自己組織化の方程式
～走化性と腫瘍形成

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A Simplified System of Chemotaxis

$$
\begin{aligned}
&u_t = \nabla \cdot (\nabla u - u\nabla v) \\
&-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \qquad\qquad \text{in } \Omega \times (0, t) \\
&\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad\qquad \text{on } \partial\Omega \times (0, T) \\
&\int_\Omega v = 0
\end{aligned}
$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, ν the outer normal vector.

Theorem 1 (formation of collapse)

If $n = 2$ and $T = T_{\max} < +\infty$, then it holds that

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

as $t \uparrow T$ in $\mathcal{M}(\overline{\Omega})$, where

$$0 \le f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$$
$$\mathcal{S} = \left\{ x_0 \in \overline{\Omega} \mid \text{there exists } (x_k, t_k) \to (x_0, T) \right.$$
$$\left. \text{such that } u(x_k, t_k) \to +\infty \right\}.$$

Theorem 2 (mass quantization)

We have

$$m(x_0) = m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega) \end{cases}$$

and hence it holds that

$$2\sharp(\mathcal{S} \cap \Omega) + \sharp(\mathcal{S} \cap \partial\Omega) \leq \|u_0\|_1 /(4\pi)$$

by $\|u(t)\|_1 = \|u_0\|_1$.

1. S. (2005)

   Free Energy and Self-Interacting Particles,
   Birkhäuser, Boston.

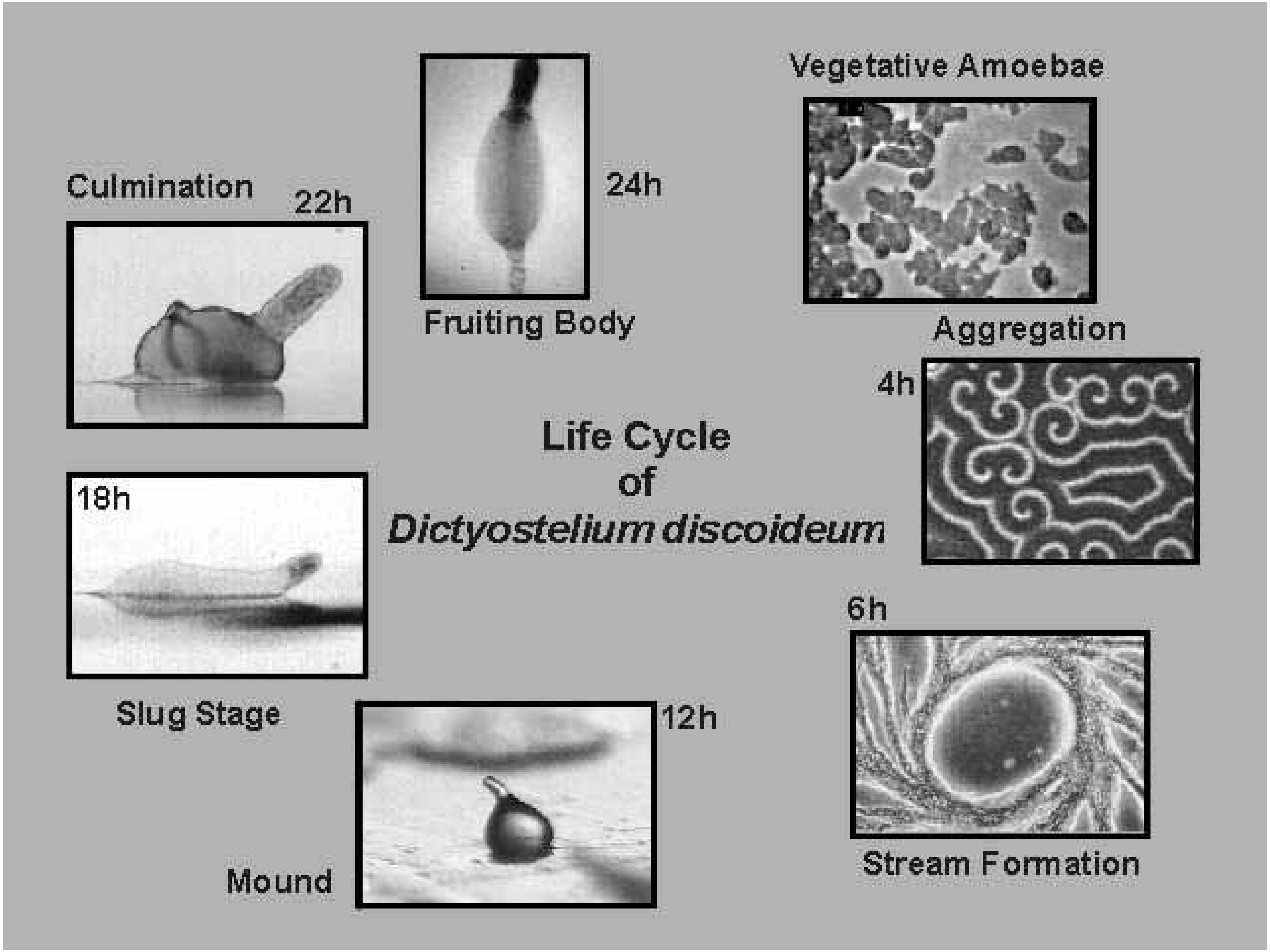
2. Nagai (1995)

3. Jäger-Luckhaus (1992)

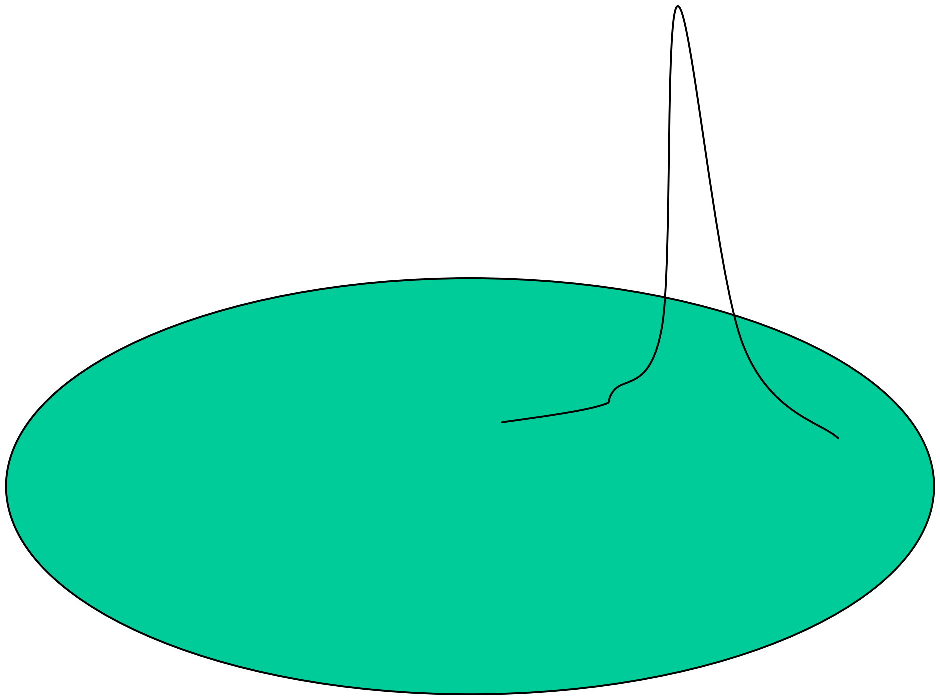
4. Childress-Percus (1981)

5. Nanjundiah (1973)

6. Keller-Segel (1970)


Vegetative Amoebae
Culmination
22h
24h
Fruiting Body
Aggregation
4h
Life Cycle
of
Dictyostelium discoideum
18h
6h
Slug Stage
12h
Mound
Stream Formation

"Formation of the quantized chemotactic collapse is proven".



Modellings

a) In the macroscopic description, the first equation indicates the mass conservation

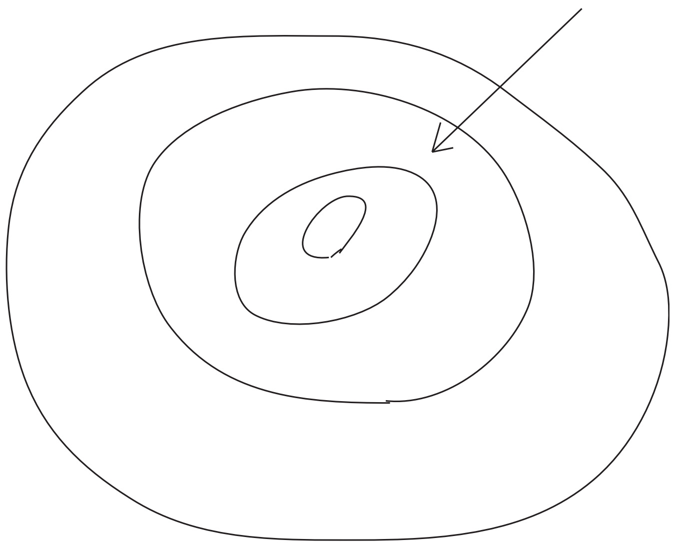
$$u_t = -\nabla \cdot j,$$

where $j = -\nabla u + u\nabla v$ denotes the flux of u.

Thus, we impose the null flux boundary condition.

Here, $u = u(x,t)$ denotes the density of cellular slime molds, while $v = v(x,t)$ is the concentration of the chemical subsequence, and therefore, $v = v(x,t)$ is a carrier; the diffusion $-\nabla u$ is competing the chemotaxis $u\nabla v$ derived from the *phenomenological relation*:

$$j = -\nabla u + u\nabla v.$$



This system also arises in the theory of material transport, called the *Smoluchowski equation*. There, $u = u(x,t)$ stands for the particle distribution, and the second equation describes the formation of self-attractive field $v = v(x,t)$ derived from this:

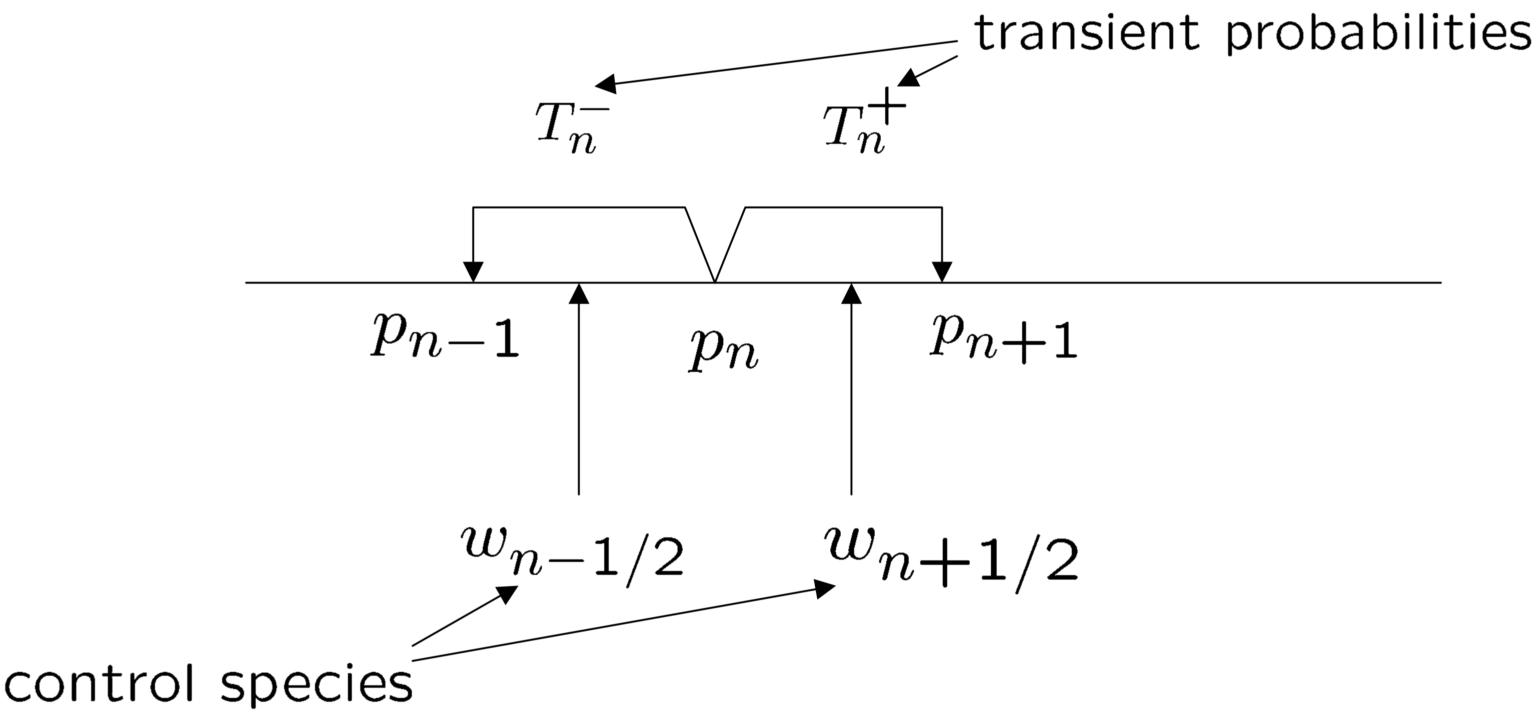
$$\begin{aligned} v(x,t) &= \int_\Omega G(x,x')u(x',t)dx' \\ G(x,x') &\approx \Gamma(x-x') \\ &= \begin{cases} \frac{1}{4\pi}\cdot\frac{1}{|x-x'|} & (n=3) \\ \frac{1}{2\pi}\log\frac{1}{|x-x'|} & (n=2), \end{cases} \end{aligned}$$

where $G = G(x,x')$ is the Green's function.

This law stands for the formation of a chemical gradient in the context of biology.

b) A microscopic modelling uses the master equation, which is reduced to the Smoluchowski equation via the Kramers-Moyal expansion derived from the master equation, e.g.,

$$\frac{\partial p_n}{\partial t} = T^+_{n-1} p_{n-1} + T^-_{n+1} p_{n+1} - \left(T^+_n + T^-_n\right) p_n.$$



Kramers equation is a kinetic model under the presense of the friction-fluctuation. Thus,

$$\mu^N(dx,dv,t) = m\sum \delta_{x_i(t)}dx \otimes \delta_{v_i(t)}(dv)$$
$$\rightharpoonup f(x,v,t)dxdv$$

holds as $N \to +\infty$ with $Nm \approx 1$, where m denotes the mass of particles. The adiabatic limit of the self-repulsive particle is the drift-diffusion model on the semi-conductor devise, while the vortex equation

$$\omega_t = \nabla \cdot (\nabla\omega - \omega\nabla^\perp\psi)$$
$$-\Delta\psi = \omega \qquad \text{in } \mathbf{R}^2 \times (0,T),$$

is derived from the Newton equation, where

$$\nabla^\perp = \begin{pmatrix} \partial/\partial x_2 \\ -\partial/\partial x_1 \end{pmatrix}$$

for $x = (x_1, x_2)$.

The *second moment* is taken in these Kramers-Moyal expansions to deduce the kinetic equation.

ODE	Newton	Langevin
Kinetic	Jeans-Vlasov	Kramers-Poisson
PDE	Euler	Keller-Segel (Smoluchowski)
Stationary	Semilinear elliptic eigenvalue problem (non-linearity unknown)	Semilinear elliptic eigenvalue problem (exponential non-linearity)
Clustered	Hamiltonian flow	Gradient flow
Physics	Conservation laws	Free energy
Mathematics	Chaotic motion	Quantized blow-up mechanism

c) Mesoscopic modelling uses Helmholtz' free energy,

$$A = U - TS,$$

defined by the inner energy minus entropy. Since $\mu(dx,t) = u(x,t)dx$ denotes the particle distribution, it holds that

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\left\langle (-\Delta_{JL})^{-1}u, u \right\rangle,$$

where $v = (-\Delta_{JL})^{-1}u$ if and only if

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

and

$$\int_\Omega v = 0.$$

This simplified system of chemotaxis is a model B equation,

$$
\begin{aligned}
&u_t = \nabla \cdot (u \nabla \delta \mathcal{F}(u)) \qquad \text{in } \Omega \times (0,T) \\
&u \frac{\partial}{\partial \nu} \delta \mathcal{F}(u) = 0 \qquad \text{on } \partial\Omega \times (0,T)
\end{aligned}
$$

and consequently, we obtain the **total mass conservation** and the **free energy decreasing**:

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega u = 0 \\
&\frac{d}{dt} \mathcal{F}(u) = - \int_\Omega u \left| \nabla \delta \mathcal{F}(u) \right|^2 \leq 0.
\end{aligned}
$$

Actually, phenomenological equations of *non-equlibrium thermodynamics* are derived from several free energies.

Model A (Allen-Cahn type)

$$\alpha\varphi_t = -\delta\mathcal{F}(\varphi)$$

$\Rightarrow$

$$\frac{d}{dt}\mathcal{F}(\varphi(\cdot,t)) = -\alpha\int_\Omega \varphi_t^2 \leq 0,$$

The decrease of total free energy is realized, and the stationary state is described by

$$\delta\mathcal{F}(\varphi) = 0.$$

Model B (Cahn-Hilliard type)

$$\varphi_t = \nabla \cdot (M \nabla \delta \mathcal{F}(\varphi))$$
$$M \frac{\partial}{\partial \nu} \delta \mathcal{F}(\varphi) \Big|_{\partial \Omega} = 0,$$

$\Rightarrow$

$$\frac{d}{dt} \int_\Omega \varphi(\cdot, t) = 0$$
$$\frac{d}{dt} \mathcal{F}(\varphi(\cdot, t)) = -\int_\Omega M \left| \nabla \delta \mathcal{F}(\varphi(\cdot, t)) \right|^2 \leq 0.$$

The conservation of the total "order parameter" is realized besides the decrease of the total free energy, and consequently, the stationary state is described by

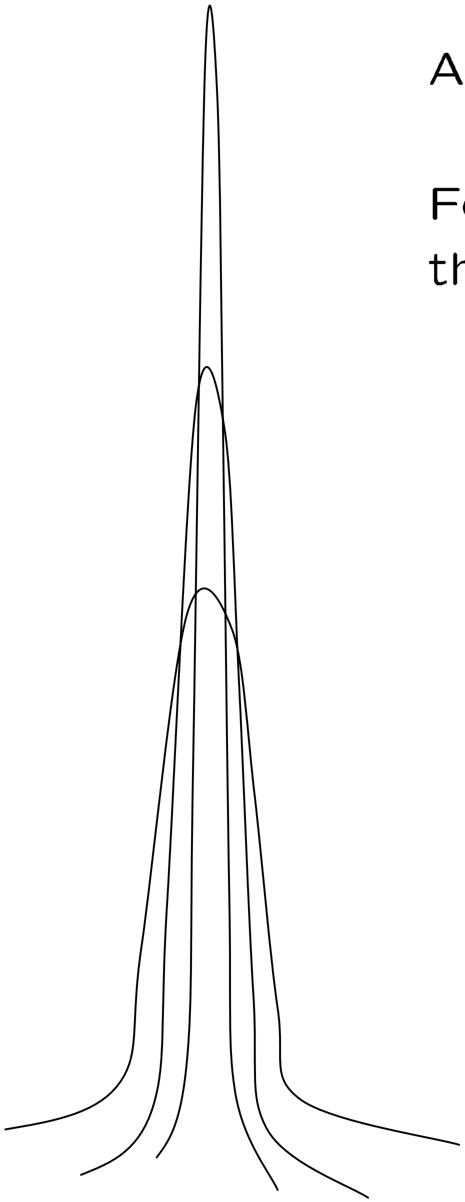
$$\delta \mathcal{F}(\varphi) = 0, \qquad \int_\Omega \varphi = \lambda.$$

Structures

a) Duality: Weak formulation, Hardy-BMO pair, and convex conjugate are hidden in this system.

a1) The first two of the following Riesz' representation theorems are not reflexive.

1. $C(\overline{\Omega})' \cong \mathcal{M}(\overline{\Omega})$
2. $L^1(\Omega)' \cong L^\infty(\Omega)$
3. $L^2(\Omega)' \cong L^2(\Omega)$

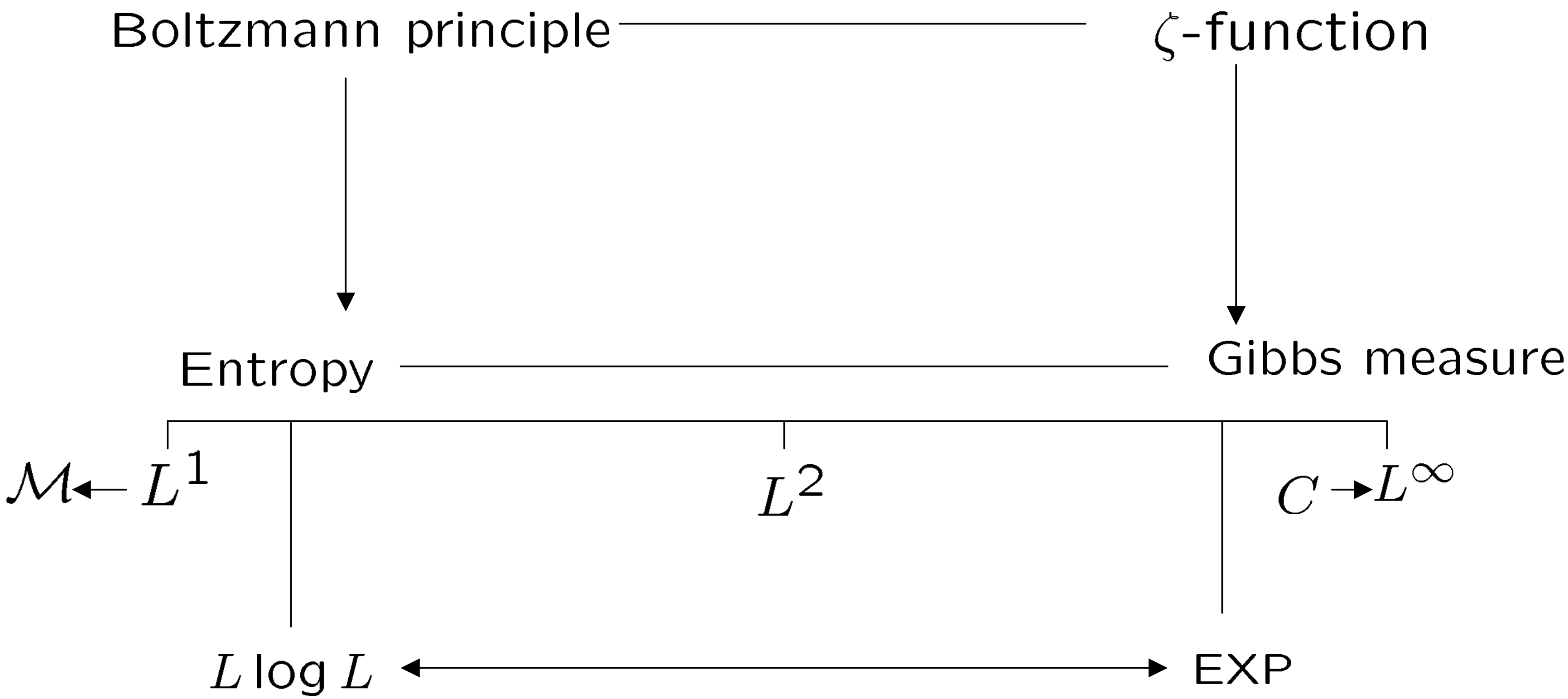


An L^1 control: concentration

Formation of the delta function (collapse) has the origin in

$$C'(\overline{\Omega}) \cong \mathcal{M}(\overline{\Omega}).$$

a2) The reflexive Hardy-BMO, or the Zygmund - John-Nirenberg duality is observed in the mean field hierarchy with the entropy - Gibbs measure paring.



a3) Entropy and the Gibbs measure are the convex functionals on the Hardy and BMO spaces, respectively. Then, there is a variational duality derived from the Legendre transformation.

Fundamentals of Convex Analysis

1. X is a Banach space over $\mathbf{R}$

2. $F : X \to (-\infty, +\infty]$ is proper, convex, lower semi-continuous.

$\Rightarrow$

1. The Legendre transformation $F^* : X^* \to (-\infty, +\infty]$ is proper, convex, lower semi-continuous, where

$$F^*(p) = \sup_{x \in X} \{\langle x, p \rangle - F(x)\}$$

2. Fenchel-Moreau's duality holds as $F^{**} = F$, where

$$F^{**}(x) = \sup_{p \in X^*} \{\langle x, p \rangle - F^*(p)\}$$

Toland duality (1978, 1979):

$$F, G : X \to (-\infty, +\infty] \quad \text{prop., c'x, l.s.c.}$$

$$J(x) = \begin{cases} G(x) - F(x) & (x \in D(G)) \\ +\infty & (\text{otherwise}) \end{cases}$$

$$J^*(p) = \begin{cases} F^*(p) - G^*(p) & (p \in D(F^*)) \\ +\infty & (\text{otherwise}) \end{cases}$$

$$L(x, p) =$$

$$\begin{cases} F^*(p) + G(x) - \langle x, p \rangle & ((x, p) \in D(G) \times X^*) \\ +\infty & (\text{otherwise}) \end{cases}$$

$\Rightarrow$

$$\inf_{(x,p) \in X \times X^*} L(x, p) = \inf_{p \in X^*} J^*(p) = \inf_{x \in X} J(x).$$

b) Scaling: Typical examples

$1 < p < \infty$, $\mu > 0$

$$
\begin{aligned}
&-\Delta v = v^p \\
&v_\mu(x) = \mu^{2/(p-1)} v(\mu x) \\
&\Rightarrow -\Delta v_\mu = v_\mu^p
\end{aligned}
$$

$$
\begin{aligned}
&u_t - \Delta u = u^p \\
&u^\mu(x,t) = \mu^{2/(p-1)} u(\mu x, \mu^2 t) \\
&\Rightarrow u_t^\mu - \Delta u^\mu = u_\mu^p,
\end{aligned}
$$

which guarantees the hierarchical argumet (*blowup anlaysis*).

Self-similar transformation

1. Backward self-similar transformation

$$
\begin{aligned}
& v_t - \Delta v = v^p \\
& z(y,s) = (T-t)^{1/(p-1)} u(x,t) \\
& y = x/(T-t)^{1/2}, \quad s = -\log(T-t) \\
& \Rightarrow \\
& z_s - \Delta z + \frac{y}{2} \cdot \nabla z + \frac{z}{p-1} = z^p,
\end{aligned}
$$

induces the classification of the blowup rate

$$
\begin{aligned}
& \text{Type (I)} \Leftrightarrow O\left((T-t)^{-1/(p-1)}\right) \\
& \text{Type (II)} \Leftrightarrow \text{the other case}
\end{aligned}
$$

2. Forward self-similar transformation

$$
\begin{aligned}
&v_t - \Delta v = v^p \\
&z(y,s) = (t+1)^{\frac{1}{p-1}} v(x,t) \\
&y = x/(t+1)^{1/2}, \quad s = \log(t+1) \\
&\Rightarrow \\
&z_s - \Delta z = \frac{y}{2} \cdot \nabla z + \frac{z}{p-1} + z^p,
\end{aligned}
$$

describes the asymptotic profile of the solution globally in time.

Plan

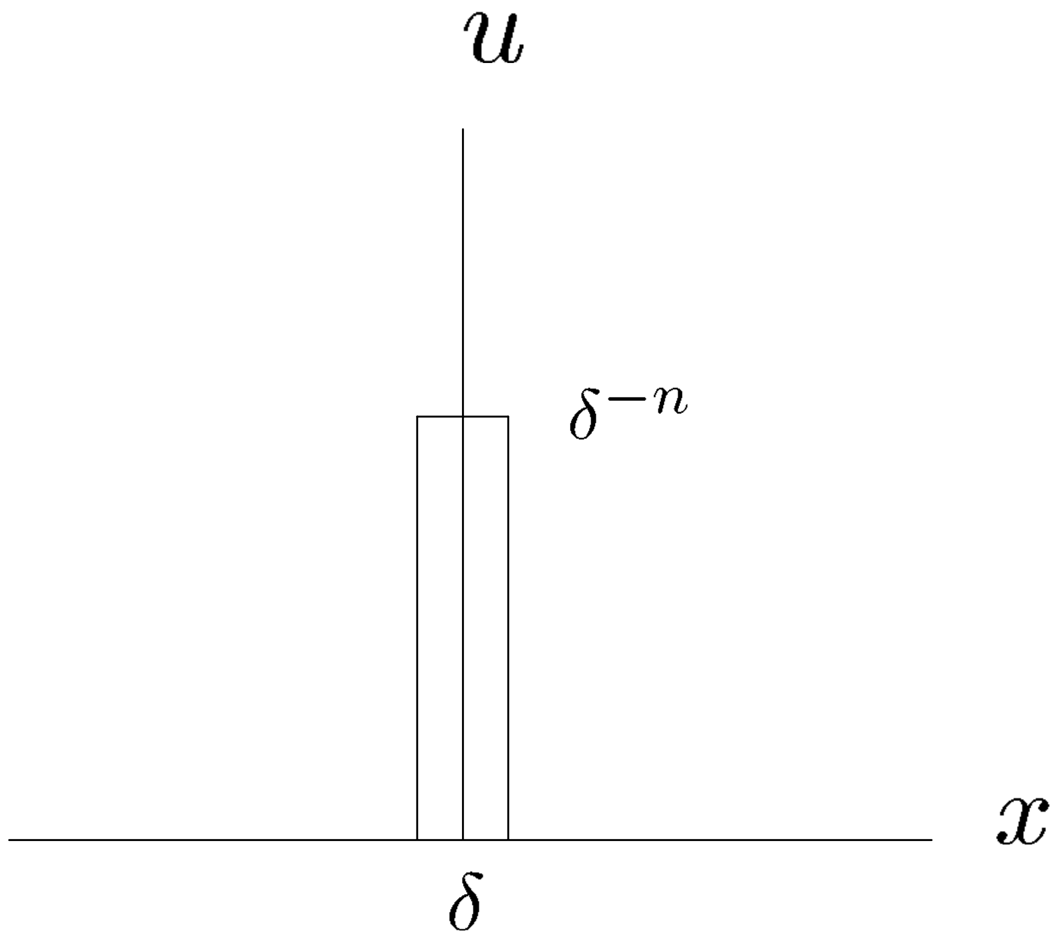
1. description of the proof
2. recent developments and related topics
3. two aspects of self-organization
4. numerical simulations
5. life of cellular slime molds (movies)

I. Description of the proof

I.1. Semi-analysis of Childress-Percus (1981)

1. Dimension 2 is selected for the formation of collapse.

2. Threshold mass is realized in the stationary state.


u
δ^{-n}
x
δ

Since total mass conservation holds as

$$\|u\|_1 = \lambda,$$

if $x \sim \delta$, then it holds that $u \sim \delta^{-n}$. Putting $\nabla \sim \delta^{-1}$, we obtain

$$\begin{aligned}
&\delta^{-1+3n/2}\left(\delta^0, \delta^0, \delta^{-1+n/2}\right) = 0 \\
&\delta^{-n}\left(\delta^0, \cdot, \delta^{-1+n/2}\right) = 0
\end{aligned}$$

by

$$\begin{aligned}
&u_t - \nabla \cdot (u\nabla v) - \Delta u = 0 \\
&-u + \frac{1}{|\Omega|}\int_\Omega u - \Delta v = 0
\end{aligned}$$

under the agreement of $v \sim \delta^{1-n/2}$, $t \sim \delta^{1+n/2}$. Then, $n = 2$ is selected from the balance of the above relations i.e., the *scaling*.

From the *model (B)* profile, the stationary state is defined by

$$\delta \mathcal{F}(u) = 0, \quad \int_\Omega u = \lambda$$

for $\lambda = \|u_0\|_1$. This means

$$\log u - (-\Delta_{JL})^{-1} u = \text{constant}, \quad \int_\Omega u = \lambda$$

by

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2} \left\langle (-\Delta_{JL})^{-1} u, u \right\rangle.$$

Using

$$v = (-\Delta_{JL})^{-1} u, \qquad u = \frac{\lambda e^v}{\int_\Omega e^v},$$

this results in the *dual* form,

$$\begin{aligned} -\Delta v &= \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right) \qquad \text{in } \Omega \\ \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega v = 0, \end{aligned}$$

a nonlinear elliptic eigenvalue problem with nonlocal term.

Such *exponential nonlinearity competing two-dimensional diffusion* arises in several areas:

1. system of chemotaxis

2. fundamental equation of material transport

3. vortex point mean field

4. self-dual gauge theory

5. normalized Ricci flow.

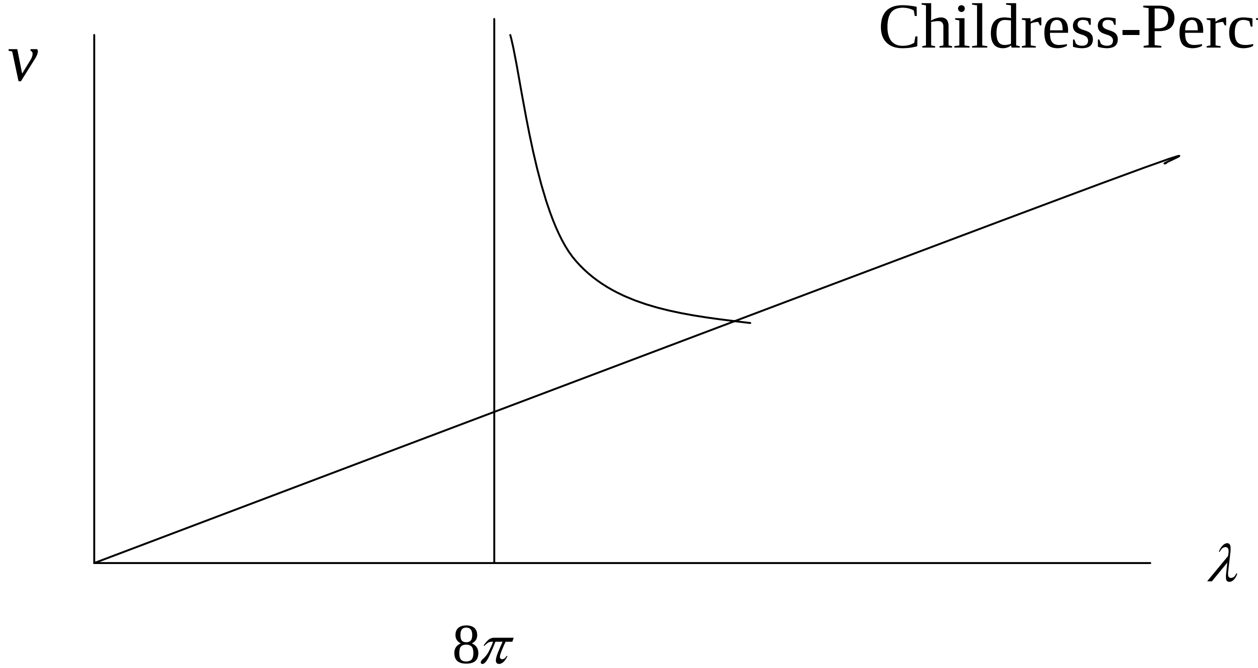
Mathematical principle of these elliptic problems is summarized by the quantized blowup mechanism. Actually, Childress-Percus picked up the *thereshold* value 8π by the (radially symmetric) exact solution of the vortex point mean field equation.

$0 < a \ll 1$

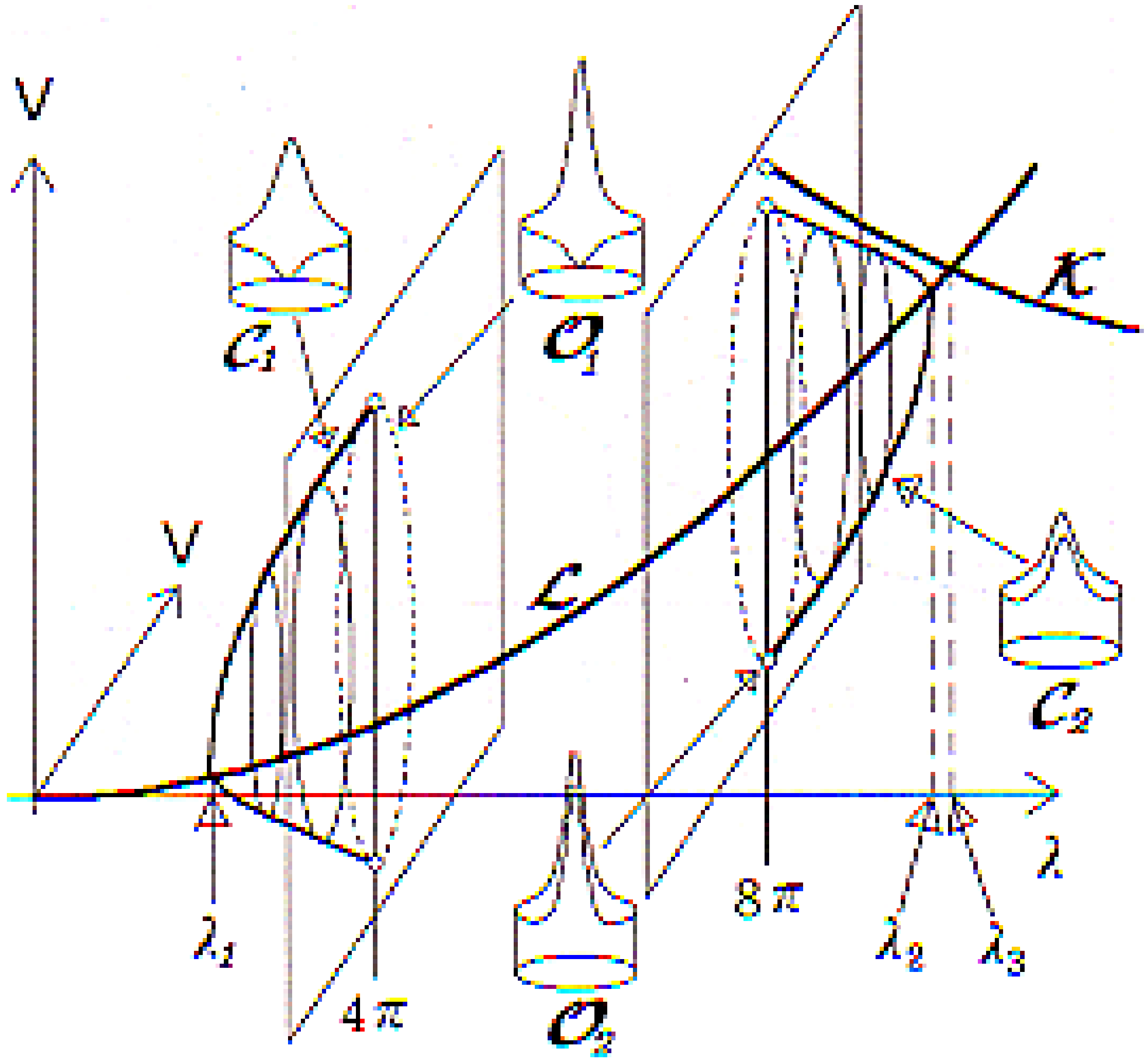
$$-\Delta v + av = \lambda \frac{e^v}{\int_\Omega e^v} \qquad \Omega$$

$$\frac{\partial v}{\partial \nu} = 0 \qquad \partial\Omega$$

Numerical computation of
Childress-Percuss (1981)



$\lambda = \| u \|_1$


V
V
$\mathcal{C}_1$
$\mathcal{O}_1$
$\mathcal{K}$
$\mathcal{C}$
$\mathcal{C}_2$
λ
λ_1
4π
$\mathcal{O}_2$
8π
λ_2
λ_3

Senba-S. (00)

Correction of the **Nanjundiah - Childress - Percus conjecture** from the study of the stationary state:

1. If $n = 2$, a quantized blowup mechanism is realized in the family of stationary states.

2. It not only suggests the L^1 threshold, but also is the origin of the quantized blowup mechanism of the non-stationary state.

3. First, 8π had been conjectured as a global L^1 threshold from the study of the stationary problem. Then, it was noticed that the boundary blowup reduces it to 4π.

In spite of this motivation based on the mean field hierarchy, mathematical treatments of the non-stationary state are *completely different* from those of the stationary states.

In other words, mean field hierarchy is a similarity rather than the continuity, e.g., **atoms are never infinitessimally small moleculars**.

I.2. Localization - Symmetrization

Global existence criteiron

Theorem. (Nagai-Senba-Yoshida 1997, Biler 1998, Gajewski-Zacharias 1998)

$$\|u_0\|_1 < 4\pi \ \Rightarrow T_{\mathsf{max}} = +\infty.$$

This criterion is proven by the decrease of the free energy and the Trudinger-Moser inequality.

First, the *dual Trudinger-Moser inequality*

$$\inf\left\{\mathcal{F}(u) \mid u \geq 0, \quad \|u\|_1 = 4\pi\right\} > -\infty,$$

e.g., guarantees

$$\sup_{t\geq 0} \|u(t)\|_{L\log L} < +\infty$$

by $\lambda = \|u_0\|_1 < 4\pi$, and then

$$T = +\infty, \qquad \sup_{t\geq 0} \|u(t)\|_\infty < +\infty$$

follows from Moser's iteration or the maximal regularity.

Proof of Theorem 1 (Formation of collapse):

Formation of collapse is proven by the anti-blowup criterion of the solution locally in space-time.

1. First, using a nice cut-off function, we prove this at each *isolated* blowup point using the Trudinger-Moser inequality and Moser's iteration scheme.

2. On the other hand, ε-regularity holds as

$$\lim_{R\downarrow 0} \limsup_{t\uparrow T_{\max}} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \varepsilon_0 \\ \Rightarrow x_0 \notin \mathcal{S},$$

i.e.,

$$x_0 \in \mathcal{S} \Rightarrow \\ \lim_{R\downarrow 0} \limsup_{t\uparrow T_{\max}} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0,$$

where $\varepsilon_0 > 0$ is an absolute constant.

3. Then, we replace

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T_{\max}} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0$$

by

$$\lim_{R\downarrow 0}\liminf_{t\uparrow T_{\max}} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0.$$

In fact, if this is the case then we obtain

$$\sharp\mathcal{S} < +\infty$$

by $\|u(\cdot,t)\|_1 = \|u_0\|_1$, and therefore, any blowup point is isolated and we are done.

4. This replacement is justified by the method of symmetrization applied to the *weak formulation* of the problem:

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u(\cdot,t)\varphi = \int_\Omega u(\cdot,t)\Delta\varphi \\
&+\frac{1}{2}\int\int_{\Omega\times\Omega}\rho_\varphi(x,x')u(x,t)u(x',t)dxdx',
\end{aligned}
$$

where $\varphi\in C^2(\overline{\Omega})$ satisfies $\frac{\partial\varphi}{\partial\nu}\Big|_{\partial\Omega}=0$ and

$$
\begin{aligned}
&\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') \\
&+\nabla\varphi(x')\cdot\nabla_{x'}G(x,x')\in L^\infty(\Omega\times\Omega).
\end{aligned}
$$

Here, we emphasize the role of the symmetry of the Green's function,

$$
G(x',x) = G(x,x'),
$$

describing the law of the action and the reaction.

5. In more detail, we obtain

$$\left|\frac{d}{dt}\int_\Omega \varphi(x)u(x,t)dx\right| \leq C_\varphi$$

from this formulation, which means that any local mass is of bounded variation in time.

Then we can replace the lim sup condition by the liminf condition:

$$x_0 \in \mathcal{S} \quad \Rightarrow$$
$$\liminf_{t\uparrow T} \|u(t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0 \quad (0 < {}^\forall R \ll 1),$$

which implies the **formation of collapse**,

$$u(x,t) \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0} + f(x)$$

with

$$m(x_0) \geq m_*(x_0).$$

1.3. Weak continuation - self-similarity

Mass quantization (Theorem 2) is a "local" blowup criterion, while the global blowup criterion is also obtained by the above mentioned weak formulation, using the second moment.

A simple case (Biler-Hilhorst-Nadzieja 1994)

$$u_t = \nabla \cdot (\nabla u - u\nabla v), \quad v = \Gamma * u \qquad \text{in } \Omega \times (0,T)$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T),$$

where $\Gamma = \frac{1}{2\pi}\log\frac{1}{|x|}$. We obtain $\|u(t)\|_1 = \|u_0\|_1$ and if Ω is star-shaped, then

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |x|^2 u(x,t)dx = -\int_\Omega 2x \cdot (\nabla u - u\nabla v) \\
&= -2\int_{\partial\Omega}(x\cdot\nu)u + \int_\Omega 4u + 2ux\cdot\nabla v \; dx \\
&\leq 4\lambda + 2\int\int_{\Omega\times\Omega} u(x,t)x\cdot\nabla\Gamma(x-x')u(x',t)dx' \\
&= 4\lambda + \int\int_{\Omega\times\Omega}(x-x')\cdot\nabla\Gamma(x-x')u(x,t)u(x',t)dxdx' \\
&= 4\lambda - \frac{\lambda^2}{2\pi}.
\end{aligned}$$

Thus, $\lambda > 8\pi \Rightarrow T = T_{\max} < +\infty$. This argument is adopted by several authors for the study of hyperbolic, Euler-Poisson, and Schrödeinger equations.

In our problem, high concentration with locally over mass quantization implies blowup, where fine profiles of the Green's function near the boundary are used.

Theorem. (Senba-S. 01)

${}^{\exists}\eta > 0$: an absolute constant, s.t.

$$
\begin{aligned}
& x_0 \in \overline{\Omega}, \quad 0 < R \ll 1 \\
& \frac{1}{R^2} \int_{\Omega \cap B_{2R}(x_0)} |x - x_0|^2 \, u_0(x) < \eta \\
& \int_{\Omega \cap B_R(x_0)} u_0(x) > m_*(x_0) \\
& \Rightarrow \\
& T = T_{\max} = o(R^2) < +\infty.
\end{aligned}
$$

From the proof, this criterion is valid even to the weak solution.

Weak solution

$0 \leq \mu = \mu(dx,t) \in C_*([0,T), M(\overline{\Omega}))$, $0 \leq {}^\exists\nu = \nu(t) \in L^\infty_*(0,T : \mathcal{E}')$ s.t.

$$\begin{aligned}
&\nu(t)|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu \otimes \mu(dxdx',t) \qquad \text{a.e. } t \\
&\varphi \in X \Rightarrow t \in [0,T) \mapsto \langle \varphi, \mu(dx,t)\rangle \quad \text{loc. a.c.} \\
&\frac{d}{dt}\langle \varphi, \mu(dx,t)\rangle = \langle \Delta\varphi, \mu(dx,t)\rangle + \frac{1}{2}\langle \rho_\varphi, \nu(t)\rangle \qquad \text{a.e. } t,
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{E} = \{\rho_\varphi \mid \varphi \in X\} + C(\overline{\Omega}\times\overline{\Omega}) \subset L^\infty(\Omega\times\Omega) \\
&X = \left\{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0\right\} \\
&\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x').
\end{aligned}$$

Generation of the weak solution

$\{u_k\}$ classical solutions on $\overline{\Omega} \times [0,T)$, $\|u_{0k}\|_1 \le C \Rightarrow {}^{\exists}$sub-sequence s.t.

$$u_k(x,t)dx \rightharpoonup {}^{\exists}\mu(dx,t) \ \text{ in } C_*([0,T), \mathcal{M}(\overline{\Omega})),$$

a weak solution.

Blowup of the weak solution

${}^{\exists}\eta > 0$ absolute constant s.t.

$$\begin{aligned} &\langle \varphi_{x_0,R}, \mu(dx,0) \rangle > m_*(x_0) \\ &\frac{1}{R^2} \left\langle |x - x_0|^2 \varphi_{x_0,2R}, \mu(dx,0) \right\rangle < \eta \end{aligned}$$

$\Rightarrow T = T_{*\max} = o(R^2)$. Here and henceforth, $\varphi_{x_0,R}$ denotes a nice cut-off function aroud x_0 with the support radius R, and $T_{*\max}$ denotes the blowup time of the weak solution.

If $u_0 = u_0(x) \geq 0$ is smooth and $T = T_{\max} < +\infty$, then Theorem 1 (formation of collapse) guarantees

$$
\begin{aligned}
&\mu(dx,t) = u(x,t)dx \in C_*([0,T], \mathcal{M}(\overline{\Omega})) \\
&\mu(dx,T) = \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx
\end{aligned}
$$

with $m(x_0) \geq m_*(x_0)$.

In case $m(x_0) > m_*(x_0)$ for ${}^\exists x_0 \in \mathcal{S}$, therefore, this $\mu(dx,t)$ is not continued after $t = T$ as a weak solution, because

$$
\begin{aligned}
&\langle \varphi_{x_0,R}, \mu(dx,T) \rangle > m_*(x_0) \\
&\frac{1}{R^2} \left\langle \left| x - x_0 \right|^2 \varphi_{x_0,2R}, \mu(dx,T) \right\rangle < \eta
\end{aligned}
$$

for any $0 < R \ll 1$.

1. Thus, the formation of an over-quantized collapse
$$m(x_0) > m_*(x_0)$$
in finite time implies the instant blowup of the solution.

2. i.e., *Weak post-blowup continuation* assures mass quantization,
$$m(x_0) = m_*(x_0).$$

3. The **Kramers-Poisson equation**, on the other hand, admits a weak solution globally in time, when the initial value is bounded.

4. In spite of this, we do not obtain the well-posedness in the space of measure.

Based on these observations, we use the rescaled variables and hierarchical argument to complete the proof of mass quantization. The *parabolic envelope* justifies this rescaling.

1. Parabolic envelope

1. For the nice cut-off function $\varphi_{x_0,R} = \varphi_{x_0,R}(x)$ we confirm

$$\left| \frac{d}{dt} \int_\Omega u(\cdot,t)\varphi_{x_0,R} \right| \leq CR^{-2},$$

where $0 < R < 1$. This implies

$$\begin{aligned} &\left| \left\langle \varphi_{x_0,R}, \mu(\cdot,T) \right\rangle - \left\langle \varphi_{x_0,R}, \mu(\cdot,t) \right\rangle \right| \\ &\quad \leq CR^{-2}(T-t) \end{aligned}$$

by

$$\mu(dx,t) = u(x,t)dx \in C_*([0,T], M(\overline{\Omega})).$$

2. Since $0 < R < 1$ is arbitrary, we can put

$$R = bR(t)$$

for given $b > 0$ in the above inequality, provided that $0 < R(t) = (T-t)^{1/2} < b^{-1}$:

$$\begin{aligned} &\left| \left\langle \varphi_{x_0,bR(t)}, \mu(\cdot,T) \right\rangle - \left\langle \varphi_{x_0,bR(t)}, \mu(\cdot,t) \right\rangle \right| \\ &\qquad \le Cb^{-2}. \end{aligned}$$

3. This implies

$$\limsup_{t \uparrow T} \left| m(x_0) - \left\langle \varphi_{x_0,bR(t)}, \mu(\cdot,t) \right\rangle \right| \le Cb^{-2}$$

for $x_0 \in \mathcal{S}$ by

$$\mu(dx,T) = \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx.$$

4. In particular,

$$\lim_{b\uparrow+\infty} \limsup_{t\uparrow T} \left| \left\langle \varphi_{x_0,bR(t)}, \mu(\cdot,t) \right\rangle - m(x_0) \right| = 0$$

and thus, infinitely *large* parabolic region in terms of the backward self-similar transformation around the blowup point (=parabolic envelope) contains the whole blowup mechanism, where

$$R(t) = (T-t)^{1/2}$$
$$\mu(dx,t) = u(x,t)dx$$

for $0 < t < T$.

2. Backward scaling

If $T = T_{\mathrm{max}} < +\infty$, $x_0 \in \mathcal{S}$, and

$$\begin{aligned} &z(y,s) = (T-t)u(x,t) \\ &y = (x-x_0)/(T-t), \quad s = -\log(T-t), \end{aligned}$$

then

$$\begin{aligned} z_s &= \nabla \cdot (\nabla z - z\nabla w - yz/2) \\ 0 &= \Delta w + z - e^{-s}\lambda / |\Omega| \end{aligned}$$

in $\cup_{s>-\log T} e^{s/2}(\Omega - \{x_0\}) \times \{s\}$ with

$$\frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0$$

on $\cup_{s>-\log T} e^{s/2}(\partial\Omega - \{x_0\}) \times \{s\}$.

3. Generation of the weak solution

${}^\forall s_k \to +\infty$, ${}^\exists\{s'_k\} \subset \{s_k\}$ such that

$$z(y, s + s'_k)dy \rightharpoonup \zeta(dy, s)$$

in $C_*(-\infty, +\infty; \mathcal{M}(\overline{L}))$, where $\zeta = \zeta(dy, s)$ is a (rescaled) weak solution to

$$\begin{aligned}
& z_s = \nabla \cdot (\nabla z - z\nabla(w + |y|^2/4)) \\
& \left.\frac{\partial z}{\partial \nu}\right|_L = 0 \\
& \nabla w(y, s) = \int_L \nabla\Gamma(y - y')z(y', s)dy',
\end{aligned}$$

and L is $\mathbf{R}^2$ if $x_0 \in \Omega$ and a half space with ∂L parallel to the tangent line of $\partial\Omega$ at x_0 if $x_0 \in \partial\Omega$, and

$$\Gamma(y) = \frac{1}{2\pi}\log\frac{1}{|y|}.$$

All the cases are reduced to $L = \mathbf{R}^2$ by the reflection:

$$
\begin{aligned}
& z_s = \nabla \cdot (\nabla z - z\nabla(w + |y|^2/4)) \\
& \nabla w = \nabla\Gamma * z \qquad \text{in } \mathbf{R}^2 \times (-\infty, +\infty).
\end{aligned}
$$

The parabolic envelope, on the other hand, guarantees

$$
\begin{aligned}
\hat{m}(x_0) & \equiv \zeta(\mathbf{R}^2, s) \\
& = \begin{cases} m(x_0) & (x_0 \in \Omega) \\ 2m(x_0) & (x_0 \in \partial\Omega) \end{cases} \geq 8\pi,
\end{aligned}
$$

and therefore, we have only to derive

$$
\hat{m}(x_0) \leq 8\pi
$$

to complete the proof of mass quantization (Theorem 2).

4. Second moment applied to the rescaled system

We use the argument of Kurokiba-Ogawa (2003) hierarchically.

Since this (rescaled) weak solution is global in time, method of the second moment assures that the sufficient concetration at the origin implies

$$\hat{m}(x_0) = \zeta(\mathbf{R}^2, 0) \leq 8\pi.$$

e.g.,

If $c = c(s)$ satisfies

$$0 \leq c'(s) \leq 1, \quad -1 \leq c(s) \leq 0, \qquad (s \geq 0)$$
$$c(s) = \begin{cases} s - 1 & (0 \leq s \leq 1/4) \\ 0 & (s \geq 4), \end{cases}$$

then, $^\forall \varepsilon > 0$, $^\exists \eta > 0$ such that

$$\left\langle c(|y|^2) + 1, \zeta(dy, 0) \right\rangle < \eta$$
$$\Rightarrow \zeta(\mathbf{R}^2, 0) < 8\pi + \varepsilon.$$

In fact, we obtain

$$\begin{aligned}
&\frac{d}{ds}\left\langle c(|y|^2)+1,\zeta(dy,s)\right\rangle \\
&\leq C\left\langle c(|y|^2)+1,\zeta(dy,s)\right\rangle+\delta\hat{m}(x_0)\left\{4-\frac{\hat{m}(x_0)}{2\pi}\right\}
\end{aligned}$$

for $\hat{m}(x_0)=\zeta(\mathbf{R}^2,s)\geq 8\pi$, and this implies the above criterion.

5. Self-similarity

The above mentioned concentration condition is moved by the *self-similarity* and the *translation invariance* of the rescaled system, because any initial data "looks like" concenrated at the origin by this transformation.

Actual *self-similarity* to the rescaled limit equation is achieved by the transformation

$$z(y,s) = e^{-s}A(y',s'), \quad w(y,s) = B(y',s'), \quad y' = e^{-s/2}y, \quad s' = -e^{-s}.$$

This implies the pre-scaled semi-orbit in the whole space,

$$A_{s'} = \nabla' \cdot (\nabla' A - A\nabla' B), \quad \nabla' B = \nabla\Gamma * A \qquad \text{in } \mathbf{R}^2 \times (-\infty, 0),$$

which is invariant under

$$A^\mu(y,s) = \mu^2 A(\mu y, \mu^2 s), \quad B^\mu(y,s) = B(\mu y, \mu^2 s)$$

for $\mu > 0$. This structure, together with the *translation invariance* in time of the scaling limit equation, removes the concentration condition

$$\left\langle c(|y|^2) + 1, \zeta(dy, 0) \right\rangle < \delta$$

to guaratee $\zeta(\mathbf{R}^2, 0) \le 8\pi$.

More precisely, we obtain

$$\left\langle c(|y|^2) + 1, \zeta^\mu(dy, s) \right\rangle \geq \eta, \qquad (-\infty < s < +\infty)$$

for

$$\zeta^\mu(dy, s) = \mu^2 e^{-s} \zeta(\mu e^{-s/2} dy, -\mu^2 e^{-s})$$

in case $\hat{m}(x_0) = \zeta(\mathbf{R}^2, s) = \zeta^\mu(\mathbf{R}^2, s) > 8\pi + \varepsilon$, and then

$$\left\langle c(\tilde{s}\,|y|^2) + 1, \zeta(dy, -\tilde{s}) \right\rangle \geq \eta$$

for any $\tilde{s} > 0$. We obtain a contrdiction by applying this criterion to $\zeta(dy, s + \tilde{s})$,

$$\left\langle c(\tilde{s}\,|y|^2) + 1, \zeta(dy, 0) \right\rangle \geq \eta$$

and then making $\tilde{s} \downarrow 0$.

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F. Blowup in infinite time

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G. Mass quantization

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H. Blowup rate, formation of sub-collapse

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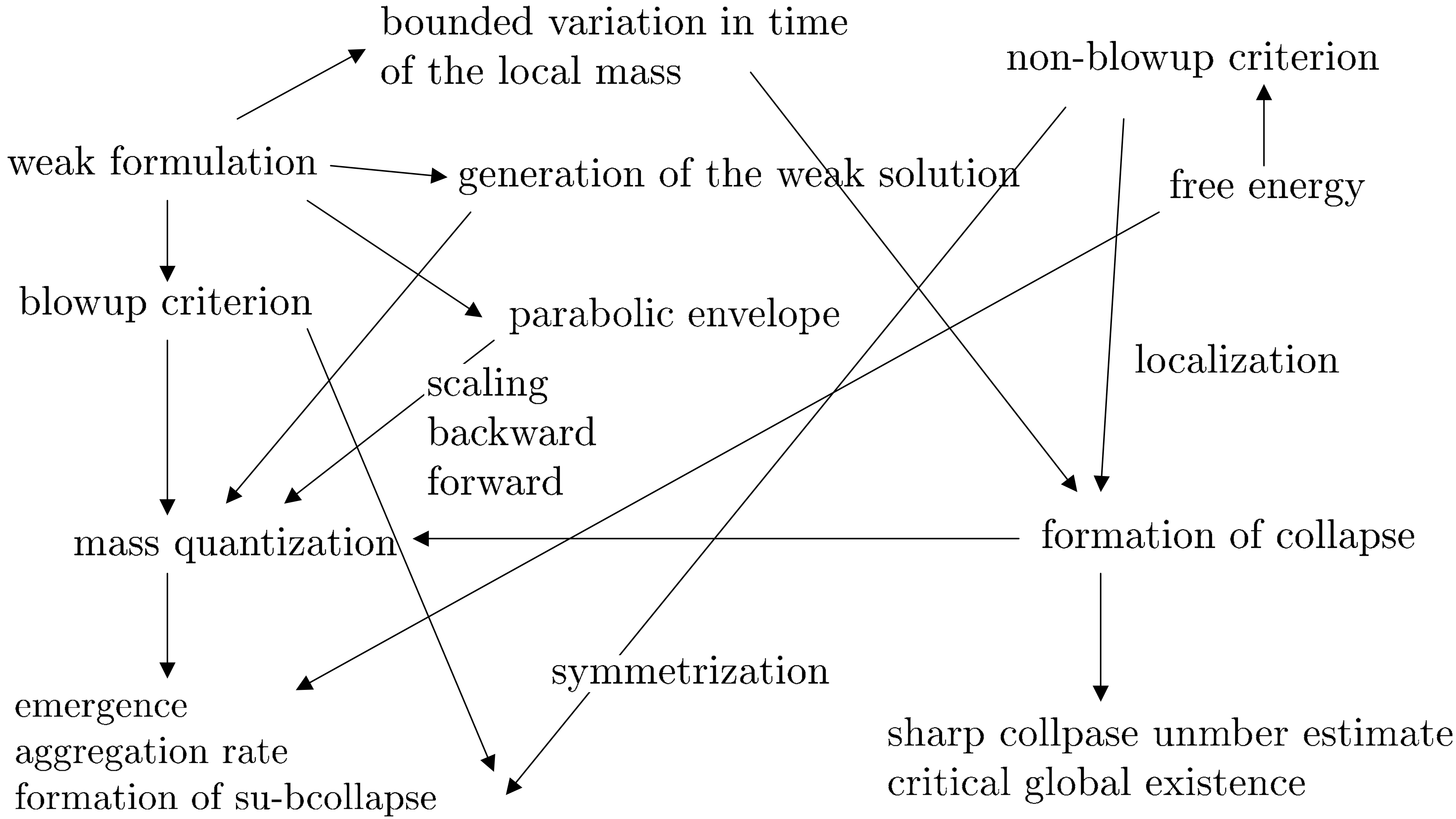
Structure of the simplified system



II. Recent developements and related topics

II-1. Formation of sub-collapse

Similarly to the parabolic envelope, it follows that

$$I(s) = \left\langle |y|^2, \zeta(dy,s) \right\rangle \leq C \qquad (-\infty < s < \infty),$$

and then we obtain

$$\frac{dI}{ds} = I \qquad (-\infty < s < \infty)$$

by the mass quantization, $\zeta(\mathbf{R}^2, s) = 8\pi$. This implies $I(s) = 0$, while

$$\zeta(dy,s) = \sum_{y_0 \mathcal{B}_s} 8\pi\delta_{y_0}(dy) + g(y,s)dy$$

holds with $0 \leq g(\cdot, s) \in L^1(\mathbf{R}^2)$ for each s. Consequently, we obtain the *formation of sub-collapse*

$$z(y, s+s')dy \rightharpoonup m_*(x_0)\delta_0(dy)$$

in $\mathcal{M}(\mathbf{R})^{\mathbf{2}})$ as $s' \uparrow +\infty$. Consequently, the whole blowup mechanism is enveloped in the *hyper-parabola*, the infinitely small parabolic region, and in particular, any $x_0 \in \mathcal{S}$ is of type (II):

$$\lim_{t \uparrow T} R(t)^2 \left\| u(t) \right\|_{L^\infty(\Omega \cap B(x_0, bR(t))} = +\infty$$

for any $b > 0$, wherer $R(t) = (T-t)^{1/2}$.

II-2. Blowup in infinite time

The collapse numer estimate is improved by

$$2\sharp(\Omega \cap \mathcal{S}) + \sharp(\partial\Omega \cap \mathcal{S}) < \|u_0\|_1 / (4\pi),$$

and hence $\|u_0\|_1 = 4\pi \Rightarrow T_{\max} = +\infty$.

If

$$\lim_{k\to\infty} \|u(t_k)\|_\infty = +\infty$$

in this case, there is $\left\{t'_k\right\} \subset \{t_k\}$ such that $u(x, t+t'_k)dx \rightharpoonup 4\pi\delta_{x(t)}dx$ with $t \in (-\infty, +\infty) \mapsto x(t) \in \partial\Omega$ satisfying

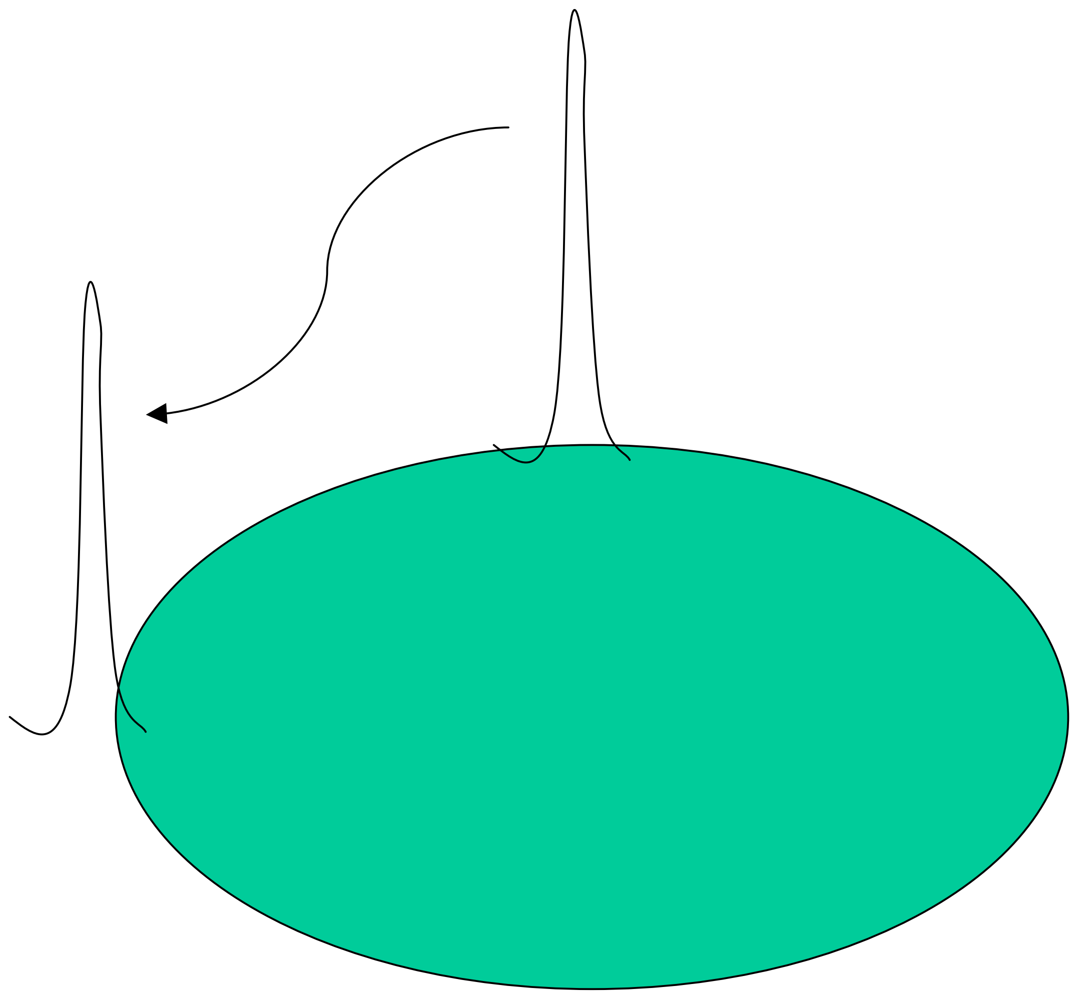
$$\frac{dx}{dt} = 4\pi\nabla_\tau R(x),$$

and hence it holds that

$$\frac{d}{dt}R(x(t)) \geq 0,$$

where

$$R(x) = \left[G(x, x') + \frac{1}{2\pi}\log\left|x - x'\right|\right]_{x'=x}.$$



The chemotactic collapse is formed in infinite time at a local maximum point of R on the boundary and then moves to that of local minimum.

In the radially symmetric case, the blowup in infinite time does not occur to $\lambda > 8\pi$. Thus, the whole dynamics has been clarified in this case.

II-3. Higher-dimensional case

1. There are type (I) and type (II) blowup in $3 \leq n \leq 9$ and $n \geq 11$, repsectively (Herrero-Medina-Velazquez, Senba, Mizoguchi-Senba, ...).

2. There are critical exponent and threshold-like phenomena for the porous medium diffusion (Luckhaus-Sugiyama, ...).

II-4. Perturbed nonlinearity

1. There is a formation of collapse (Kurokiba-Senba-S.).

2. There is an ill-posedness because of the multiple existence of the stationary state (Ishiwata-Kurokiba-Ogawa).

II-5. The other reduction

Full system of chemotaxis is involved by the relazation time $\tau > 0$:

$$\begin{aligned}
&\varepsilon u_t = \nabla \cdot (\nabla u - u\nabla v) \\
&\tau v_t = \Delta v + u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T) \\
&\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \\
&\int_\Omega v = 0,
\end{aligned}$$

where

$\Omega \subset \mathbf{R}^2$: bounded domain

$\partial\Omega$: smooth

$\varepsilon, \tau > 0$: constans

ν : the outer normal vector.

There may be two reductions, $\tau = 0$ (simplified system) and $\varepsilon = 0$ (non-local parabolic equation).

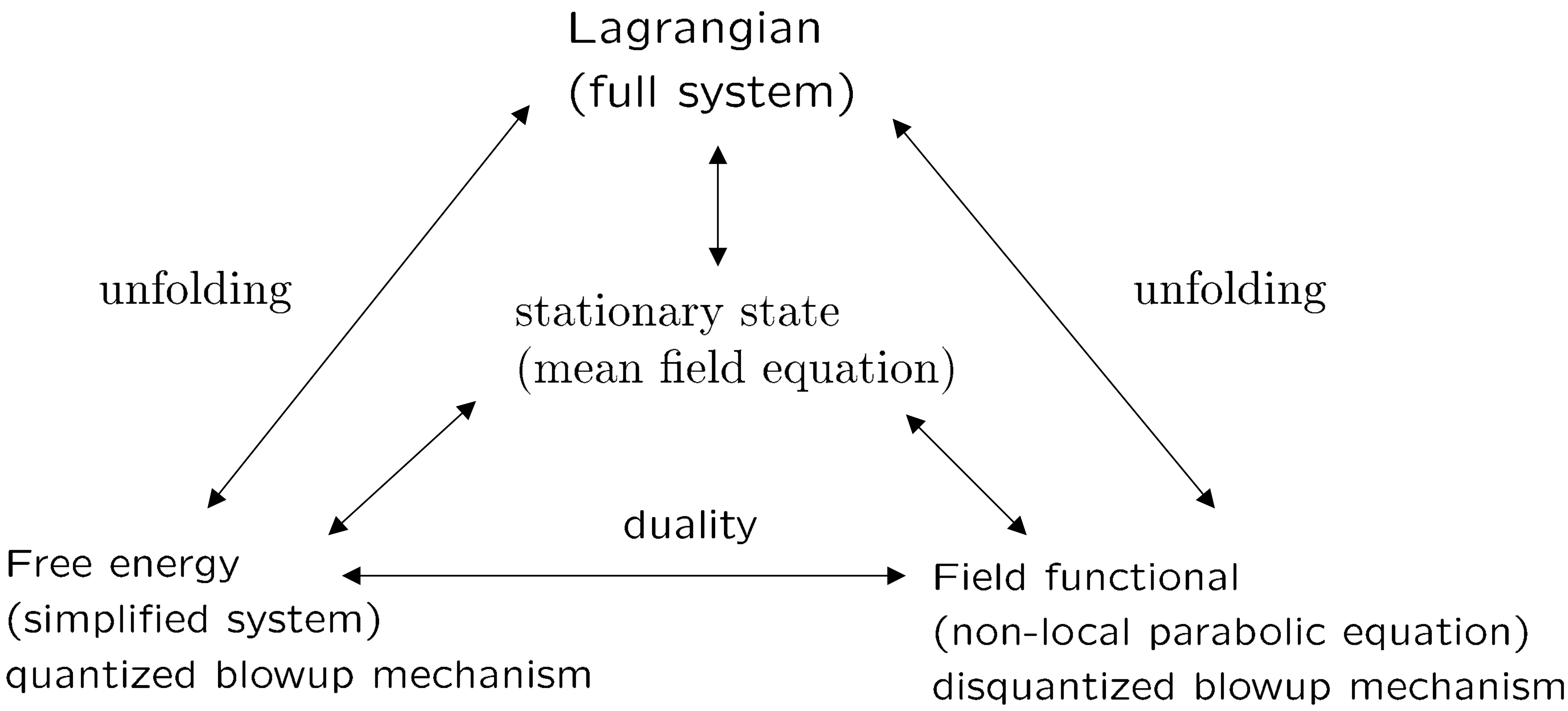
$\varepsilon = 0$: *non-local parabolic equation* (Wolansky 1997)

$$\begin{aligned} & v_t = \Delta v + \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right) \qquad \text{in } \Omega \times (0,T) \\ & \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \\ & \int_\Omega v = 0. \end{aligned}$$

1. lack of the weak formulation, i.e., compensated compactness via symmetrization is invalid.

2. dis-quantized blowup mechanism in some case (Kavallalis-S. preprint).

Dual variation guarantees the equivalence of the particle density and the field distribution in the stationary state, while this is not the case of non-stationary formulations.



Some other systems share the same stationary state. If $\Omega = S^2$, $\lambda = 8\pi$,

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda \left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|} \right) \qquad \text{in } \Omega \times (0,T)$$

is a **normalized Ricci flow**. The general case of the compact Riemann surface Ω and $\lambda > 0$ still describes several physical phenomena:

1. expansion of a thermolized electron cloud
2. central limit approximation to Carleman's model of the Boltzmann equation
3. thin liquid film.

We obtain the global existence for any $\lambda > 0$ and ω-limit convergence to the stationary state for $\lambda < 8\pi$ (Kavallais-S. in preparation).

II-6. Tumour growth models

Since the formation of field is restricted to the cells, the elliptic equation is replaced by ODE in tumour growth equations.

Othmer-Stevens' model

$$
\begin{aligned}
& p_t = D\nabla \cdot (\nabla p - p\nabla \log \Phi(w)) \\
& w_t = F(w,p) && \text{in } \Omega \times (0,T) \\
& \frac{\partial p}{\partial \nu} - p\frac{\partial}{\partial \nu}\log w = 0 && \text{on } \partial\Omega \times (0,T)
\end{aligned}
$$

on the chemotaxis aggregation of myxobacteria reads;

$$
\begin{aligned}
& p_t = D\nabla \cdot (\nabla p - ap\nabla v) \\
& v_t = p && \text{in } \Omega \times (0,T) \\
& \frac{\partial p}{\partial \nu} - p\frac{\partial v}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T)
\end{aligned}
$$

if $F = wp$ and $\Phi = w^a$, where $v = \log w$. It has a Lyapunov function, and the solution exists gloablly in time if $n = 1$ (Rascle 1979).

It is also reduced to the evolution equation with strong dissipation,

$$v_{tt} = D\Delta v_t - aD\nabla \cdot (v_t \nabla v).$$

In the case of $n = 1$, there are solutions satisfying

1. $\lim_{t\uparrow\infty} \inf w(\cdot, t) = +\infty$ if $a = -1$ (Levine-Sleenman 1997)
2. $T_{\max} < +\infty$ if $a = 1$ (Levine-Sleenman 1997, Yang-Chen-Liu 2001).

The first case occurs to any n if $\Phi(w)$ is saturated (Kubo-S. 2004).

Anderson-Chaplain model

$$
\begin{aligned}
& n_t = D\Delta n - \nabla \cdot (\chi(c) n \nabla c) - \rho_0 \nabla \cdot (n \nabla f) \\
& f_t = \beta n - \gamma_0 n f \\
& c_t = -\eta n c \qquad\qquad \text{in } \Omega \times (0,T) \\
& \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial f}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)
\end{aligned}
$$

describes the **tumour angiogenesis**, where

$$
\chi(c) = \frac{\chi_0}{1+\alpha c}.
$$

For small initial data the solution converges to the ODE solution, while there is a Lyapunov function in the case $\beta - \gamma_0 f < 0$ (Kubo-S.-Hoshino 2005). Then, we obtain the global existence of the solution if $n = 1$ (S.-R.Takahashi).

II-7. Structure of the dual variation

1. Several model (C) equations of the phase field theory are combinations of the model (A) and model (B) equations using *Lagrangian* ...Landau-Ginzburg and Penrose-Fife theories for **critial phenomena** such as phase transition, phase separation, hysteresis.

2. There is *semi-unfolding-minimality* in several fundamental equations in **fluid mechanics** ... Euler-Poisson, MHD.

T.S., *Mean Field Theories and Dual Variation*, Elsevier, submitted.

II-8. Quantization

1. Energy quantization...harmonic map, H-surface, critical Sobolev exponent.

2. Mass quantization...self-dual gauge theory, stationary vortex point/filament mean field, plasma confinement.

3. L^2 quantization...gauge-invariant Schrödinger equation.

There are different principles with common phenomena?

S.-F.Takahashi, *Nonlinear eigenvalue problem with quantization*, preprint.

II-9. Emergence observed in the simplified system of chemotaxis for $n = 2$

If $t_k \uparrow T$, $b > 0$, and $r(t_k) \downarrow 0$ satisfy

$$\limsup_{k\to\infty} r(t_k)^2 \left\| u(t_k) \right\|_{L^\infty(\Omega\cap B(x_0,2br(t_k)))} < +\infty,$$

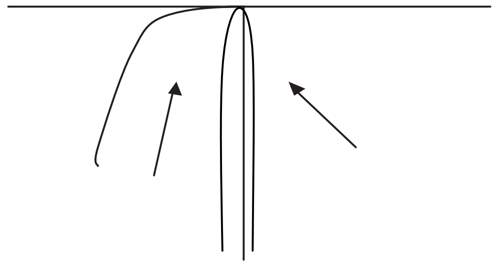
then it follows that

$$\lim_{k\to\infty} \mathcal{F}_{x_0,br(t_k)}(u(t_k)) = +\infty,$$

where

$$\mathcal{F}_R(u) = \int_{B(x_0,R)} u(\log u - 1) \\ -\frac{1}{2} \int\int_{\Omega\cap B(x_0,R)\times\Omega\cap B(x_0,R)} G(x,x')u \otimes u dx dx'$$

stand for the local free energy around the blowup point x_0. Thus, the **free energy is enclosed around the collapse in space and time** (emergence).



Parabolic envelope =
infinitely large parabolic region

Hyper-parabola =
infinitely small parabolic region

Definig the blowup rate $r(t_k) \downarrow 0$ by

$$\lim_{k\to\infty} r(t_k)^2 \, \|u(t)\|_{L^\infty(\Omega\cap B(x_0,br(t_k))} < +\infty$$

for any $b > 0$ and

$$\lim_{b\uparrow\infty} \limsup_{k\to\infty} \left| \|u(t_k)\|_{L^1(\Omega\cap B(x_0,br(t_k))} - m_*(x_0) \right|$$
$$= 0,$$

we have summarized that

"**Mass and entropy are exchanged at the wedge of the blowup envelope, creating a clean, quantized self**".

- S., *Free Energy and Self-Interacting Particles*, Birkhauser, 2005.

III. Two aspects of self-organization

Original Keller-Segel system:

$$
\begin{aligned}
u_t &= \nabla \cdot (d_1(u,v)\nabla v) - \nabla \cdot (d_2(u,v)\nabla v) \\
v_t &= d_v \Delta v - k_1 uw + k_{-1} p + f(v)u \\
w_t &= d_w \Delta w - k_1 vw + (k_{-1} + k_2)p + g(v,w)u \\
p_t &= d_p \Delta p + k_1 vw - (k_{-1} + k_2)p
\end{aligned}
$$

Tsujikawa-Mimura model:

$$
\begin{aligned}
u_t &= d_u \Delta u - \nabla \cdot (u \nabla \chi(\rho)) + f(u) \\
\rho_t &= d_\rho \Delta \rho - c\rho + du
\end{aligned}
$$

are not provided with the Lyapunov function.

Model (A) or (B) equations describe thermally closed systems and are different from **Prigogine's** entropy, inner energy, and material density **balance equation** describing the dissipative structure:

$$\frac{\partial s_v}{\partial t} = -\nabla \cdot J_s + \sigma$$
$$\frac{\partial c_i}{\partial t} = -\nabla \cdot J_i + \nu_i J_{ch}$$
$$\frac{\partial q_v}{\partial t} = -\nabla \cdot J_q.$$

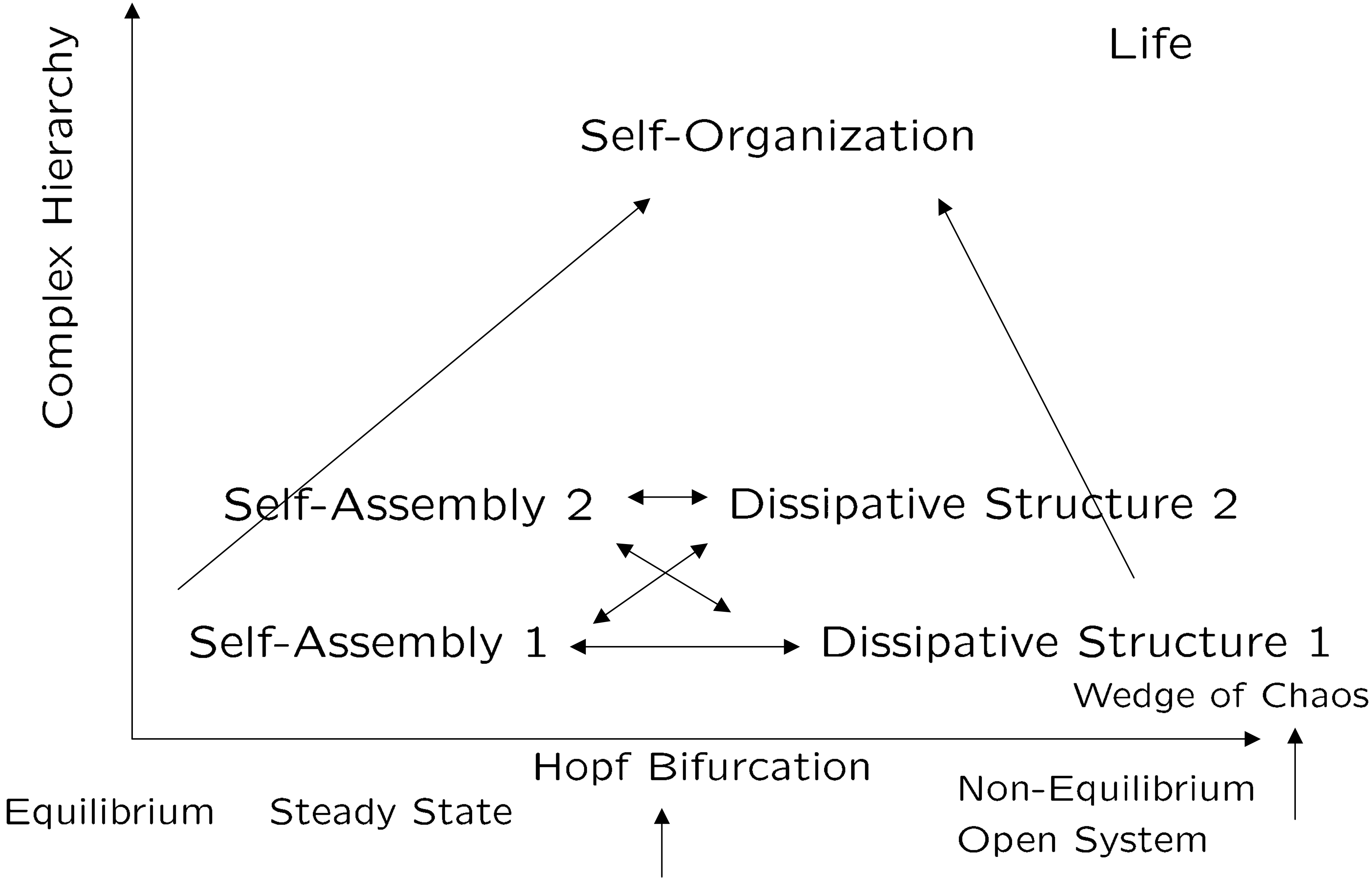
Dissipative Structure

Far-From-Equilibrium
Top Down
Travelling wave
Spiral Wave
Periodic Strcture
Self-Similar Evolution
Open

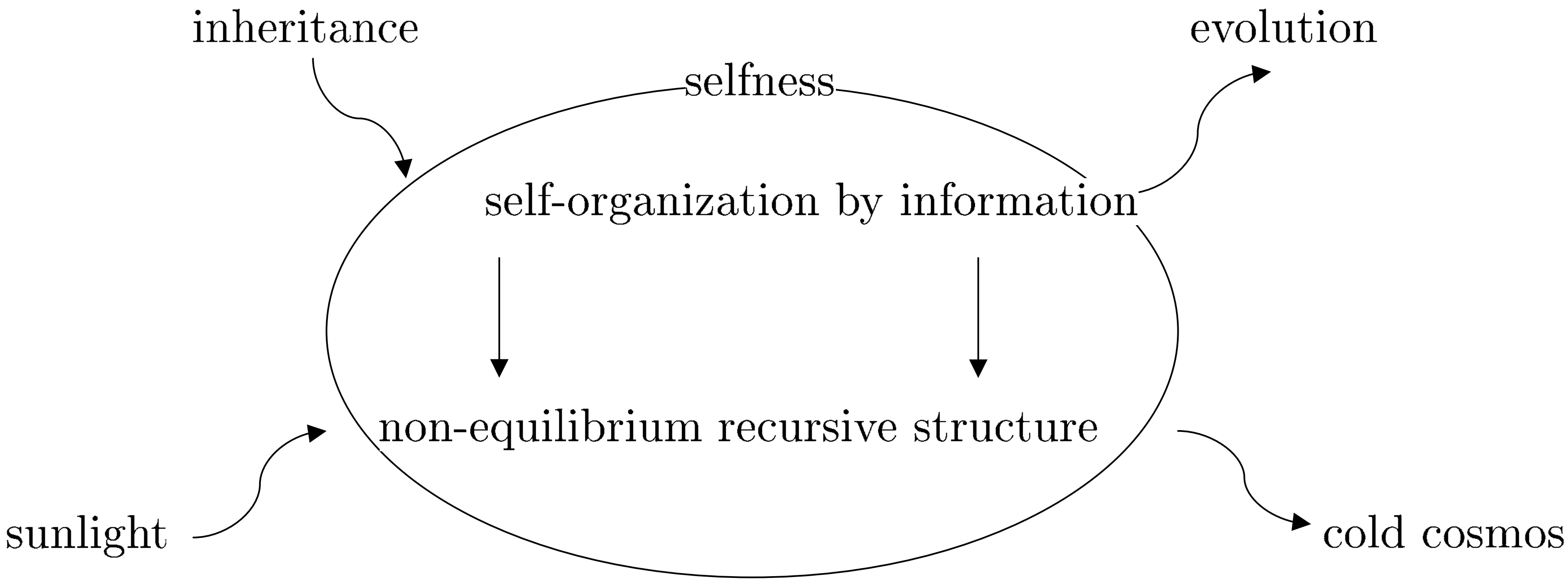
Self-assembly

Near-From-Equilibrium
Botom up
Condensate
Collapse
Stationary
Closed
Variational Structure

T. Yamaguchi et. al. (2004)



The above described physically recursive structure sustains the informatic organization and the complex evolution of life.



H. Tanaka, *Life and Complex Systems*, Baifukan, Tokyo, 2002.

Summary: self-assembly induced by chemotaxis

1. Formation of collapse with quantized mass is suggested by the mean field hiearchy.

2. It is a localization of the blowup threshold, while the proof is done by the blowup analysis using self-similarity and the parabolic envelope derived from the weak formulation.

3. Here, an important factor of self-assembly, the free energy transmission (emergence) is observed as a consequence of the collapse mass quantization.

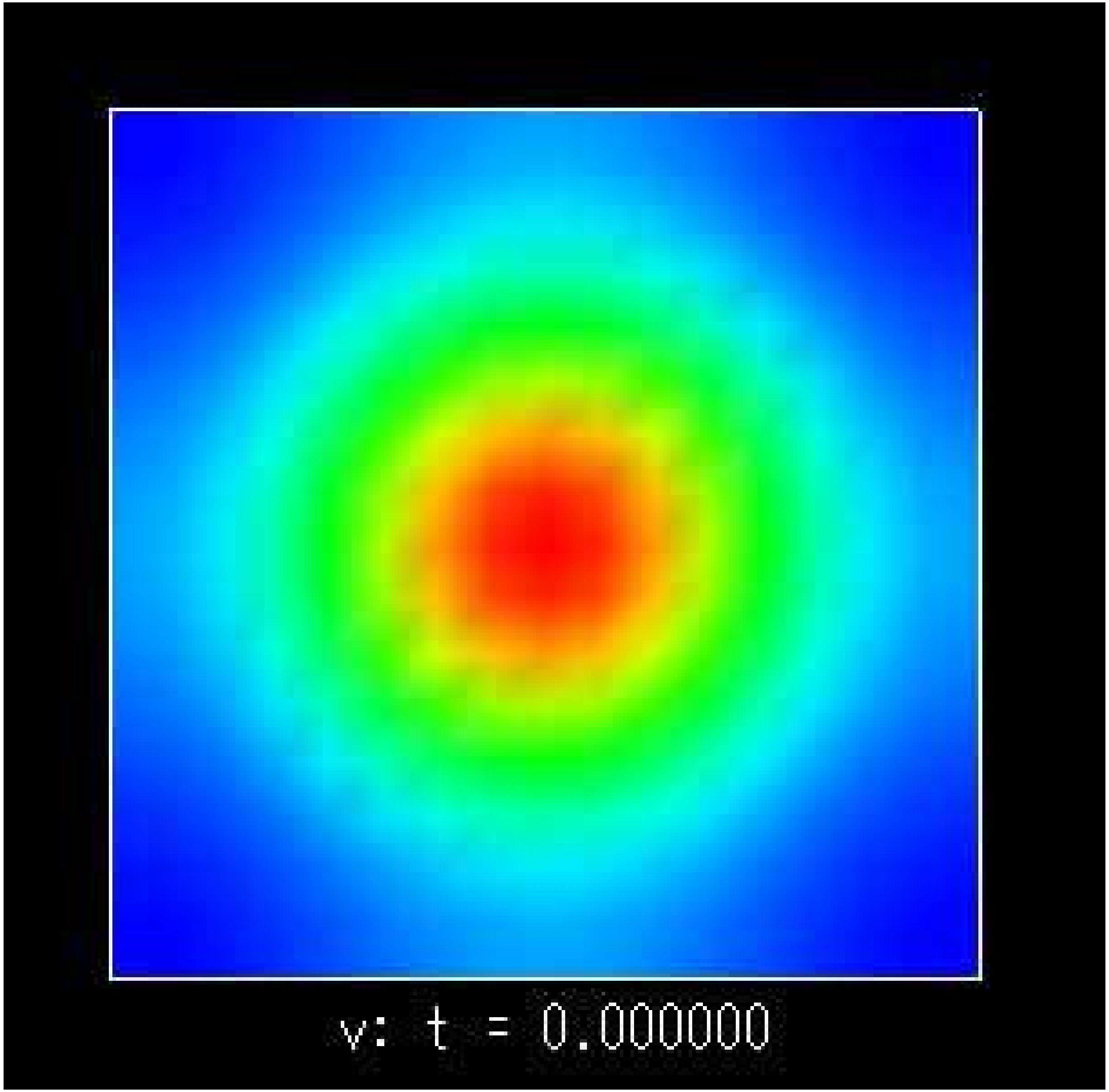
IV Numerical simulations

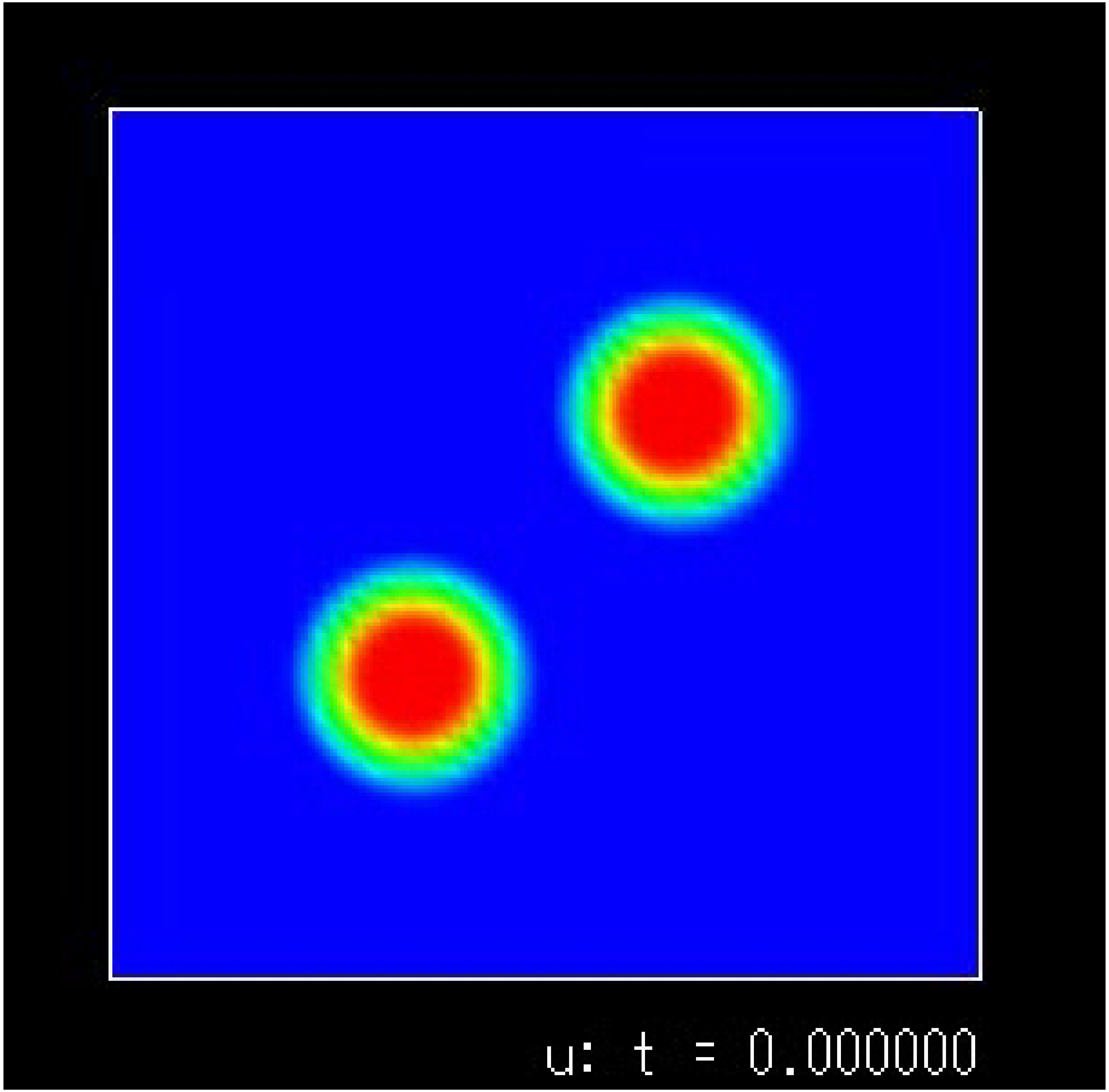
1. simplified system of chemotaxis (N. Saito)

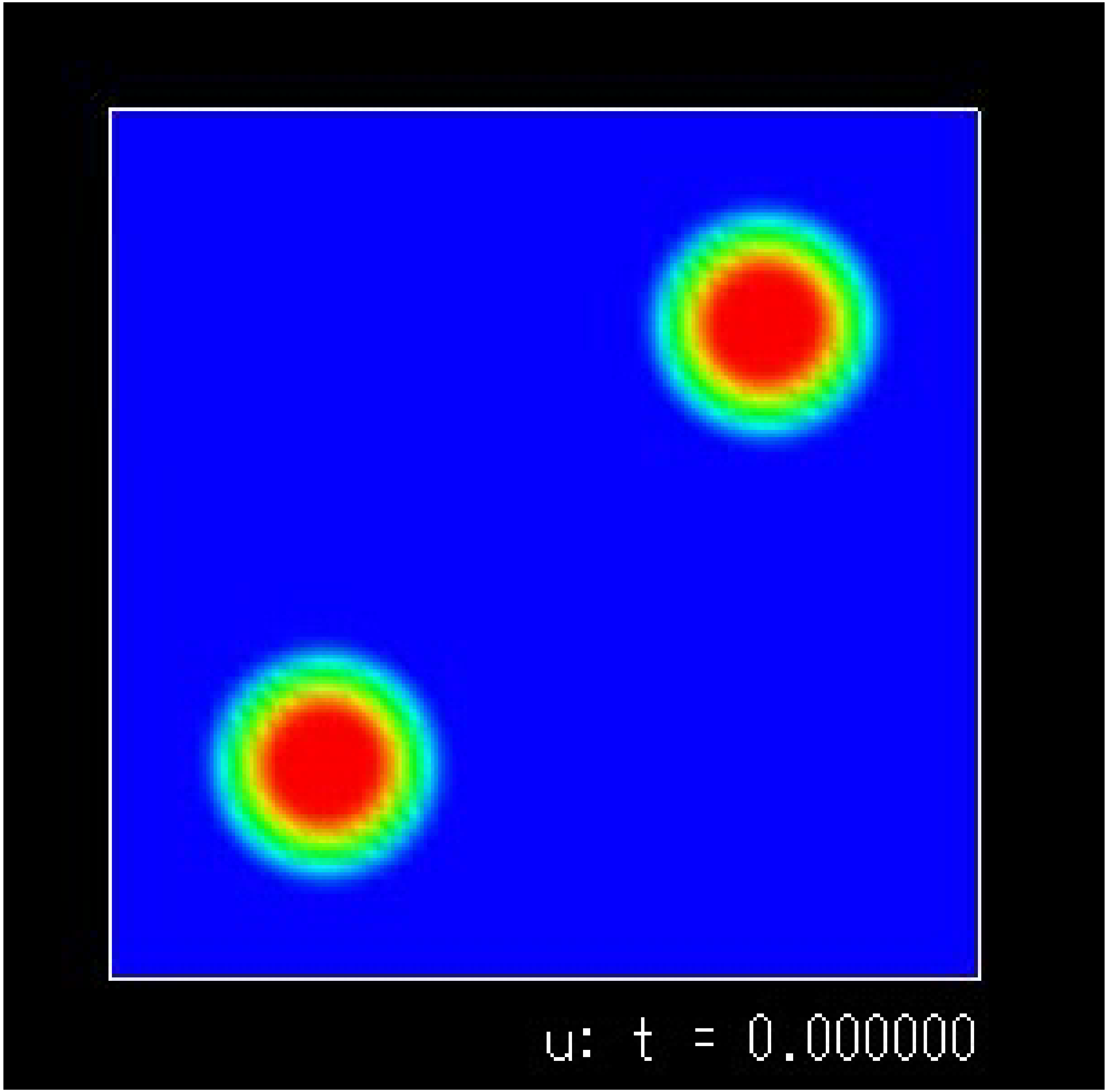
2. Chaplain-Anderson's avascular tumour model (K. Hayashi)

$$
\begin{aligned}
& n_t = \nabla \cdot (d_n \nabla n - \gamma n \nabla c) \\
& f_t = d_f \Delta f + \alpha n - \beta f \\
& c_t = -\eta f c \qquad \text{in } \Omega \times (0,T) \\
& d_n \frac{\partial n}{\partial \nu} - \gamma \frac{\partial n}{\partial \nu} = d_f \frac{\partial f}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T),
\end{aligned}
$$

where n, f, and c are the tumour cell density, the MDE (matrix degrading), and ECM (extra cell matrix), respectively.


v: t = 0.000000


u: t = 0.000000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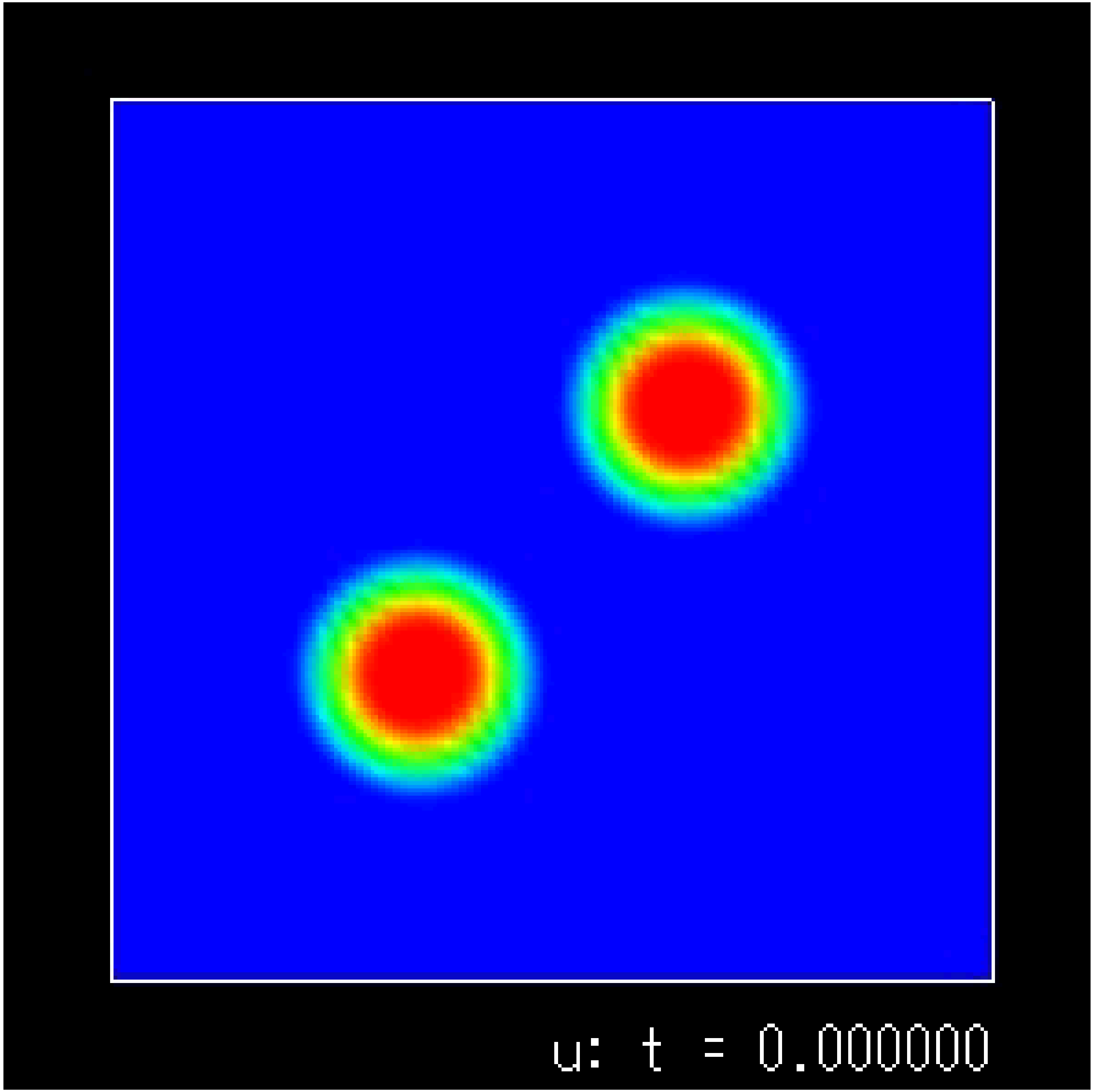
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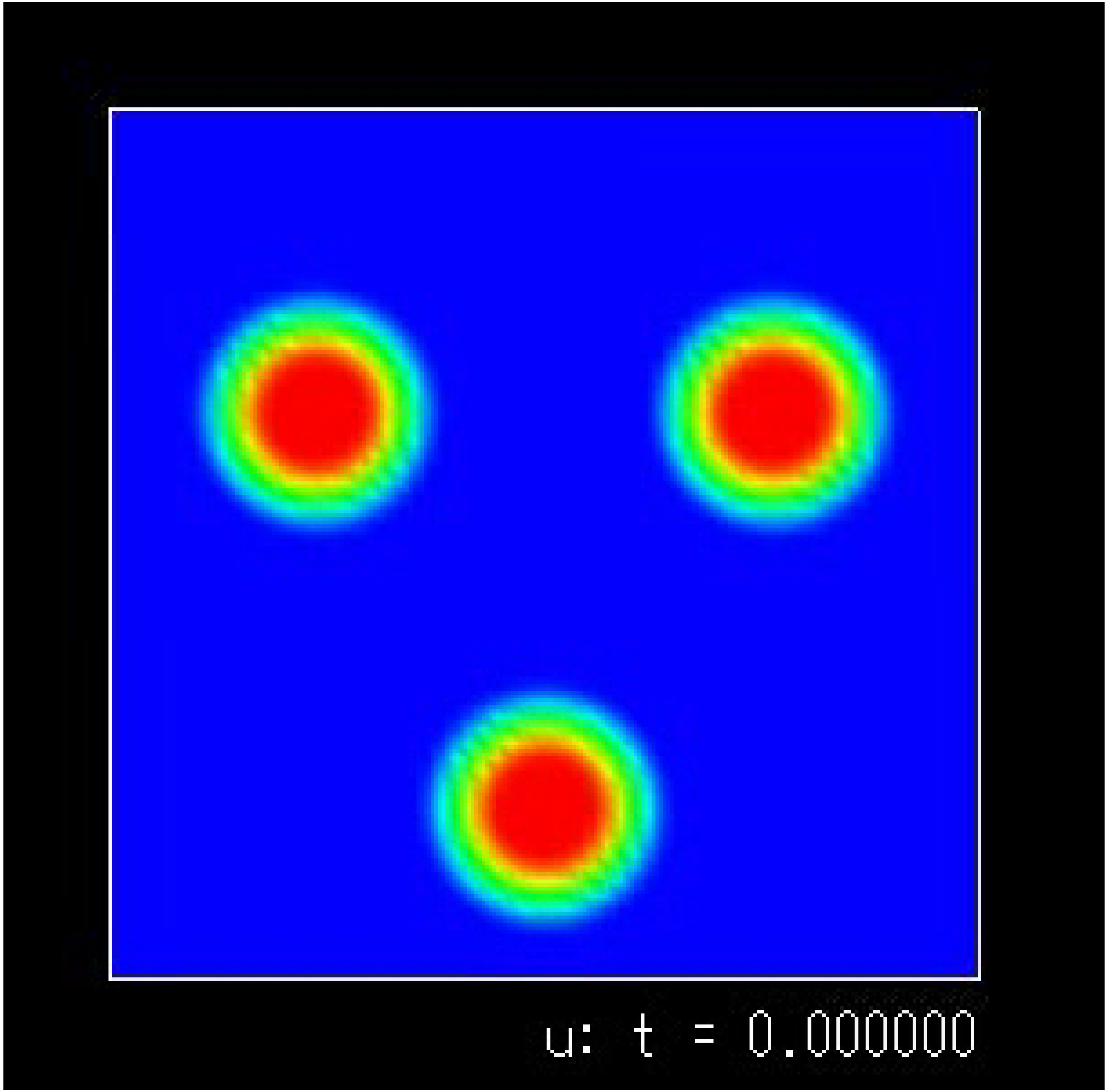
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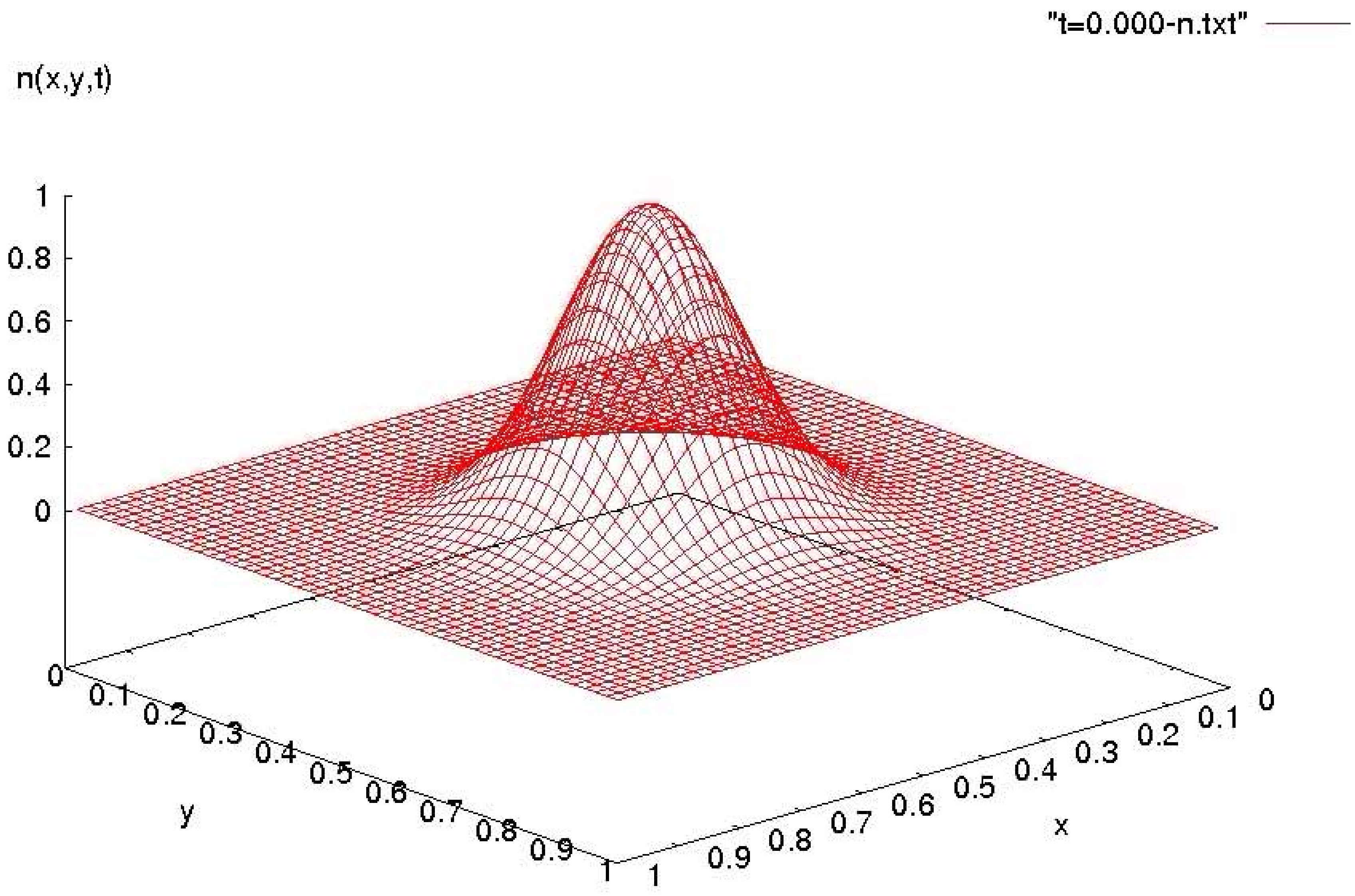
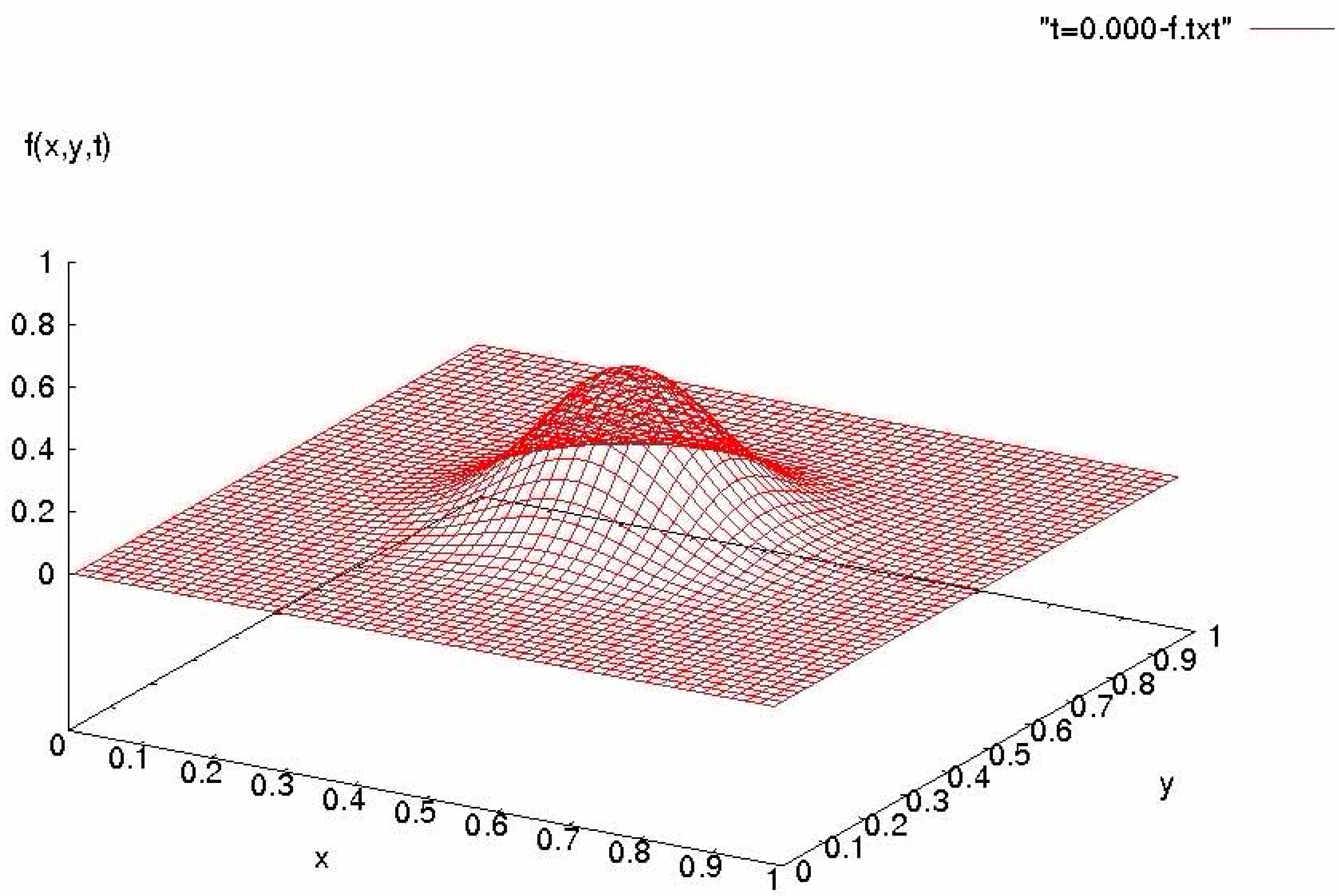


"t=0.000-n.txt"
n(x,y,t)
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x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f(x,y,t)
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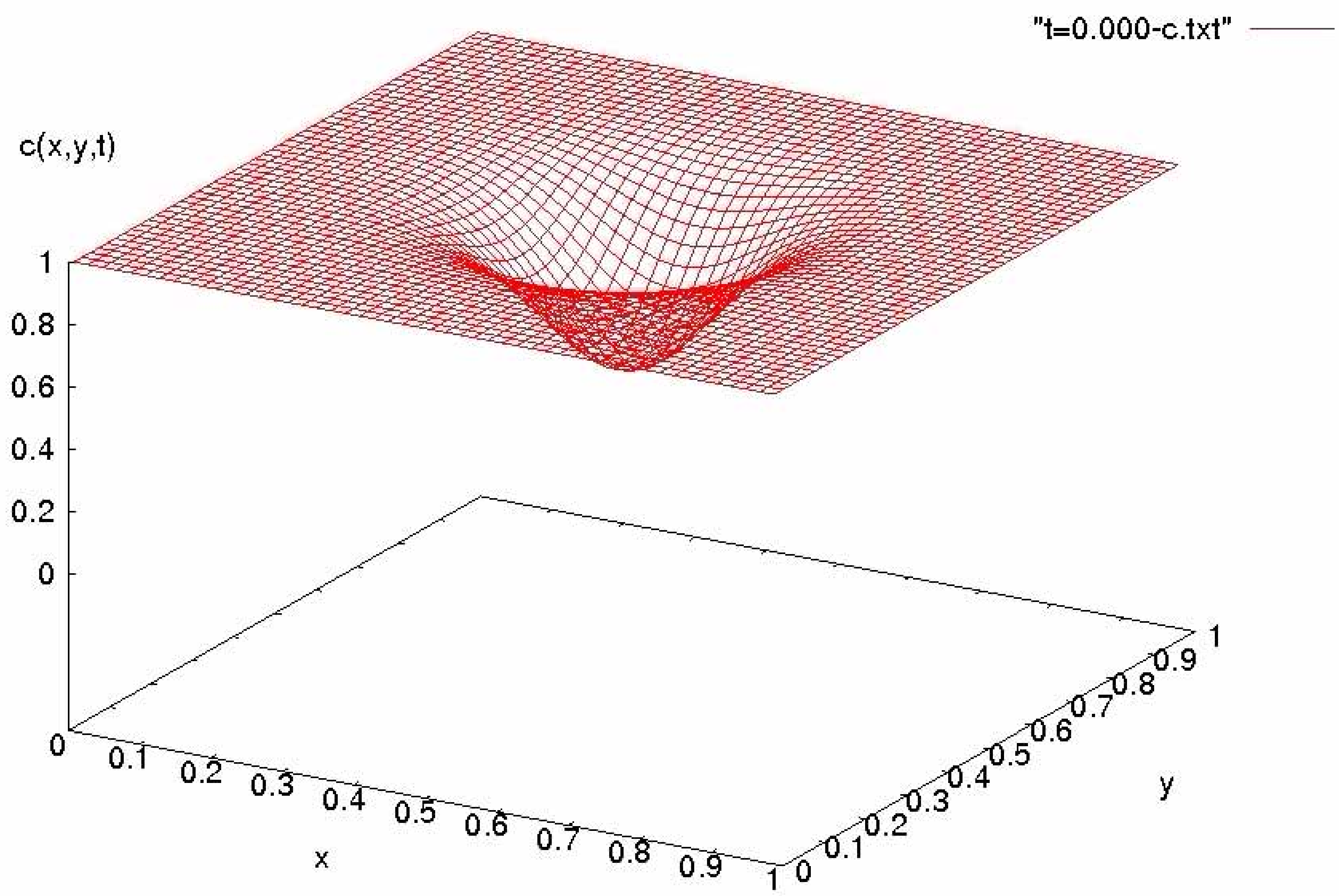
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
y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0.8
0.6
0.4
0.2
0
0
0.1
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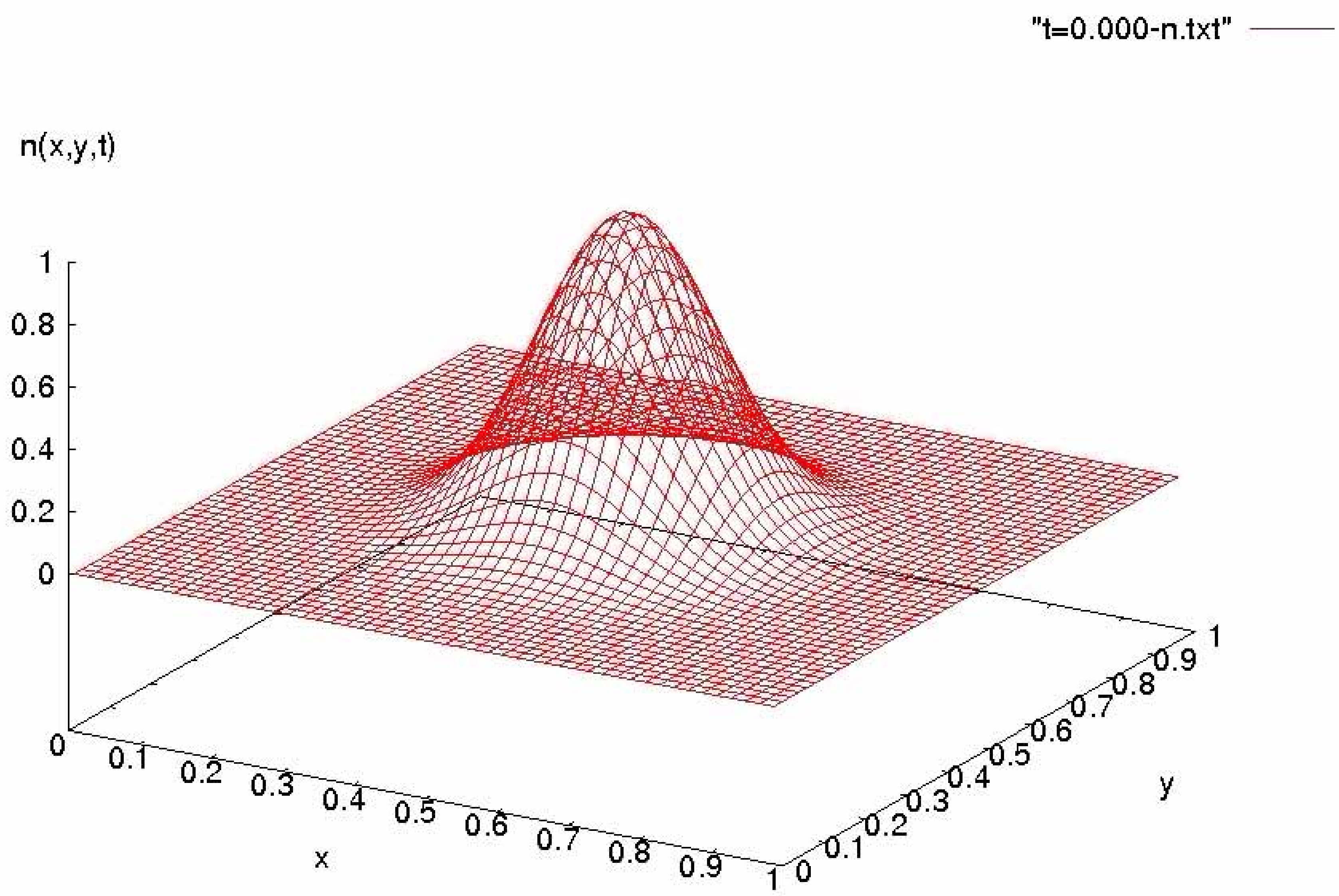
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
y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n(x,y,t)
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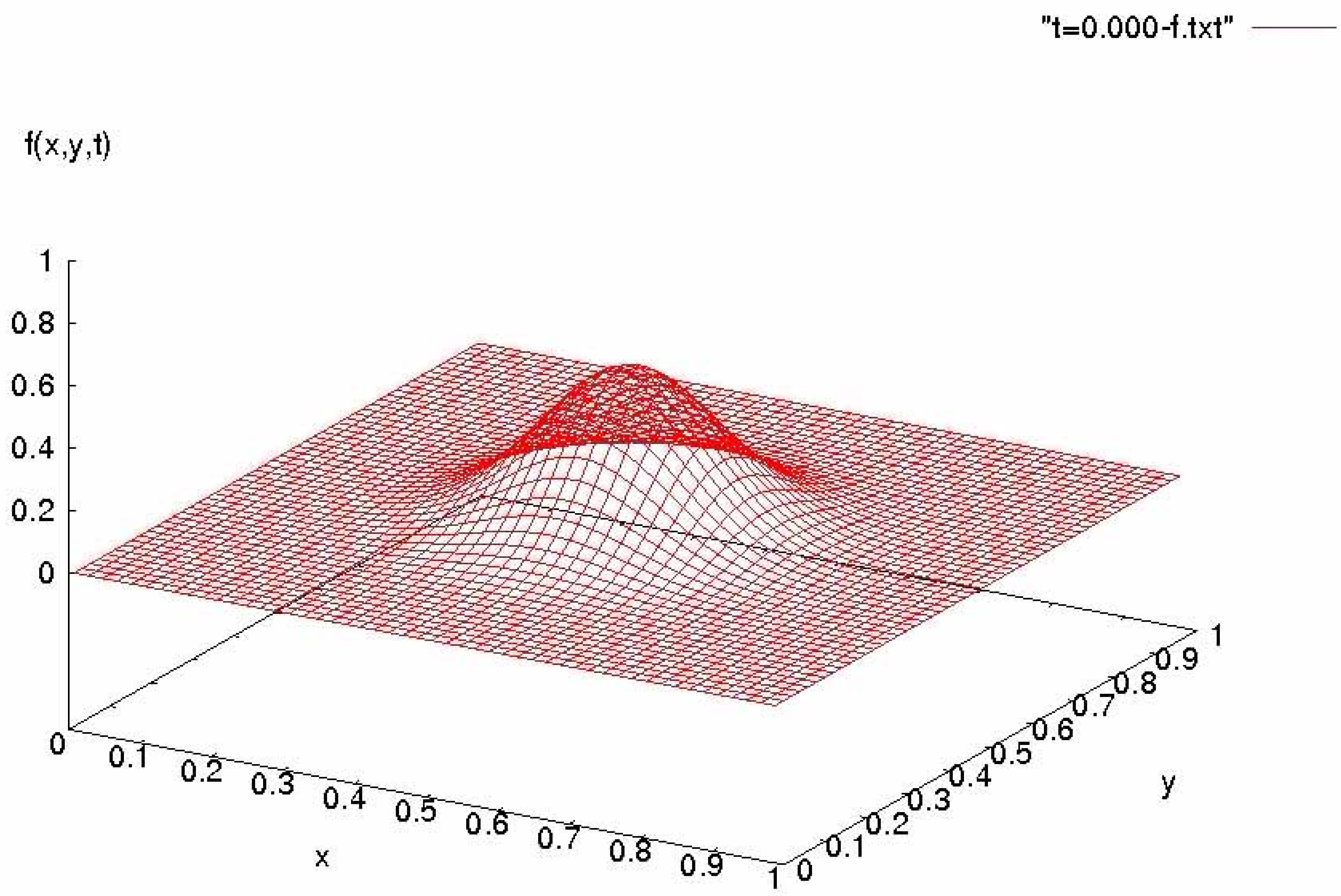
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
y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f(x,y,t)
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0.8
0.6
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0.2
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0.4
0.5
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0.9
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x
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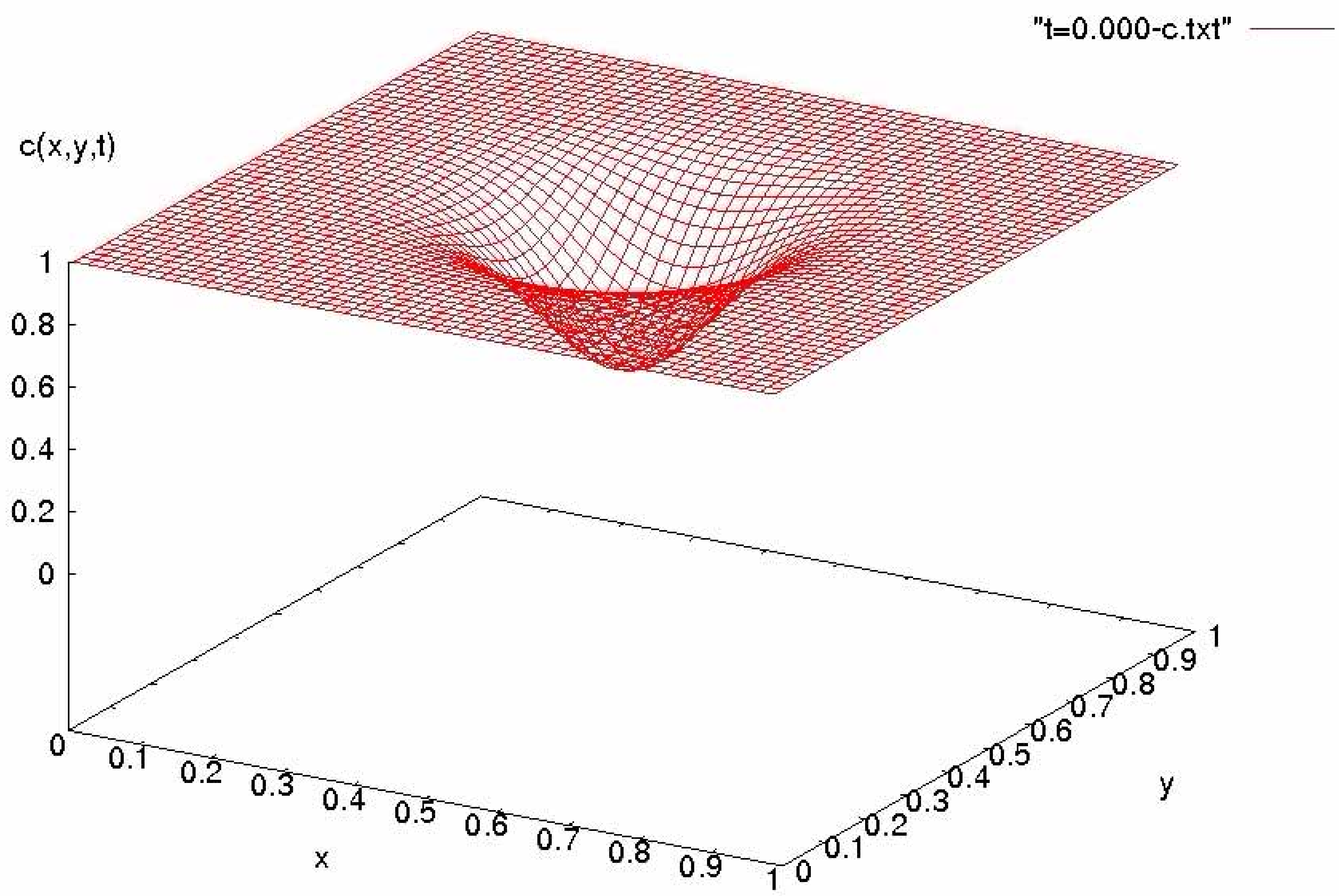
0.6
0.7
0.8
0.9
1
y



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c(x,y,t)
1
0.8
0.6
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0.9
1
x
0
0.1
0.2
0.3
0.4
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0.6
0.7
0.8
0.9
1
y