ラプラシアンのGreen関数のHadamard変分

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数学協働プログラム　工学と現代数学の接点を求めて (1)

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The Green Function

$\Omega \subset \mathbb{R}^n$: a bounded domain with the Liptschitz boundary $\partial\Omega$ ($n \geq 2$).
$f \in L^2(\Omega)$: given.

Consider the following Poisson problem:

$$-\Delta w = f \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

The unique solution of the Poisson problem is written by

$$w(x) = \int_\Omega G(x, y) f(y) \mathrm{d}y,$$

where $G(x, y)$ is the **Green function** of Δ on Ω.
The **Green function** of the above BVP is defined so that

- $-\Delta G(x, y) = \delta(x - y)$, $x, y \in \Omega$. δ is the Dirac's delta function.
- $G(x, y) = 0$, $x \in \partial\Omega$, $y \in \Omega$.

Define the fundamental solution $\Gamma(x)$ of Δ by

$$\Gamma(x) := \begin{cases} -\frac{1}{2\pi}\log|x|, & n=2 \\ \frac{1}{(n-2)\omega_n}|x|^{2-n}, & n \geq 3. \end{cases}$$

Then, we have

$$-\Delta\Gamma(x-y) = \delta(x-y),$$

or, for $w \in C^2(\overline{\Omega})$,

$$\begin{aligned} w(y) = &\int_{\Omega}(-\Delta w(x))\Gamma(x-y)\mathrm{d}x \\ &+ \int_{\partial\Omega}\left[\frac{\partial w}{\partial \nu}(x)\Gamma(x-y) - w(x)\frac{\partial}{\partial \nu_x}\Gamma(x-y)\right]\mathrm{d}s_x. \end{aligned}$$

This formula is called Green's representation formula.

For a harmonic function $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$, we have the Green's formula:

$$0 = \int_\Omega (-\Delta w) u \mathrm{d}x + \int_{\partial\Omega} \left[\frac{\partial w}{\partial \nu} u - w \frac{\partial u}{\partial \nu} \right] \mathrm{d}s.$$

Therefore, we obtain a more general version of Green's representation formula:

$$\begin{aligned} w(y) = &\int_\Omega (-\Delta w)(x)(\Gamma(x-y) + u(x))\mathrm{d}x \\ &+ \int_{\partial\Omega} \Big[\frac{\partial w(x)}{\partial \nu} (\Gamma(x-y) + u(x)) - w(x) \frac{\partial}{\partial \nu_x} (\Gamma(x-y) + u(x)) \Big] \mathrm{d}s_x. \end{aligned}$$

Then, the Green function $G(x,y)$ is defined by

$$G(x,y) := \Gamma(x-y) + K(x,y),$$

where $u(x) = K(x,y)$ is a harmonic function of x for fixed $y \in \Omega$ such that

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = -\Gamma(\cdot - y) \quad \text{on } \partial\Omega.$$

$$w(y) = \int_\Omega (-\Delta w)(x) G(x,y) \mathrm{d}x - \int_{\partial\Omega} w(x) \frac{\partial}{\partial \nu_x} G(x,y) \mathrm{d}s_x.$$

Therefore, the solution of the BVP

$$-\Delta w = f \quad \text{in } \Omega, \qquad w = g \quad \text{on } \partial\Omega$$

is written by

$$w(y) = \int_{\Omega} f(x) G(x,y) \mathrm{d}x - \int_{\partial\Omega} g(x) \frac{\partial}{\partial \nu_x} G(x,y) \mathrm{d}s_x.$$

Domain Purtabations

Let $\widetilde{\Omega} \subset \mathbb{R}^n$ be a sufficiently large domain such that $\overline{\Omega} \subset \widetilde{\Omega}$.
We consider a family of bi-Lipschitz homeomorphisms

$$\mathcal{T}_t(x) : \Omega \to \mathcal{T}_t(\Omega) \subset \widetilde{\Omega}, \qquad \mathrm{supp}(\mathcal{T}_t) \subset \widetilde{\Omega}.$$

We assume that $\overline{\Omega} \ni x \mapsto \mathcal{T}_t(x)$ is twice differentiable with respect to t. We also assume that, for $i = 1, 2$,

$$\frac{\partial^i}{\partial t^i}\mathcal{T}_t(x), \quad \frac{\partial^i}{\partial t^i}\nabla(\mathcal{T}_t(x)), \quad \frac{\partial^i}{\partial t^i}\mathcal{T}_t^{-1}(x), \quad \frac{\partial^i}{\partial t^i}\nabla(\mathcal{T}_t^{-1}(x))$$

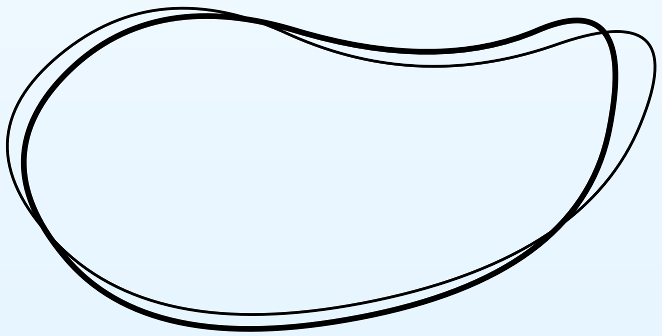
are uniformly bounded on $\overline{\Omega} \times (-\varepsilon, \varepsilon)$ for sufficiently small $\varepsilon > 0$.

Define

$$\mathsf{S} := \frac{\partial \mathcal{T}_t}{\partial t}\Big|_{t=0}, \qquad \mathsf{R} := \frac{\partial^2 \mathcal{T}_t}{\partial t^2}\Big|_{t=0}.$$

Then, $\mathcal{T}_t$ has the Taylor expansion

$$\mathcal{T}_t(x) = x + t\mathsf{S}(x) + \frac{1}{2}t^2\mathsf{R}(x) + o(t^2).$$



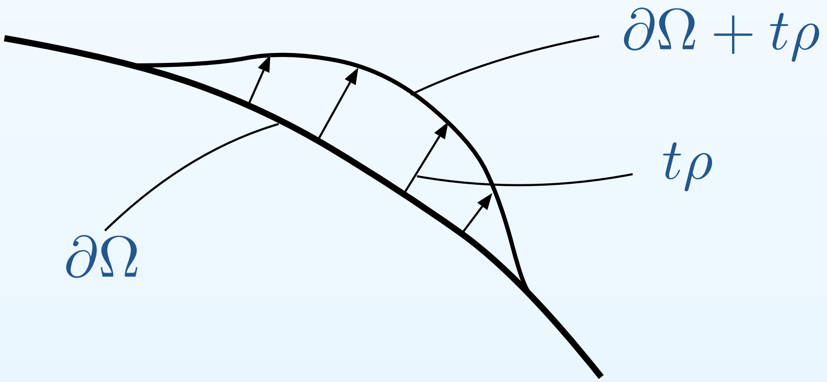
Ω and Ω_t.

Normal Perturbations

Let $\partial\Omega$ be sufficiently smooth and $\rho(x)$ be a smooth function defined on $\partial\Omega$. Then, define

$$\partial\Omega + t\rho : x + t\rho(x)\boldsymbol{\nu}_x, \quad x \in \partial\Omega.$$

The domain Ω_t is the domain with $\partial\Omega_t = \partial\Omega + t\rho$.



Dynamical Perturbations

Let a vector field S be given on $\widetilde{\Omega}$ with $\mathrm{supp}\mathsf{S} \subset \widetilde{\Omega}$.
Then, the domain transformations

$$\mathcal{T}_t(x) : \Omega \to \mathcal{T}_t(\Omega) \subset \mathbb{R}^n$$

are defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_t(x) = \mathsf{S}(\mathcal{T}_t(x)), \quad \mathcal{T}_0(x) = x.$$

In this case, $\mathcal{T}_t(x)$ is called a **dynamical perturbation**.

Note that, for dynamical perturbations, we have

$$(\mathsf{S} \cdot \nabla)\mathsf{S} = \mathsf{R}.$$

Let $G(x,y,t)$: the Green ft. of $-\Delta$ on Ω_t.
Hadamard obtained the derivative of the Green ft. $G(x,y,t)$ with respect $t \geq 0$:

$$\delta G(x,y) := \lim_{t\to 0+} \frac{G(x,y,t) - G(x,y)}{t}, \quad x,y \in \Omega.$$

Theorem 1 (Hadamard's variational formula) *Let $\partial\Omega$ be of $C^{1,1}$-class. For the differentiable perturbation $\mathcal{T}_t$, we have*

$$\delta G(w,y) = \left\langle \delta\rho \frac{\partial}{\partial \boldsymbol{\nu}} G(\cdot, y), \frac{\partial}{\partial \boldsymbol{\nu}} G(\cdot, w) \right\rangle_{1/2,\partial\Omega},$$

where $\delta\rho := \mathsf{S} \cdot \boldsymbol{\nu}$.

$\langle \cdot, \cdot \rangle_{1/2,\partial\Omega}$: the duality pairing of $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

History of the Hadamard Variational Formula

Hadamard, Mémoire sur le probleme d'analyse relatif à l'equilibre des plaques élastiques encastrées, Oeuvres, 2 (1968), 515–631.

Garabedian, Partial Differential Equation, Chelsea,1964.

小沢真, 領域の摂動と固有値問題、「数学」, 33 (1981), 248 –261.

Suzuki, Tsuchiya, First and second Hadamard Variational formulae of the Green function for general domain perturbations, Journal of Mathematical Society of Japan, *to appear*.

Hadamard (1908):
$\partial\Omega$ and $\mathsf{S}(x)$ are of C^ω-class and $\mathcal{T}_t(x)$ is a normal perturbation.

Schiffer (1946), Garabedian-Schiffer (1952-53):
$\partial\Omega$ and $\mathsf{S}(x)$ are of C^k-class with sufficiently large k and $\mathcal{T}_t(x)$ is a normal perturbation.

Suzuki–Tsuchiya:
$\partial\Omega$ and $\mathsf{S}(x)$ are of $C^{1,1}$-class and $\mathcal{T}_t(x)$ is a general perturbation.

Eulerian Derivative and Lagrangian Derivative

$\Omega \subset \mathbb{R}^n$: $C^{k,1}$ domain, $k = 1, 2$.

$\Omega_t := \mathcal{T}_t(\Omega)$, $t \geq 0$.

φ : a function defined in the nbd of Ω.

Consider the Dirichlet problem:

$$\Delta u(\cdot, t) = 0 \text{ in } \Omega_t, \qquad u(\cdot, t) = \varphi \text{ on } \partial\Omega_t.$$

Now, we consider the two kinds of derivatives of $u(\cdot, t)$ with respect of t.

$$\dot{u}_{\mathcal{L}}(x) := \frac{\mathrm{d}}{\mathrm{d}t}\left(u(\mathcal{T}_t(x), t)\right)\Big|_{t=0}, \quad \ddot{u}_{\mathcal{L}}(x) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(u(\mathcal{T}_t(x), t)\right)\Big|_{t=0}$$

$$\dot{u}_{\mathcal{E}}(x) := \frac{\partial}{\partial t}u(x, t)\Big|_{t=0}, \quad \ddot{u}_{\mathcal{E}}(x) := \frac{\partial^2}{\partial t^2}u(x, t)\Big|_{t=0}.$$

$\dot{u}_{\mathcal{L}}$ and $\ddot{u}_{\mathcal{L}}$ are called the Lagrangian derivatives,
$\dot{u}_{\mathcal{E}}$ and $\ddot{u}_{\mathcal{E}}$ are called the Eulerian derivatives.

Eulerian Derivative

Differentiating $u(\mathcal{T}_t(x), t) = \varphi(\mathcal{T}_t(x))$ with respect to t and letting $t \to 0+$, we have

$$\dot{u}_{\mathcal{E}} + \mathsf{S} \cdot \nabla u = \mathsf{S} \cdot \nabla \varphi \qquad \text{on } \partial\Omega.$$

Therefore, $\dot{u}_{\mathcal{E}}$ satisfies

$$\begin{aligned} \Delta \dot{u}_{\mathcal{E}} &= 0 \quad \text{in } \Omega, \\ \dot{u}_{\mathcal{E}} &= \mathsf{S} \cdot (\nabla \varphi - \nabla u) \quad \text{on } \partial\Omega. \end{aligned}$$

Let $\Gamma(x)$ be the fundamental solution of the Laplacian. Then, the Green function $G(x,y)$ is defined by

$$G(x,y) := \Gamma(x-y) + K(x,y),$$

where $u(x) = K(x,y)$ is a harmonic function of x for fixed $y \in \Omega$ such that

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = -\Gamma(\cdot - y) \quad \text{on } \partial\Omega.$$

$G(x,y,t)$: the Green function of Ω_t:

$$G(x,y,t) = \Gamma(x-y) + K(x,y,t),$$

where $u(x,t) = K(x,y,t)$ is a harmonic function such that

$$\Delta u(\cdot,t) = 0 \quad x \in \Omega_t, \qquad u(\cdot,t) = -\Gamma(\cdot - y) \quad x \in \partial\Omega_t.$$

Therefore, the Eulerian derivative $\dot{u}_{\mathcal{E}}$ of u satisfies

$$\begin{aligned}\Delta \dot{u}_{\mathcal{E}} &= 0 \quad \text{in } \Omega \\ \dot{u}_{\mathcal{E}} &= \mathsf{S} \cdot \nabla_x(-\Gamma(x-y) - u(x)) = -\mathsf{S} \cdot \nabla_x G(x,y) \\ &= -(\mathsf{S} \cdot \boldsymbol{\nu}) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,y) \quad \text{on } \partial\Omega.\end{aligned}$$

The solution of the BVP

$$\Delta v = 0 \quad \text{in } \Omega, \qquad v = g \quad \text{on } \partial\Omega$$

is written by

$$v(w) = -\left\langle g, \frac{\partial}{\partial \boldsymbol{\nu}_x} G(\cdot, w) \right\rangle_{1/2,\partial\Omega}.$$

Therefore, we have

$$\dot{u}_{\mathcal{E}}(w) = \left\langle (\mathsf{S}\cdot\boldsymbol{\nu})\frac{\partial}{\partial\boldsymbol{\nu}}G(\cdot,y), \frac{\partial}{\partial\boldsymbol{\nu}}G(\cdot,w)\right\rangle_{1/2,\partial\Omega}.$$

$$\left(\dot{u}_{\mathcal{E}}(w) = \int_{\partial\Omega}\frac{\partial}{\partial\boldsymbol{\nu}_x}G(x,w)\frac{\partial}{\partial\boldsymbol{\nu}_x}G(x,y)(\mathsf{S}\cdot\boldsymbol{\nu})\mathrm{d}s_x\right)$$

The first variation $\delta G(x,y)$ of the Green function is written by, for $w,y\in\Omega$

$$\begin{aligned}\delta G(w,y) &:= \lim_{t\to 0+}\frac{G(w,y,t)-G(w,y)}{t} = \dot{u}_{\mathcal{E}}(w)\\ &= \left\langle (\mathsf{S}\cdot\boldsymbol{\nu})\frac{\partial}{\partial\boldsymbol{\nu}}G(\cdot,y), \frac{\partial}{\partial\boldsymbol{\nu}}G(\cdot,w)\right\rangle_{1/2,\partial\Omega}.\end{aligned}$$

Hadamard's Variational Formula — the Second Variation

Consider the boundary value problem $\Delta u = 0$ in Ω, $u = \varphi$ on $\partial\Omega$. Then, the solution's Eularian derivative $\dot{u}_{\mathcal{E}}$ satisfies

$$\Delta \dot{u}_{\mathcal{E}} = 0 \text{ in } \Omega, \qquad \dot{u}_{\mathcal{E}} = \mathsf{S} \cdot (\nabla\varphi - \nabla u) \text{ on } \partial\Omega.$$

Also, $\ddot{u}_{\mathcal{E}}$ satisfies

$$\begin{aligned} \Delta \ddot{u}_{\mathcal{E}} &= 0 \quad \text{in } \Omega, \\ \ddot{u}_{\mathcal{E}} &= -2\mathsf{S} \cdot \nabla \dot{u}_{\mathcal{E}} + \mathsf{R} \cdot (\nabla\varphi - \nabla u) + (\mathcal{H}_x\varphi - \mathcal{H}_x u) \cdot (\mathsf{S})^2 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\mathcal{H}_x\varphi$ is the Hesse matrix of φ.

On the case of the Green function, $\varphi = -\Gamma$.

Recall that the Green function G is defined by

$$G(x,y) := \Gamma(x-y) + u(x).$$

The harmonic function u satisfies $\Delta u = 0$ in Ω, $u = -\Gamma$ on $\partial\Omega$. The second variation of the Green function is defined by

$$\delta^2 G(x,y) := \frac{\partial^2}{\partial t^2} G(x,y,t)\Big|_{t=0} = \ddot{u}_{\mathcal{E}}(x).$$

The function $\ddot{u}_{\mathcal{E}}$ is written by

$$\begin{aligned}\delta^2 G(x,y) = \ddot{u}_{\mathcal{E}}(x) =& 2\int_{\partial\Omega} \mathsf{S}\cdot\nabla_w \delta G(w,y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,w)\mathrm{d}s_w \\ &+ \int_{\partial\Omega} \mathsf{R}\cdot\nabla_w G(w,y) \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,w)\mathrm{d}s_w \\ &+ \int_{\partial\Omega} \mathcal{H}_w G(w,y)\cdot(\mathsf{S})^2 \frac{\partial}{\partial \boldsymbol{\nu}_x} G(x,w)\mathrm{d}s_w.\end{aligned}$$

Theorem 2 (Hadamard's second variational formula)
Let $\partial\Omega$ be of $C^{2,1}$-class. For the twice differentiable perturbation $\mathcal{T}_t$, we have

$$\delta^2 G(x,y) := \frac{\partial^2}{\partial t^2} G(x,y,t)\Big|_{t=0} = \ddot{u}_{\mathcal{E}}(x)$$
$$= \left\langle \chi \frac{\partial}{\partial \boldsymbol{\nu}} G(\cdot,x), \frac{\partial}{\partial \boldsymbol{\nu}} G(\cdot,y) \right\rangle_{1/2,\partial\Omega} - 2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y))_\Omega$$
$$\chi := \delta^2\rho - ((\mathsf{S}\cdot\nabla)\mathsf{S})\cdot\boldsymbol{\nu} - (\delta\rho)^2(\nabla\cdot\boldsymbol{\nu}) - (\mathsf{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial\boldsymbol{\nu}},$$

where $\delta\rho = \mathsf{S}\cdot\boldsymbol{\nu}$, $\delta^2\rho := \mathsf{R}\cdot\boldsymbol{\nu}$ and $\nabla\cdot\boldsymbol{\nu}$ is the ***mean curvature*** *of $\partial\Omega$, that is, $\nabla\cdot\boldsymbol{\nu} = \sum_{i=1}^{n-1}\kappa_i$.*

Suzuki, Tsuchiya, First and second Hadamard Variational formulae of the Green function for general domain perturbations, Journal of Mathematical Society of Japan, *to appear*.

Corollary 3 (Garabedian-Schiffer formula)
Let $\partial\Omega$ be of $C^{2,1}$-class. If $\mathcal{T}_t$ is a normal perturbation, we have

$$\delta^2 G(x,y) = -\left\langle (\delta\rho)^2 (\nabla\cdot\boldsymbol{\nu}) \frac{\partial}{\partial \boldsymbol{\nu}_w} G(\cdot,x), \frac{\partial}{\partial \boldsymbol{\nu}_w} G(\cdot,y) \right\rangle_{1/2,\partial\Omega} - 2(\nabla\delta G(\cdot,x), \nabla\delta G(\cdot,y))_\Omega.$$

Garabedian, Schiffer, Convexity of domain functionals,
J. Anal. Math., 2 (1952-53) 281–368.

Corollary 4 *Let $\partial\Omega$ be of $C^{2,1}$-class. If $\mathcal{T}_t$ is a dynamical perturbation, we have*

$$\delta^2 G(x,y) := \frac{\partial^2}{\partial t^2} G(x,y,t)\Big|_{t=0} = \ddot{u}_{\mathcal{E}}(x)$$

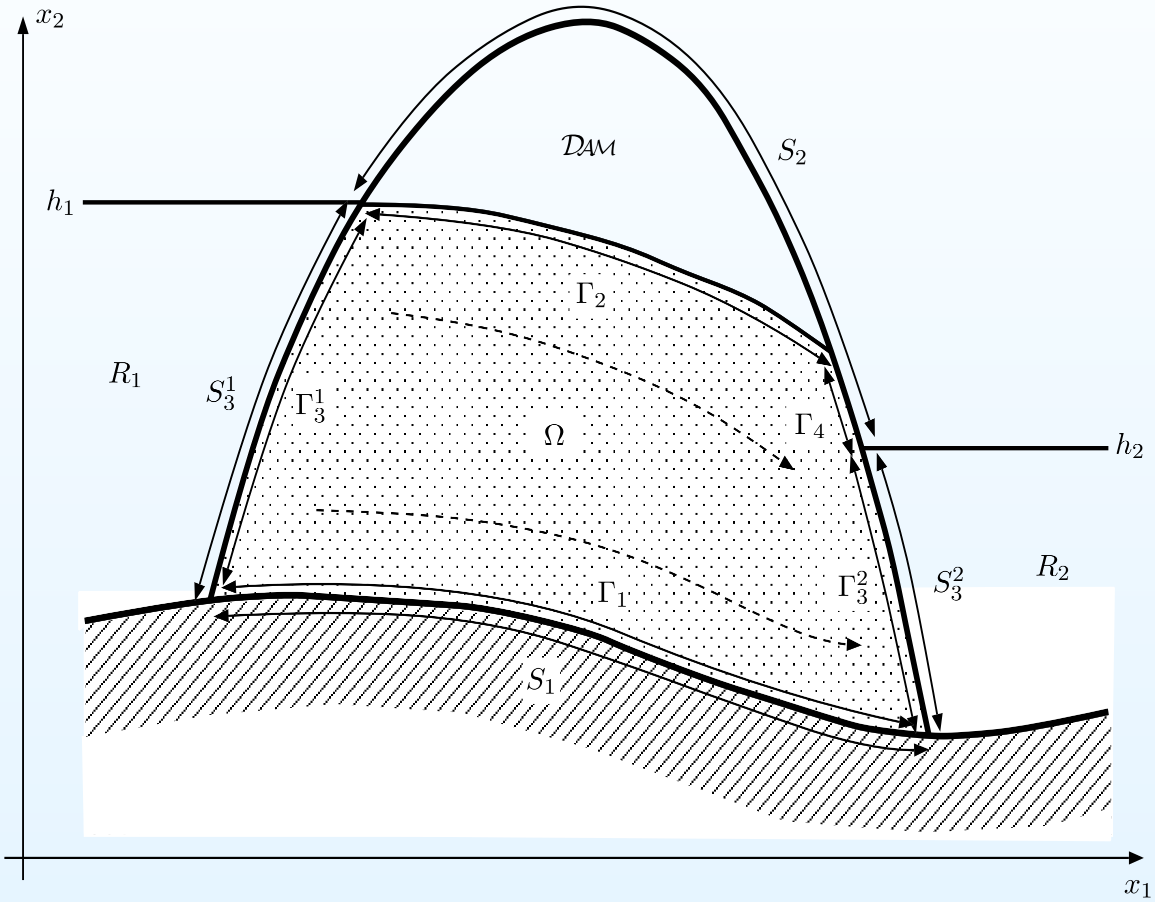
$$= \left\langle \sigma \frac{\partial}{\partial \boldsymbol{\nu}} G(\cdot,x), \frac{\partial}{\partial \boldsymbol{\nu}} G(\cdot,y) \right\rangle_{1/2,\partial\Omega} - 2(\nabla \delta G(\cdot,x), \nabla \delta G(\cdot,y))_\Omega$$

$$\sigma := -(\delta\rho)^2(\nabla \cdot \boldsymbol{\nu}) - (\mathsf{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial \boldsymbol{\nu}}.$$

$\delta^2\rho = \mathsf{R}\cdot\boldsymbol{\nu}$, $\mathsf{R} = (\mathsf{S}\cdot\nabla)\mathsf{S}$.

$$\chi := \delta^2\rho - ((\mathsf{S}\cdot\nabla)\mathsf{S})\cdot\boldsymbol{\nu} - (\delta\rho)^2(\nabla\cdot\boldsymbol{\nu}) - (\mathsf{S}\cdot\nabla)\delta\rho + \frac{\partial(\delta\rho)^2}{\partial\boldsymbol{\nu}}.$$

The Dam Problem



$\mathcal{DAM}$: the region of the dam.

Ω: the portion of water flow in $\mathcal{DAM}$.

Then, the boundary $\partial\Omega$ of Ω consists of four parts:

$$\begin{aligned}
\Gamma_1 &= S_1 && \text{(the impervious part)} \\
\Gamma_2 &\subset \mathcal{DAM} && \text{(the free boundary)} \\
\Gamma_3 &= S_3 && \text{(the part in contact with water reservoirs)} \\
\Gamma_4 &\subset S_2 && \text{(the part in contact with air)}
\end{aligned}$$

R_j, $(j = 1, 2)$: two disjoint reservoirs.
h_j ($h_1 > h_2$): the height of the water in R_j, $(j = 1, 2)$.

The Dam (Filtration) Problem

Find the flow region $\Omega \subset \mathcal{DAM}$ and the potential function u defined on Ω which satisfies

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ u &= x_2 && \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\ \frac{\partial u}{\partial \boldsymbol{\nu}} &= 0 && \text{on } \Gamma_1 \cup \Gamma_2, \\ \frac{\partial u}{\partial \boldsymbol{\nu}} &\leq 0 && \text{on } \Gamma_4, \end{aligned}$$

where $\boldsymbol{\nu} := (\nu_1, \nu_2)$ is the unit outer normal vector of $\partial\Omega$.

Note that on the free boundary Γ_2, both Dirichlet's and Neumann's conditions are imposed.

Trial (Fictitious) Free Boundary Methods

$\Omega^{(0)}$: an initial guess, $u^{(0)}$: solution of

$$\Delta u^{(0)} = 0 \quad \text{in } \Omega^{(0)} \quad \text{with, say, } \frac{\partial u^{(0)}}{\partial \boldsymbol{\nu}} = 0 \text{ on the F.B.}$$

In general, $u^{(0)} \neq x_2$ on the F.B.

From the computed data, define $\Omega^{(1)}$ and compute the solution $u^{(1)}$ of ....

To obtain a better understanding on TFBM, we need to have “**Calculus of Variation of the Dam Problem**” w.r.t. *domain transformations*.

Variational Principle for the Dam Problem

Suzuki, Tsuchiya, Convergence analysis of trial free boundary methods for the two-dimensional filtration problem, Numerische Mathematik, 100 (2005) 537–564.

Suppose that we have a set of **admissible domains** (candidates of solution) of the dam problem. We denote it by $\mathcal{A}_{\mathcal{D}}$. Let $\Omega \in \mathcal{A}_{\mathcal{D}}$. Consider the functions u_Ω, w_Ω which satisfy

$$\Delta u_\Omega = \Delta w_\Omega = 0 \quad \text{in } \Omega,$$

$$u_\Omega = x_2 \quad \text{on } \Gamma_2, \qquad \frac{\partial w_\Omega}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \Gamma_2.$$

We introduce a variational principle of the dam problem defined by the difference between u_Ω and w_Ω.

Let $D_\Omega(v)$ be the Dirichlet integral defined by

$$D_\Omega(v) := \frac{1}{2}\int_\Omega |\nabla v|^2, \qquad \text{for } v \in H^1(\Omega).$$

Define the *domain functionals* $a : \mathcal{A}_\mathcal{D} \to \mathbb{R}$, $b : \mathcal{A}_\mathcal{D} \to \mathbb{R}$ by

$$a(\Omega) := D_\Omega(u_\Omega), \qquad b(\Omega) := D_\Omega(w_\Omega).$$

Note that, since $A(\Omega) \subset B(\Omega)$, we have

$$J(\Omega) := a(\Omega) - b(\Omega) \geq 0.$$

Moreover, we have

$$J(\Omega) = 0 \iff \Omega \text{ is the sol. of the Dam Problem.}$$

The first variation of $a(\Omega)$

Consider domain perturbations $\mathcal{T}_t$ of $\mathcal{DAM}$ such that $\Omega_t := \mathcal{T}_t(\Omega) \in \mathcal{A}_{\mathcal{D}}$ for any sufficiently small $t > 0$. Define the first variation of $a(\Omega)$ by

$$\delta a(\Omega) := \lim_{t \to 0+} \frac{a(\Omega_t) - a(\Omega)}{t}.$$

Lemma[Suzuki-Tsuchiya]

Let $p_\Omega := u_\Omega - x_2$. Then, we have

$$\delta a(\Omega) = \frac{1}{2} \int_{\Gamma_2 \cup \Gamma_4} \left(1 - \left(\frac{\partial p_\Omega}{\partial \boldsymbol{\nu}} \right)^2 \right) \delta \rho \, \mathrm{d}s,$$

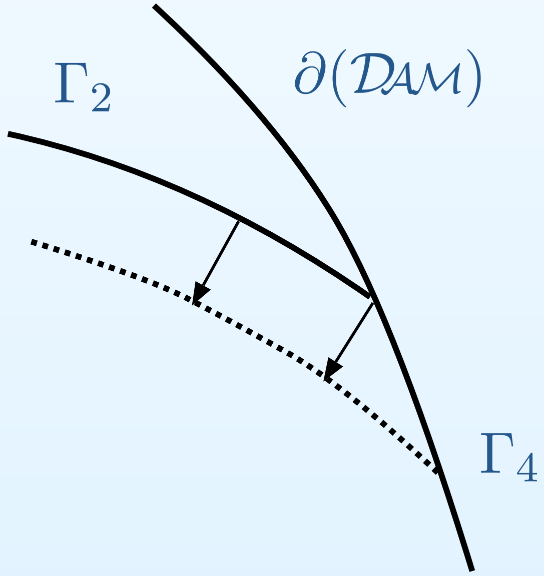
where $\delta \rho := \mathsf{S} \cdot \boldsymbol{\nu}$ is the normal component of S.

The first variation of $b(\Omega)$.

We would also like to obtain the **first variation** $\delta b(\Omega)$ of $b(\Omega)$ with respect to the domain transformation $\mathcal{T}_t$:

$$\delta b(\Omega) := \lim_{t \to 0+} \frac{b(\Omega_t) - b(\Omega)}{t}$$

The first variation of b is more difficult than thta of a, because domain transformation may cause changes of the mixed boundary condition.



Note that for Ω_t, Γ_2 and Γ_4 depend on t. So we denote them by Γ_2^t and Γ_4^t. Define

$$V_1(\Omega_t) := \left\{ v \in H^1(\Omega_t) \;\middle|\; v = 0 \text{ on } \Gamma_3 \cup \Gamma_4^t \right\}.$$

The difficulty here comes from the fact that

$$w \in V_1(\Omega_t) \Longleftrightarrow w \circ \mathcal{T}_t \in V_1(\Omega) \tag{1}$$

is not valid in general. If (1) holds for all sufficiently small $t \geq 0$, the perturbation $\mathcal{T}_t$ is said to satisfy **NPO condition** (Non-Peeling-Off).

Lemma (Suzuki-Tsuchiya)

Suppose that $\mathcal{T}_t$ satisfies the NPO condition. Then, we have

$$\delta b(\Omega) = \frac{1}{2}\int_{\Gamma_2} |\nabla_s w_\Omega|^2 \delta\rho \mathrm{d}s$$

where $\delta\rho := \mathsf{S}\cdot\boldsymbol{\nu}$ is the normal component of S, and ∇_s stands for the gradient on the tangential space of Γ_2.

Theorem (Suzuki-Tsuchiya)

Suppose that $\mathcal{T}_t$ satisfies the NPO condition. Then, we have

$$\delta J(\Omega) = \frac{1}{2}\int_{\Gamma_2} \left(1 - \left(\frac{\partial p_\Omega}{\partial \boldsymbol{\nu}}\right)^2 - |\nabla_s w_\Omega|^2\right) \delta\rho \mathrm{d}s.$$

Moreover, $\delta J(\Omega) = 0$ for any sufficiently small $\delta\rho$ if and only if $\Omega \in \mathcal{A}_{\mathcal{D}}$ is the solution of the filtration problem.

Suzuki, Tsuchiya, Weak formulation of Hadamard variation applied to the filtration problem, Japan J. Indust. Appl. Math., 28 (2011) 327–350.

TFBM using the Hadamard Variation I

The Steepest Descent Method (?)

$\Gamma_2^{(k)}$: the k-th geuss of the free boundary

$FV(x) := 1 - \left(\dfrac{\partial p_\Omega}{\partial \boldsymbol{\nu}}\right)^2 - \left(\dfrac{\partial w_\Omega}{\partial s}\right)^2$, $x \in \Gamma_2^{(k)}$

A naive iterative scheme is

$$\Gamma_2^{(k+1)} := \Gamma_2^{(k)} - \epsilon FV(x)\boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ is the outer unit normal vector at $x \in \Gamma_2^{(k)}$ and ϵ is a positive constant.
However, this iteration dose not work at all.

TFBM using the Hadamard Variation II

The Traction Method or **The H^1 Gradient Method**

Let $z^{(k)} \in H^1(\Omega^{(k)})$ be the solution of the following problem:

$$\Delta z^{(k)} = 0 \quad \text{in } \Omega^{(k)}, \qquad z^{(k)} = 0 \quad \text{on } \Gamma_3 \cup \Gamma_4^{(k)},$$
$$\frac{\partial z^{(k)}}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \Gamma_1, \qquad \frac{\partial z^{(k)}}{\partial \boldsymbol{\nu}} = FV \quad \text{on } \Gamma_2^{(k)},$$

The traction method is to define iteration by

$$\Gamma_2^{(k+1)} := \Gamma_2^{(k)} - z^{(k)}(x)\boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ is the outer unit normal vector at $x \in \Gamma_2^{(k)}$.
This method was proposed by Azegami;
H. Azegami, A solution to domain optimization problems (in Japanese), Trans. of Japan Society of Mech Engs., Ser. A, 60 (1994) 1479–1486

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