) \frac{\partial}{\partial \nu_x} \Gamma(x-y) \right] \, ds_x.$$

This formula is called Green's representation formula.
For a harmonic function \( u \in C^1(\overline{\Omega}) \cap C^2(\Omega) \), we have the Green's formula:

\[
0 = \int_{\Omega} (-\Delta w) u \, dx + \int_{\partial \Omega} \left[ \frac{\partial w}{\partial \nu} u - w \frac{\partial u}{\partial \nu} \right] \, ds.
\]

Therefore, we obtain a more general version of Green's representation formula:

\[
w(y) = \int_{\Omega} (-\Delta w)(x)(\Gamma(x - y) + u(x)) \, dx \\
+ \int_{\partial \Omega} \left[ \frac{\partial w(x)}{\partial \nu}(\Gamma(x - y) + u(x)) - w(x) \frac{\partial}{\partial \nu_x}(\Gamma(x - y) + u(x)) \right] \, ds_x.
\]
Then, the Green function $G(x, y)$ is defined by

$$G(x, y) := \Gamma(x - y) + K(x, y),$$

where $u(x) = K(x, y)$ is a harmonic function of $x$ for fixed $y \in \Omega$ such that

$$\Delta u = 0 \quad \text{in} \, \Omega, \quad u = -\Gamma(\cdot - y) \quad \text{on} \, \partial \Omega.$$

$$w(y) = \int_{\Omega} (-\Delta w)(x)G(x, y)dx - \int_{\partial \Omega} w(x) \frac{\partial}{\partial \nu_x} G(x, y)ds_x.$$
Therefore, the solution of the BVP

\[-\Delta w = f \quad \text{in } \Omega, \quad w = g \quad \text{on } \partial \Omega\]

is written by

\[w(y) = \int_{\Omega} f(x) G(x, y) \, dx - \int_{\partial \Omega} g(x) \frac{\partial}{\partial \nu_x} G(x, y) \, ds_x.\]
Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be a sufficiently large domain such that $\Omega \subset \widetilde{\Omega}$.

We consider a family of bi-Lipschitz homeomorphisms

$$\mathcal{T}_t(x) : \Omega \rightarrow \mathcal{T}_t(\Omega) \subset \widetilde{\Omega}, \quad \text{supp}(\mathcal{T}_t) \subset \widetilde{\Omega}.$$ 

We assume that $\Omega \ni x \mapsto \mathcal{T}_t(x)$ is twice differentiable with respect to $t$. We also assume that, for $i = 1, 2$,

$$\frac{\partial^i}{\partial t^i} \mathcal{T}_t(x), \quad \frac{\partial^i}{\partial t^i} \nabla(\mathcal{T}_t(x)), \quad \frac{\partial^i}{\partial t^i} \mathcal{T}_t^{-1}(x), \quad \frac{\partial^i}{\partial t^i} \nabla(\mathcal{T}_t^{-1}(x))$$

are uniformly bounded on $\Omega \times (-\varepsilon, \varepsilon)$ for sufficiently small $\varepsilon > 0$. 
Define

\[ S := \left. \frac{\partial T_t}{\partial t} \right|_{t=0}, \quad R := \left. \frac{\partial^2 T_t}{\partial t^2} \right|_{t=0}. \]

Then, \( T_t \) has the Taylor expansion

\[ T_t(x) = x + tS(x) + \frac{1}{2} t^2 R(x) + o(t^2). \]

\( \Omega \) and \( \Omega_t \).
Normal Perturbations

Let $\partial \Omega$ be sufficiently smooth and $\rho(x)$ be a smooth function defined on $\partial \Omega$. Then, define

$$\partial \Omega + t \rho : x + t \rho(x) \nu_x, \quad x \in \partial \Omega.$$ 

The domain $\Omega_t$ is the domain with $\partial \Omega_t = \partial \Omega + t \rho$. 
Dynamical Perturbations

Let a vector field $S$ be given on $\tilde{\Omega}$ with $\text{supp} S \subset \tilde{\Omega}$. Then, the domain transformations

$$\mathcal{T}_t(x) : \Omega \to \mathcal{T}_t(\Omega) \subset \mathbb{R}^n$$

are defined by

$$\frac{d}{dt} \mathcal{T}_t(x) = S(\mathcal{T}_t(x)), \quad \mathcal{T}_0(x) = x.$$

In this case, $\mathcal{T}_t(x)$ is called a dynamical perturbation.

Note that, for dynamical perturbations, we have

$$(S \cdot \nabla)S = R.$$
Let $G(x, y, t)$: the Green ft. of $-\Delta$ on $\Omega_t$.
Hadamard obtained the derivative of the Green ft. $G(x, y, t)$ with respect $t \geq 0$:
\[
\delta G(x, y) := \lim_{t \to 0^+} \frac{G(x, y, t) - G(x, y)}{t}, \quad x, y \in \Omega.
\]

**Theorem 1 (Hadamard’s variational formula)** Let $\partial \Omega$ be of $C^{1,1}$-class. For the differentiable perturbation $T_t$, we have
\[
\delta G(w, y) = \left\langle \delta \rho \frac{\partial}{\partial \nu} G(\cdot, y), \frac{\partial}{\partial \nu} G(\cdot, w) \right\rangle_{1/2, \partial \Omega},
\]
where $\delta \rho := S \cdot \nu$.

$\left\langle \cdot, \cdot \right\rangle_{1/2, \partial \Omega}$: the duality pairing of $H^{-1/2}(\partial \Omega)$ and $H^{1/2}(\partial \Omega)$. 
History of the Hadamard Variational Formula


Hadamard (1908):
\[ \partial \Omega \text{ and } S(x) \text{ are of } C^\omega \text{-class and } T_t(x) \text{ is a normal perturbation.} \]

Schiffer (1946), Garabedian-Schiffer (1952-53):
\[ \partial \Omega \text{ and } S(x) \text{ are of } C^k \text{-class with sufficiently large } k \text{ and } T_t(x) \text{ is a normal perturbation.} \]

Suzuki–Tsuchiya:
\[ \partial \Omega \text{ and } S(x) \text{ are of } C^{1,1} \text{-class and } T_t(x) \text{ is a general perturbation.} \]
Eulerian Derivative and Lagrangian Derivative

\[ \Omega \subset \mathbb{R}^n: C^{k,1} \text{ domain, } k = 1, 2. \]

\[ \Omega_t := \mathcal{T}_t(\Omega), \ t \geq 0. \]

\( \varphi \): a function defined in the nbd of \( \Omega \).

Consider the Dirichlet problem:

\[ \Delta u(\cdot, t) = 0 \text{ in } \Omega_t, \quad u(\cdot, t) = \varphi \text{ on } \partial \Omega_t. \]

Now, we consider the two kinds of derivatives of \( u(\cdot, t) \) with respect of \( t \).

\[ \dot{u}_L(x) := \frac{d}{dt} (u(T(x), t)) \bigg|_{t=0}, \quad \ddot{u}_L(x) := \frac{d^2}{dt^2} (u(T(x), t)) \bigg|_{t=0} \]

\[ \dot{u}_E(x) := \frac{\partial}{\partial t} u(x, t) \bigg|_{t=0}, \quad \ddot{u}_E(x) := \frac{\partial^2}{\partial t^2} u(x, t) \bigg|_{t=0}. \]

\( \dot{u}_L \) and \( \ddot{u}_L \) are called the Lagrangian derivatives,
\( \dot{u}_E \) and \( \ddot{u}_E \) are called the Eulerian derivatives.
Eulerian Derivative

Differentiating $u(\mathcal{T}_t(x), t) = \varphi(\mathcal{T}_t(x))$ with respect to $t$ and letting $t \to 0^+$, we have

$$\dot{u}_\varepsilon + S \cdot \nabla u = S \cdot \nabla \varphi \quad \text{on } \partial \Omega.$$ 

Therefore, $\dot{u}_\varepsilon$ satisfies

$$\Delta \dot{u}_\varepsilon = 0 \quad \text{in } \Omega,$$

$$\dot{u}_\varepsilon = S \cdot (\nabla \varphi - \nabla u) \quad \text{on } \partial \Omega.$$
Let $\Gamma(x)$ be the fundamental solution of the Laplacian. Then, the Green function $G(x, y)$ is defined by

$$G(x, y) := \Gamma(x - y) + K(x, y),$$

where $u(x) = K(x, y)$ is a harmonic function of $x$ for fixed $y \in \Omega$ such that

$$\Delta u = 0 \quad \text{in } \Omega, \quad u = -\Gamma(\cdot - y) \quad \text{on } \partial \Omega.$$

$G(x, y, t)$: the Green function of $\Omega_t$:

$$G(x, y, t) = \Gamma(x - y) + K(x, y, t),$$

where $u(x, t) = K(x, y, t)$ is a harmonic function such that

$$\Delta u(\cdot, t) = 0 \quad x \in \Omega_t, \quad u(\cdot, t) = -\Gamma(\cdot - y) \quad x \in \partial \Omega_t.$$
Therefore, the Eulerian derivative $\dot{u}_\varepsilon$ of $u$ satisfies

$$
\Delta \dot{u}_\varepsilon = 0 \quad \text{in } \Omega
$$

$$
\dot{u}_\varepsilon = S \cdot \nabla_x ( -\Gamma(x-y) - u(x)) = -S \cdot \nabla_x G(x,y)
$$

$$
= -(S \cdot \nu) \frac{\partial}{\partial \nu_x} G(x,y) \quad \text{on } \partial \Omega.
$$

The solution of the BVP

$$
\Delta v = 0 \quad \text{in } \Omega, \quad v = g \quad \text{on } \partial \Omega
$$

is written by

$$
v(w) = - \left\langle g, \frac{\partial}{\partial \nu_x} G(\cdot, w) \right\rangle_{1/2, \partial \Omega}.
$$
Therefore, we have

\[ \dot{u}_\varepsilon(w) = \left\langle (S \cdot \nu) \frac{\partial}{\partial \nu} G(\cdot, y), \frac{\partial}{\partial \nu} G(\cdot, w) \right\rangle_{1/2, \partial \Omega} \]

\[
\left( \dot{u}_\varepsilon(w) = \int_{\partial \Omega} \frac{\partial}{\partial \nu_x} G(x, w) \frac{\partial}{\partial \nu_x} G(x, y) (S \cdot \nu) ds_x \right)
\]

The first variation \( \delta G(x, y) \) of the Green function is written by, for \( w, y \in \Omega \)

\[ \delta G(w, y) := \lim_{t \to 0^+} \frac{G(w, y, t) - G(w, y)}{t} = \dot{u}_\varepsilon(w) \]

\[ = \left\langle (S \cdot \nu) \frac{\partial}{\partial \nu} G(\cdot, y), \frac{\partial}{\partial \nu} G(\cdot, w) \right\rangle_{1/2, \partial \Omega} \]
Consider the boundary value problem $\Delta u = 0$ in $\Omega$, $u = \varphi$ on $\partial\Omega$. Then, the solution’s Eulerian derivative $\dot{u}_\varepsilon$ satisfies

$$\Delta \dot{u}_\varepsilon = 0 \text{ in } \Omega, \quad \dot{u}_\varepsilon = S \cdot (\nabla \varphi - \nabla u) \text{ on } \partial\Omega.$$  

Also, $\ddot{u}_\varepsilon$ satisfies

$$\Delta \ddot{u}_\varepsilon = 0 \quad \text{in } \Omega,$$

$$\ddot{u}_\varepsilon = -2S \cdot \nabla \dot{u}_\varepsilon + R \cdot (\nabla \varphi - \nabla u) + (\mathcal{H}_x \varphi - \mathcal{H}_x u) \cdot (S)^2 \quad \text{on } \partial\Omega,$$

where $\mathcal{H}_x \varphi$ is the Hesse matrix of $\varphi$.

On the case of the Green function, $\varphi = -\Gamma$. 
Recall that the Green function $G$ is defined by

$$G(x, y) := \Gamma(x - y) + u(x).$$

The harmonic function $u$ satisfies $\Delta u = 0$ in $\Omega$, $u = -\Gamma$ on $\partial \Omega$. The second variation of the Green function is defined by

$$\delta^2 G(x, y) := \frac{\partial^2}{\partial t^2} G(x, y, t) \bigg|_{t=0} = \ddot{u}_\mathcal{E}(x).$$

The function $\ddot{u}_\mathcal{E}$ is written by

$$\delta^2 G(x, y) = \ddot{u}_\mathcal{E}(x) = 2 \int_{\partial \Omega} S \cdot \nabla_w \delta G(w, y) \frac{\partial}{\partial \nu_x} G(x, w) ds_w + \int_{\partial \Omega} R \cdot \nabla_w G(w, y) \frac{\partial}{\partial \nu_x} G(x, w) ds_w + \int_{\partial \Omega} H_w G(w, y) \cdot (S)^2 \frac{\partial}{\partial \nu_x} G(x, w) ds_w.$$
Theorem 2 (Hadamard’s second variational formula)

Let $\partial \Omega$ be of $C^{2,1}$-class. For the twice differentiable perturbation $T_t$, we have

$$
\delta^2 G(x, y) := \frac{\partial^2}{\partial t^2} G(x, y, t) \bigg|_{t=0} = \ddot{u} \mathcal{E}(x)
$$

$$
= \left\langle \chi \frac{\partial}{\partial \nu} G(\cdot, x), \frac{\partial}{\partial \nu} G(\cdot, y) \right\rangle_{1/2, \partial \Omega} - 2(\nabla \delta G(\cdot, x), \nabla \delta G(\cdot, y))\Omega
$$

$$
\chi := \delta^2 \rho - ((S \cdot \nabla) S) \cdot \nu - (\delta \rho)^2 (\nabla \cdot \nu) - (S \cdot \nabla) \delta \rho + \frac{\partial (\delta \rho)^2}{\partial \nu},
$$

where $\delta \rho = S \cdot \nu$, $\delta^2 \rho := R \cdot \nu$ and $\nabla \cdot \nu$ is the mean curvature of $\partial \Omega$, that is, $\nabla \cdot \nu = \sum_{i=1}^{n-1} \kappa_i$.

Corollary 3 (Garabedian-Schiffer formula)

Let $\partial \Omega$ be of $C^{2,1}$-class. If $T_t$ is a normal perturbation, we have

$$\delta^2 G(x, y) = - \left\langle (\delta \rho)^2 (\nabla \cdot \nu) \frac{\partial}{\partial \nu_w} G(\cdot, x), \frac{\partial}{\partial \nu_w} G(\cdot, y) \right\rangle_{1/2, \partial \Omega}$$

$$- 2(\nabla \delta G(\cdot, x), \nabla \delta G(\cdot, y))_{\Omega}.$$

**Corollary 4** Let $\partial \Omega$ be of $C^{2,1}$-class. If $T_t$ is a dynamical perturbation, we have

$$\delta^2 G(x, y) := \left. \frac{\partial^2}{\partial t^2} G(x, y, t) \right|_{t=0} = \bar{u} \varepsilon(x)$$

$$= \left\langle \sigma \frac{\partial}{\partial \nu} G(\cdot, x), \frac{\partial}{\partial \nu} G(\cdot, y) \right\rangle_{1/2, \partial \Omega} - 2(\nabla \delta G(\cdot, x), \nabla \delta G(\cdot, y))$$

$$\sigma := -((\delta \rho)^2 (\nabla \cdot \nu) - (S \cdot \nabla) \delta \rho + \frac{\partial((\delta \rho)^2}{\partial \nu}.$$ 

$$\delta^2 \rho = R \cdot \nu, R = (S \cdot \nabla)S.$$ 

$$\chi := \delta^2 \rho - ((S \cdot \nabla)S) \cdot \nu - (\delta \rho)^2 (\nabla \cdot \nu) - (S \cdot \nabla) \delta \rho + \frac{\partial((\delta \rho)^2}{\partial \nu}.$$
The Dam Problem

\(\mathcal{D}:\) the region of the dam.
\(\Omega: \) the portion of water flow in \(\mathcal{D}\).
Then, the boundary $\partial \Omega$ of $\Omega$ consists of four parts:

- $\Gamma_1 = S_1$ (the impervious part)
- $\Gamma_2 \subset DAM$ (the free boundary)
- $\Gamma_3 = S_3$ (the part in contact with water reservoirs)
- $\Gamma_4 \subset S_2$ (the part in contact with air)

$R_j, (j = 1, 2)$: two disjoint reservoirs.
$h_j (h_1 > h_2)$: the height of the water in $R_j, (j = 1, 2)$.
The Dam (Filtration) Problem

Find the flow region $\Omega \subset D$ and the potential function $u$ defined on $\Omega$ which satisfies

\[
\Delta u = 0 \quad \text{in } \Omega,
\]
\[
u = x_2 \quad \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,
\]
\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2,
\]
\[
\frac{\partial u}{\partial \nu} \leq 0 \quad \text{on } \Gamma_4,
\]

where $\nu := (\nu_1, \nu_2)$ is the unit outer normal vector of $\partial \Omega$.

Note that on the free boundary $\Gamma_2$, both Dirichlet’s and Neumann’s conditions are imposed.
Trial (Fictitious) Free Boundary Methods

\[ \Omega^{(0)}: \text{an initial guess,} \quad u^{(0)}: \text{solution of} \]

\[ \Delta u^{(0)} = 0 \quad \text{in} \quad \Omega^{(0)} \quad \text{with, say,} \quad \frac{\partial u^{(0)}}{\partial \nu} = 0 \quad \text{on the F.B.} \]

In general, \( u^{(0)} \neq x_2 \) on the F.B.

From the computed data, define \( \Omega^{(1)} \) and compute the solution \( u^{(1)} \) of ....

To obtain a better understanding on TFBM, we need to have “Calculus of Variation of the Dam Problem” w.r.t. domain transformations.
Variational Principle for the Dam Problem


Suppose that we have a set of admissible domains (candidates of solution) of the dam problem. We denote it by $\mathcal{A}_D$. Let $\Omega \in \mathcal{A}_D$. Consider the functions $u_\Omega, w_\Omega$ which satisfy

$$
\Delta u_\Omega = \Delta w_\Omega = 0 \quad \text{in } \Omega,
$$

$$
u u_\Omega = x_2 \quad \text{on } \Gamma_2, \quad \frac{\partial w_\Omega}{\partial \nu} = 0 \quad \text{on } \Gamma_2.
$$

We introduce a variational principle of the dam problem defined by the difference between $u_\Omega$ and $w_\Omega$.
Let $D_\Omega(v)$ be the Dirichlet integral defined by

$$D_\Omega(v) := \frac{1}{2} \int_\Omega |\nabla v|^2, \quad \text{for } v \in H^1(\Omega).$$

Define the domain functionals $a : A_D \to \mathbb{R}, b : A_D \to \mathbb{R}$ by

$$a(\Omega) := D_\Omega(u_\Omega), \quad b(\Omega) := D_\Omega(w_\Omega).$$

Note that, since $A(\Omega) \subset B(\Omega)$, we have

$$J(\Omega) := a(\Omega) - b(\Omega) \geq 0.$$ 

Moreover, we have

$$J(\Omega) = 0 \iff \Omega \text{ is the sol. of the Dam Problem.}$$
The first variation of $a(\Omega)$

Consider domain perturbations $\mathcal{T}_t$ of $\mathcal{DAM}$ such that $\Omega_t := \mathcal{T}_t(\Omega) \in \mathcal{A}_\mathcal{D}$ for any sufficiently small $t > 0$. Define the first variation of $a(\Omega)$ by

$$\delta a(\Omega) := \lim_{t \to 0^+} \frac{a(\Omega_t) - a(\Omega)}{t}.$$  

**Lemma**[Suzuki-Tsuchiya]

Let $p_\Omega := u_\Omega - x_2$. Then, we have

$$\delta a(\Omega) = \frac{1}{2} \int_{\Gamma_2 \cup \Gamma_4} \left( 1 - \left( \frac{\partial p_\Omega}{\partial \nu} \right)^2 \right) \delta \rho ds,$$

where $\delta \rho := S \cdot \nu$ is the normal component of $S$. 
The first variation of $b(\Omega)$.

We would also like to obtain the first variation $\delta b(\Omega)$ of $b(\Omega)$ with respect to the domain transformation $T_t$:

$$\delta b(\Omega) := \lim_{t \to 0^+} \frac{b(\Omega_t) - b(\Omega)}{t}$$

The first variation of $b$ is more difficult than that of $a$, because domain transformation may cause changes of the mixed boundary condition.
Note that for $\Omega_t$, $\Gamma_2$ and $\Gamma_4$ depend on $t$. So we denote them by $\Gamma_2^t$ and $\Gamma_4^t$.

Define

$$V_1(\Omega_t) := \{ v \in H^1(\Omega_t) \mid v = 0 \text{ on } \Gamma_3 \cup \Gamma_4^t \}.$$ 

The difficulty here comes from the fact that

$$w \in V_1(\Omega_t) \iff w \circ T_t \in V_1(\Omega)$$

(1)

is not valid in general. If (1) holds for all sufficiently small $t \geq 0$, the perturbation $T_t$ is said to satisfy **NPO condition** (Non-Peeling-Off).
**Lemma** (Suzuki-Tsuchiya)
Suppose that $\mathcal{T}_t$ satisfies the NPO condition. Then, we have

$$\delta b(\Omega) = \frac{1}{2} \int_{\Gamma_2} |\nabla_s w_\Omega|^2 \delta \rho ds$$

where $\delta \rho := S \cdot \nu$ is the normal component of $S$, and $\nabla_s$ stands for the gradient on the tangential space of $\Gamma_2$.

**Theorem** (Suzuki-Tsuchiya)
Suppose that $\mathcal{T}_t$ satisfies the NPO condition. Then, we have

$$\delta J(\Omega) = \frac{1}{2} \int_{\Gamma_2} \left( 1 - \left( \frac{\partial p_\Omega}{\partial \nu} \right)^2 - |\nabla_s w_\Omega|^2 \right) \delta \rho ds.$$

Moreover, $\delta J(\Omega) = 0$ for any sufficiently small $\delta \rho$ if and only if $\Omega \in \mathcal{A}_D$ is the solution of the filtration problem.

TFBM using the Hadamard Variation I

The Steepest Descent Method (\(\Gamma\))

\(\Gamma_2^{(k)}\) : the \(k\)-th guess of the free boundary

\[ FV(x) := 1 - \left( \frac{\partial p_\Omega}{\partial \nu} \right)^2 - \left( \frac{\partial w_\Omega}{\partial s} \right)^2, \quad x \in \Gamma_2^{(k)} \]

A naive iterative scheme is

\[ \Gamma_2^{(k+1)} := \Gamma_2^{(k)} - \epsilon FV(x) \nu, \]

where \(\nu\) is the outer unit normal vector at \(x \in \Gamma_2^{(k)}\) and \(\epsilon\) is a positive constant. However, this iteration dose not work at all.
TFBM using the Hadamard Variation II

**The Traction Method** or **The \( H^1 \) Gradient Method**

Let \( z^{(k)} \in H^1(\Omega^{(k)}) \) be the solution of the following problem:

\[
\begin{align*}
\Delta z^{(k)} &= 0 \quad \text{in } \Omega^{(k)}, \\
z^{(k)} &= 0 \quad \text{on } \Gamma_3 \cup \Gamma_4^{(k)}, \\
\frac{\partial z^{(k)}}{\partial \nu} &= 0 \quad \text{on } \Gamma_1, \\
\frac{\partial z^{(k)}}{\partial \nu} &= FV \quad \text{on } \Gamma_2^{(k)},
\end{align*}
\]

The traction method is to define iteration by

\[
\Gamma_2^{(k+1)} := \Gamma_2^{(k)} - z^{(k)}(x) \nu,
\]

where \( \nu \) is the outer unit normal vector at \( x \in \Gamma_2^{(k)} \).

This method was proposed by Azegami;
H. Azegami, A solution to domain optimization problems (in Japanese),
References

