

Semilinear parabolic equation on bounded domain with critical Sobolev exponent

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The semilinear parabolic equation

$$u_t - \Delta u = |u|^{p-1} u \quad \text{in } \Omega \times (0, T), \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) \quad (1)$$

was introduced pure mathematically, but its profiles of the solution, particularly blowup in finite and infinite time, have suggested several principles in mathematical science, and the tools developed to approach them have gained much generality because of a variety of its mathematical backgrounds; comparison principle, scaling property, and variational structure, where $\Omega \subset \mathbf{R}^n$ is bounded domain with smooth boundary $\partial\Omega$ and $p > 1$.

To (1), there is a local in time unique classical solution $u = u(\cdot, t)$, if $u_0 = u_0(x) \in C(\overline{\Omega})$ for instance, and henceforth $T = T_{\max} \in (0, +\infty]$ denotes its existence time. It is well-known that

$$\begin{aligned} T < +\infty & \Rightarrow \lim_{t \uparrow T} \|u(t)\|_{\infty} = +\infty \\ T = +\infty & \Rightarrow \|u(t)\|_2 \leq C, \quad J(u(t)) \geq 0, \end{aligned}$$

where $J(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1}$ acts as the Lyapunov function:

$$\frac{d}{dt} J(u(t)) = -\|u_t\|_2^2.$$

Up to now, three important categories have been noticed in connection with the blowup rate and the uniform boundedness of the solution.

1. Sub-critical case, $1 < p < p_s$. We obtain

$$\begin{aligned} T = +\infty & \Rightarrow \|u(t)\|_{\infty} \leq C \\ T < +\infty, \quad \Omega: \text{convex} & \Rightarrow \lim_{t \uparrow T} J(u(t)) = -\infty, \quad \text{type I}, \end{aligned}$$

where type I indicates $\limsup_{t \uparrow T} (T - t)^{\frac{1}{p-1}} \|u(t)\|_{\infty} < +\infty$ and $p_s = \begin{cases} +\infty & (n = 1, 2) \\ \frac{n+2}{n-2} & (n \geq 3). \end{cases}$

2. Monotone increasing case, $u_0 \geq 0$, $u_0 \not\equiv 0$, $-\Delta u_0 \leq u_0^p$, $-\Delta u_0 \not\equiv u_0^p$. We obtain $T < +\infty$ and type I.

3. Radially symmetric case, $\Omega = B = B(0, 1)$, $u_0(x) = U_0(|x|)$. We obtain

$$\begin{aligned} T < +\infty, \quad 1 < p < p^* & \Rightarrow \text{type I} \\ p > p^* & \Rightarrow \exists \text{ type II}, \end{aligned}$$

where type II means $\limsup_{t \uparrow T} (T - t)^{\frac{1}{p-1}} \|u(t)\|_{\infty} = +\infty$ and

$$p^* = \begin{cases} +\infty & (n \leq 10) \\ 1 + \frac{4}{n-4-2\sqrt{n-1}} & (n \geq 11). \end{cases}$$

The blowup rate is related to the problem of complete blowup of the solution, which means, roughly, $u(x, t) = +\infty$ for $x \in \Omega$, $t > T$. More precisely, given $0 \leq \psi \in C(\overline{\Omega})$ with $\psi \not\equiv 0$, we take $u_0 = \lambda\psi$, where $\lambda > 0$ is a constant. Then, it holds that

$$\lambda_0 = \sup \left\{ \lambda > 0 \mid T = +\infty, \quad \lim_{t \uparrow +\infty} \|u(t)\|_{\infty} = 0 \right\} \in (0, \infty). \quad (2)$$

For $\lambda = \lambda_0$, Ni, Sacks, and Tavantzis [7] confirmed the existence of a global in time weak solution to (1) such that $u \in C([0, \infty), L^1(\Omega, \delta(x)dx))$, where $\delta(x) = \text{dist}(x, \partial\Omega)$, and henceforth we call it the NST solution. This NST solution satisfies $T = +\infty$ if either $1 < p < \frac{n+2}{n}$ and Ω is convex or $1 < p < \frac{n+2}{n-2}$ and the solution is radially symmetric. Consequently, it is uniformly bounded in these cases. When Ω is star-shaped, $n \geq 3$, and $p \geq \frac{n+2}{n-2}$, on the other hand, the NST solution cannot be uniformly bounded globally in time, because there is no non-trivial stationary solution. Thus, the NST solution blows-up in finite or infinite time, and the first case of the NST solution, $T < +\infty$, is not consistent to the complete blowup. Later, Galaktionov and Vazquez [2] studied these alternatives in detail. First, the blowup in finite time is always complete in the case of $p = \frac{n+2}{n-2}$ and the soliton is radially symmetric, and consequently the NST solution blows-up in infinite time if

$$\Omega = B, \quad \psi(x) = \Psi(|x|) \geq 0.$$

Next, the NST solution always blows-up in finite time in the case of $\frac{n+2}{n-2} < p < 1 + \frac{6}{(n-10)_+}$ and

$$\Omega = B, \quad \psi(x) = \Psi(|x|) \geq 0, \quad \Psi_r(r) < 0 \quad (0 < r \leq 1).$$

This talk is concerned with the critical Sobolev exponent $p = \frac{n+2}{n-2}$, while the solution is not necessarily radially symmetric. The exponent $p = \frac{n+2}{n-2}$ for $n \geq 3$ is due to Sobolev's imbedding $H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$, which results in the following structures:

1. There is energy quantization in the non-compact Palais-Smale sequence to $J(u)$ ([3]).
2. Problem (1) is well-posed in $X = H_0^1$ or $X = L^{p+1}$, but the existence time T is not estimated below by $\|u_0\|_X$, which means the ill-posedness concerning the weak topology ([3]).

Under the presense of the above described critical structures of variation, the global in time behavior of the solution is seriously involved by its energy level in this case. To describe details, we introduce We take, furthermore,

$$\begin{aligned} I(v) &= \|\nabla v\|_2^2 - \|v\|_{p+1}^{p+1} \\ d &= \inf \left\{ \sup_{\lambda > 0} J(\lambda v) \mid v \in H_0^1(\Omega) \setminus \{0\} \right\} && \text{potential depth} \\ S &= \inf \left\{ \|\nabla v\|_2^2 \mid v \in H_0^1(\Omega), \|v\|_{p+1} = 1 \right\} && \text{Sobolev constant} \\ W &= \{v \in H_0^1(\Omega) \mid J(v) < d, I(v) > 0\} \cup \{0\} && \text{stable set} \\ V &= \{v \in H_0^1(\Omega) \mid J(v) < d, I(v) < 0\} && \text{unstable set,} \end{aligned}$$

where $S > 0$ depends only on n and it holds that $d = \frac{1}{n}S^{n/2}$. The stable and unstable sets W and V are invariant with respect to (1). Then, it holds that

$$\begin{aligned} u(t_0) \in W, \quad (0 \leq t_0 < T) &\Rightarrow T = +\infty, \quad \|u(t)\|_\infty \leq C \\ u(t_0) \in V, \quad (0 \leq t_0 < T) &\Rightarrow T < +\infty, \end{aligned}$$

and this implies the following theorem.

Theorem 1 *Let Ω be star-shaped, $n \geq 3$, $p = \frac{n+2}{n-2}$, and $u_0 \geq 0$. Then we obtain the following alternatives for the solution $u = u(\cdot, t)$ to (1):*

1. $T < +\infty$ and $\lim_{t \uparrow T} \|u(t)\|_\infty = +\infty$.
2. $T = +\infty$ and $\lim_{t \uparrow +\infty} \|u(t)\|_\infty = 0$.
3. $T = +\infty$ and $\lim_{t \uparrow +\infty} \|u(t)\|_\infty = +\infty$.

The third case of the above theorem is called the blowup in infinite time, which is equivalent to $T = +\infty$ and $u(t) \notin W \cup V$ for $t \geq 0$. The second theorem is concerned with the NST solution. Thus, given $0 \leq \psi \in C(\bar{\Omega})$ with $\psi \not\equiv 0$, we take $u_0 = \lambda\psi$ and define $\lambda_0 \in (0, \infty)$ by (2).

Theorem 2 *Let Ω be star-shaped, $n \geq 3$, and $p = \frac{n+2}{n-2}$. Then we have the following cases:*

1. *If $0 < \lambda < \lambda_0$, then $T = +\infty$ and $\lim_{t \uparrow \infty} \|u(t)\|_\infty = 0$.*
2. *If $\lambda = \lambda_0$, then $\lim_{t \uparrow T} J(u(t)) \geq d$, and the solution blows-up in finite or infinite time.*
3. *If $\lambda > \lambda_0$, then $T < +\infty$.*

To prove the third case of the above theorem, we use Struwe's energy quantization principle [8]. The next theorem describes the relation between the blowup rate and the energy level.

Theorem 3 *Let Ω be bounded, $n \geq 3$, $p = \frac{n+2}{n-2}$, $u_0 \geq 0$, and $T < +\infty$. Then, we have the following alternatives.*

1. $\lim_{t \uparrow T} J(u(t)) = -\infty$.
2. *If $\|u(t)\|_\infty = u(x(t), t)$ and $r(t) = \|u(t)\|_\infty^{-\frac{p-1}{2}}$, it holds that*

$$r(t)^{\frac{2}{p-1}} u(x(t) + r(t)z, t) \rightarrow v_\infty(z), \quad \text{locally uniformly in } \mathbf{R}^n,$$

where $v_\infty = v_\infty(z)$ is the solution to

$$-\Delta v_\infty = |v_\infty|^{p-1} v_\infty, \quad 0 \leq v_\infty \leq v_\infty(0) = 1 \quad \text{in } \mathbf{R}^n. \quad (3)$$

If Ω is convex, it holds that $\lim_{t \uparrow T} (T - t)^{\frac{1}{p-1}} \|u(t)\|_\infty = +\infty$, i.e., $r(t) = o((T - t)^{1/2})$.

For radially symmetric positive solution, we have always type I blowup rate by $p^* > \frac{n+2}{n-2}$, and consequently, $T < +\infty$ is excluded in the second case of Theorem 2. Thus, the NST solution blows-up in finite time if $\Omega = B$, $p = \frac{n+2}{n-2}$, and $\psi(x) = \Psi_0(|x|) \geq 0$; compare this with the proof of [2] using the complete blowup of the solution. This profile of the NST solution, however, is valid under the presense of the convexity and the symmetry of Ω .

Theorem 4 *Let $p = \frac{n+2}{n-2}$, $n \geq 3$ and Ω be convex and symmetric with respect to $x_i = 0$ ($i = 1, 2, \dots, n$). Let, furthermore, $0 \leq u_0(x) \in C(\bar{\Omega})$ be symmetric with respect to $x_i = 0$, and be decreasing in $x_i > 0$ ($i = 1, 2, \dots, n$). Then, for the solution $u = u(\cdot, t)$ to (1) it holds that*

$$T < +\infty \quad \Rightarrow \quad \lim_{t \uparrow T} J(u(t)) = -\infty,$$

and in particular, the NST solution blows-up in infinite time by Theorems 2 and 3. Thus, the third case of Theorem 1 actually occurs.

References

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