有界領域上のホッジ分解

2026. 01. 30

鈴木　貴

Differential Form

Notation

$\Omega \subset \mathbf{R}^n,\ n \geq 2$ bounded domain $\partial\Omega$ smooth

$0 \leq k \leq n$ $\mathcal{D}'_k$ k forms on Ω with distributional coefficients

$d : \mathcal{D}'_k \to \mathcal{D}'_{k+1},\ 0 \leq k \leq n-1$ exterior derivative

$\wedge : \mathcal{D}'_k \times \mathcal{D}'_\ell \to \mathcal{D}'_{k+\ell},\ k, \ell \geq 0,\ k+\ell \leq n$ exterior product

$\delta = (-1)^{n(k+1)+1} * d* : \mathcal{D}'_k \to \mathcal{D}'_{k-1},\ 1 \leq k \leq n$ co-derivative

$* : \mathcal{D}'_k \to \mathcal{D}'_{n-k},\ 0 \leq k \leq n$ Hodge operator

↔ transpose w.r.t. metric

$$*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \mathrm{sgn}\,\sigma \cdot dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_n}$$

$$** = (-1)^{k(n-k)} \qquad \sigma : (i_1, \cdots, i_k, i_{k+1}, \cdots, i_n) \mapsto (1, \cdots, n)$$

Remark $\mathcal{D}'_k = \{0\},\ k \leq -1,\ k \geq n+1$ $\quad dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0,\ k \geq n+1$

$d = 0$ on $\mathcal{D}'_n$ $\quad \delta = 0$ on $\mathcal{D}'_0$ $\quad d^2 = 0,\ \delta^2 = 0$

Hodge decomposition

Result

$\Lambda_k,\ 0 \le k \le n$ elements in $\mathcal{D}'_k$ of which coefficients are smooth on $\overline{\Omega}$

$\vec{\nu} = (\nu^i)$ outer unit normal vector on $\partial\Omega$ ⟷ $\nu = \sum_i \nu^i dx_i$ smooth 1 form on $\partial\Omega$

$$H_k = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge *h|_{\partial\Omega} = 0\}$$ harmonic forms

Theorem 1 $1 \le k \le n-1$ $\forall u \in \Lambda_k,\ \exists (h, \eta) \in H_k \times (\Lambda_k \cap H_k^{\perp})$ $\nu \wedge *\eta|_{\partial\Omega} = 0,\ \nu \wedge *d\eta|_{\partial\Omega} = 0$

$$u = h + \delta d\eta + d\delta\eta$$

tangential derivative

Proposition 1 $\eta \in \Lambda_k,\ \nu \wedge *\eta|_{\partial\Omega} = 0 \ \Rightarrow\ \nu \wedge *\delta\eta|_{\partial\Omega} = 0$

Corollary 1 (Hodge) $1 \le k \le n-1$ $\forall u \in \Lambda_k,\ \exists (h, \omega, p) \in H_k \times \Lambda_{k+1} \times \Lambda_{k-1}$

$$d\omega = 0,\ \nu \wedge *\omega|_{\partial\Omega} = 0,\quad \delta p = 0,\ \nu \wedge *p|_{\partial\Omega} = 0$$

$$u = h + \delta\omega + dp$$

Proof $\omega = d\eta,\ p = \delta\eta$

Result

Corollary 1 $(H)_k$ ⟶ $\Lambda_k = H_k \oplus \delta W_k \oplus dV_k,\ 1 \le k \le n-1$

$$H_k = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge *h|_{\partial\Omega} = 0\}$$

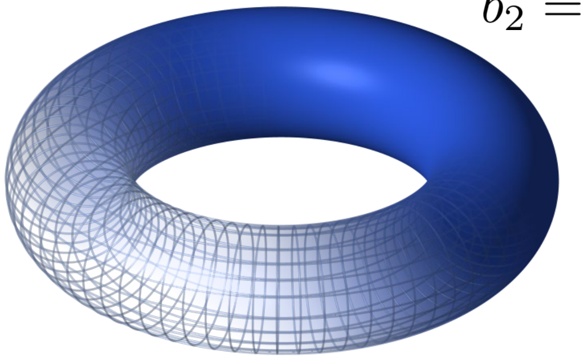
$$W_k = \{\omega \in \Lambda_{k+1} \mid d\omega = 0,\ \nu \wedge *\omega|_{\partial\Omega} = 0\}$$

$$V_k = \{p \in \Lambda_{k-1} \mid \delta p = 0,\ \nu \wedge *p|_{\partial\Omega} = 0\}$$

$$\breve{\Lambda}_k \overset{\text{def}}{=} \{\theta \in \Lambda_k \mid \nu \wedge *\theta|_{\partial\Omega} = 0\}$$

$b_1 = 1$

$b_2 = 0$



Theorem 2 $H_k \cong H^k_{DR}(\Omega)$ de Rham cohomology

Remark $H^k_{DR}(\Omega) \cong H^k(\Omega, \mathbf{R})$ (de Rham ≅ singular cohomology)

$\dim H_k = b_k$ $\quad k$ th Betti number

$$\Lambda_k = H_k^* \oplus \delta V_k^* \oplus dW_k^*,\ 1 \le k \le n-1$$

$$H_k^* = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge h|_{\partial\Omega} = 0\}$$

$$V_k^* = \{p \in \Lambda_{k+1} \mid dp = 0,\ \nu \wedge p|_{\partial\Omega} = 0\}$$

$$W_k^* = \{\omega \in \Lambda_{k-1} \mid \delta\omega = 0,\ \nu \wedge \omega|_{\partial\Omega} = 0\}$$

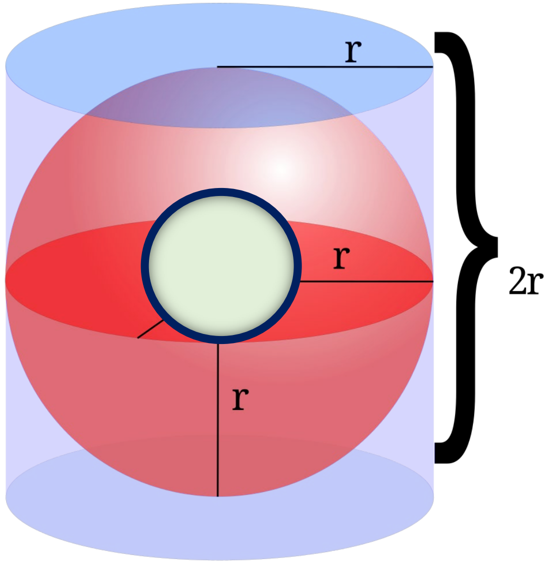
$H_k^* \cong H^{n-k}_{DR}(\Omega)$

$\dim H_k^* = b_{n-k}$

c.f. Poincare duality

$b_1 = 0$

$b_2 = 1$



Ω closed manifold ⟶ $\dot{\Lambda}_k = \dot{H}_k \oplus \delta\dot{\Lambda}_{k+1} \oplus d\dot{\Lambda}_{k-1}$

Corollary 2 (Helmholtz) $\forall u \in \Lambda_k,\ 1 \le k \le n-1$

$(Hd)_k$ $\exists (v,p) \in \Lambda_k \times \Lambda_{k-1},\ \delta v = 0,\ \nu \wedge *v|_{\partial\Omega} = 0$

$u = v + dp$

$(H\delta)_k$ $\nu \wedge *u|_{\partial\Omega} = 0 \Longrightarrow$

$\exists (z,\omega) \in \Lambda_k \times \Lambda_{k+1},\ dz = 0,\ \nu \wedge *z|_{\partial\Omega} = 0$

$u = z + \delta\omega$

Proof $u = h + \delta d\eta + d\delta\eta$ Hodge

$\nu \wedge *\eta|_{\partial\Omega} = 0,\ \nu \wedge *d\eta|_{\partial\Omega} = 0$

$v = h + \delta\omega,\ \omega = d\eta$

$z = h + dp,\ p = \delta\eta$

$\theta \in \Lambda_k,\ \nu \wedge *\theta|_{\partial\Omega} = 0 \Rightarrow \nu \wedge *\delta\theta|_{\partial\Omega} = 0$

Application $n = 3$ $u = (u^i) \in \mathcal{X}(\overline{\Omega})$ Result

$\longleftrightarrow$ $u = u^1 dx_1 + u^2 dx_2 + u^3 dx_3$

$u = h + \text{rot } \omega + \text{grad } p$

$\text{div } h = 0,\ \text{rot } h = 0$

$\vec{\nu} \cdot h|_{\partial\Omega} = 0$

$\text{div } \omega = 0,\ \vec{\nu} \times \omega|_{\partial\Omega} = 0$

$u = v + \text{grad } p$ scalar potential

$\text{div } v = 0,\ \vec{\nu} \cdot v|_{\partial\Omega} = 0$

$\longleftrightarrow$ $u = u^1 dx_2 \wedge dx_3 + u^2 dx_3 \wedge dx_1 + u^3 dx_1 \wedge dx_2$

$u = h + \text{grad } \omega + \text{rot } p$

$\text{div } h = 0,\ \text{rot } h = 0$

$\vec{\nu} \times h|_{\partial\Omega} = 0,\ \omega|_{\partial\Omega} = 0$

$\text{div } p = 0, \vec{\nu} \cdot p|_{\partial\Omega} = 0$

$u = z + \text{rot } p$ vector potential

$\text{rot } z = 0,\ \vec{\nu} \times z|_{\partial\Omega} = 0$

$u = h + \text{rot rot } \eta - \text{grad div } \eta$

$\text{div } h = 0,\ \text{rot } h = 0$

$\vec{\nu} \cdot h|_{\partial\Omega} = 0$

$\vec{\nu} \times \text{rot } \eta|_{\partial\Omega} = 0$

$\vec{\nu} \times h|_{\partial\Omega} = 0$

$\text{div } \eta|_{\partial\Omega} = 0$

$\vec{\nu} \cdot \text{rot } \eta|_{\partial\Omega} = 0$

L^r theory

Result

Hodge $L^r_k = H_k \oplus \delta W^{1,r}_k \oplus dV^{1,r}_k,\ 1 \le k \le n-1,\ 1 < r < \infty$

$n = 3$

Kozono-Yanagisawa 09

$$H_k = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge *h|_{\partial\Omega} = 0\}$$

$$W^r_k = \{\omega \in W^{1,r}_{k+1} \mid d\omega = 0,\ \nu \wedge *\omega|_{\partial\Omega} = 0\}$$

$$V^r_k = \{p \in W^{1,r}_{k-1} \mid \delta p = 0,\ \nu \wedge *p|_{\partial\Omega} = 0\}$$

$$\forall u \in L^r_k,\ \exists (h,\eta) \in H_k \times (W^{2,r}_k \cap H^\perp_k) \quad \nu \wedge *\eta|_{\partial\Omega} = 0,\ \nu \wedge *d\eta|_{\partial\Omega} = 0 \qquad u = h + \delta d\eta + d\delta\eta$$

Helmholtz 1 $\forall u \in L^r_k \quad \exists (v,p) \in L^{1,r}_k \times V^r_k,\ \delta v = 0,\ \nu \wedge *v|_{\partial\Omega} = 0 \qquad u = v + dp$

$n = 3$

Fujiwara-Morimoto 77

Helmholtz 2 $\exists (z,\omega) \in L^r_k \times W^r_k,\ dz = 0 \qquad u = z + \delta\omega$

$$u \in W^{1,r}_{k,\delta},\ \nu \wedge *u|_{\partial\Omega} = 0 \longrightarrow z \in W^{1,r}_{k,\delta},\ \nu \wedge *z|_{\partial\Omega} = 0$$

$$W^{1,r}_{k,\delta} = \{\theta \in L^r_k \mid \delta\theta \in L^r_{k-1}\}$$

weak trace

$d\delta$ regularity

$$\breve{W}^{1,r}_{k,d\delta} = \{\theta \in W^{1,r}_{k,\delta} \mid d\theta \in L^r_{k+1},\ \nu \wedge *\theta|_{\partial\Omega} = 0\} \subset W^{1,r}_k$$

strategy

$$L^2 \longrightarrow C^\infty \longrightarrow L^r$$

General result in geometry

Remark

Ω compact manifold with boundary

$$\Lambda_k = \tilde{H}_k \oplus \delta \tilde{W}_k \oplus d\tilde{V}_k,\ 1 \le k \le n-1$$

$$\tilde{H}_k = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0\}$$
$$\tilde{W}_k = \{\omega \in \Lambda_{k+1} \mid d\omega = 0,\ n\omega = 0\}$$
$$\tilde{V}_k = \{p \in \Lambda_{k-1} \mid \delta p = 0,\ tp = 0\}$$

tangential, normal, canonical inclusion

$$tp = j^* p,\ n\omega = \omega - t\omega,\ j : \partial\Omega \to \overline{\Omega}$$

$$\tilde{H}_k = \{h \in \tilde{H}_k \mid th = 0\} \oplus \{h \in \tilde{H}_k \mid h = \delta\gamma\}$$
$$\tilde{H}_k = \{h \in \tilde{H}_k \mid nh = 0\} \oplus \{h \in \tilde{H}_k \mid h = d\varepsilon\}$$

$$\forall v \in (H_k^*)^\perp,\ \exists \eta \in \Lambda_k$$
$$v = \delta d\eta + d\delta\eta,\ t\eta = 0,\ t\delta\eta = 0$$

Schwarz 85

$\Longrightarrow$

$$\Lambda_k = H_k^* \oplus \delta V_k^* \oplus dW_k^*,\ 1 \le k \le n-1$$

$$H_k^* = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ th = 0\}$$
$$V_k^* = \{p \in \Lambda_{k+1} \mid dp = 0,\ tp = 0\}$$
$$W_k^* = \{\omega \in \Lambda_{k-1} \mid \delta\omega = 0,\ t\omega = 0\}$$

dual form

c.f. $\Omega = \mathbf{R}^n_+ = \{(x', x_n) \mid x_n > 0\}$

$$u = \sum_{i_1 < \cdots < i_k} u^{i_1 \dots i_k} dx_{i_1} \wedge \cdots dx_{i_k} \Longrightarrow tu = \sum_{i_1 < \cdots < i_k < n} u^{i_1 \dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$\begin{aligned}\nu \wedge u &= (-dx_n) \wedge \sum_{i_1 < \cdots < i_k} u^{i_1 \dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= -\sum_{i_1 < \cdots < i_k < n} u^{i_1 \dots i_k} dx_n \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_k})\end{aligned}$$

$$tu = 0 \Longleftrightarrow \nu \wedge u|_{\partial\Omega} = 0$$
$$nu = 0 \Longleftrightarrow \nu \wedge *u|_{\partial\Omega} = 0$$

Gauss formula

Lemma $(\theta,\omega)\in\Lambda_k\times\Lambda_{k+1},\ 0\le k\le n-1$

$$\int_\Omega (d\theta,\omega)\ dx = \int_{\partial\Omega}(\nu\wedge\theta,\omega)\ ds + \int_\Omega(\theta,\delta\omega)\ dx$$

Proof Stokes formula

$$\int_\Omega (d\theta,\omega)\ dx = \int_{\partial\Omega}\theta *\wedge\omega + \int_\Omega(\theta,\delta\omega)\ dx$$

vector area element

$$\vec{\nu}ds = (*dx_1,\cdots,*dx_n)^T$$

S.-Tsuchiya 2023

$$\Longrightarrow\quad (\nu\wedge\theta,\omega)ds = \theta\wedge*\omega \quad \square$$

$$(\theta,\beta) = (-1)^k\sum_\ell\sum_{i_2<\cdots<i_{k+1}}\nu^\ell\tilde{\omega}^{\ell i_2\cdots i_{k+1}}\theta^{i_2\cdots i_{k+1}}$$

$$\sum_\ell\sum_{i_2<\cdots<i_{k+1}}\nu^\ell\tilde{\omega}^{\ell i_2\cdots i_{k+1}}\theta^{i_2\cdots i_{k+1}} = \sum_{i_1<i_2<\cdots<i_{k+1}}\omega^{i_1\cdots i_{k+1}}\zeta^{i_1\cdots i_{k+1}} = (\nu\wedge\theta,\omega)$$

$$\omega = \sum_{i_1<\cdots<i_{k+1}}\omega^{i_1\cdots i_{k+1}}dx_{i_1}\wedge\cdots\wedge dx_{i_{k+1}}$$

$$\zeta^{i_1 i_2\cdots i_{k+1}} = \sum_{s=1}^{k+1}\nu^{i_s}(-1)^{s-1}\theta^{i_1\cdots i_{s-1}i_{s+1}\cdots i_{k+1}} = (\nu\wedge\theta)_{i_1 i_2\cdots i_{k+1}}$$

Lemma $(\nu\wedge\theta,\omega) = (-1)^k(*\theta,\nu\wedge*\omega)$ adjoint form

Proof $\nu\wedge*\omega = *\beta$ S.-Watanabe 24

$$\beta = \sum_{i_2<\cdots<i_{k+1}}(\hat{\omega}^{i_2\cdots i_{k+1}},\nu)dx_{i_2}\wedge\cdots\wedge dx_{i_{k+1}}\in\Lambda_k$$

$$\hat{\omega}^{i_2\cdots i_{k+1}} = \sum_\ell\tilde{\omega}^{i_2\cdots i_{k+1}\ell}dx_\ell \qquad \tilde{\omega}^{i_1\cdots i_{k+1}} = \mathrm{sgn}\ \sigma\cdot\omega^{i'_1\cdots i'_{k+1}}$$

$$\sigma : (i_1,\cdots,i_{k+1})\mapsto(i'_1,\cdots,i'_{k+1})$$

$$\{i'_1,\cdots,i'_{k+1}\} = \{i_1,\cdots,i_{k+1}\} \quad i'_1<\cdots<i'_{k+1}$$

$$\Rightarrow\ (*\theta,\nu\wedge*\omega) = (*\theta,*\beta) = (\theta,\beta)$$

$$\theta = \sum_{i_1<\cdots<i_{k+1}}\theta^{i_1\cdots i_{k+1}}dx_{i_1}\wedge\cdots\wedge dx_{i_{k+1}}$$

Corollary $$\int_\Omega (d\theta,\omega)\,dx = \int_\Omega (\theta,\delta\omega)\,dx + (-1)^k \int_{\partial\Omega} (*\theta, \nu\wedge *\omega)\,ds,\ (\theta,\omega)\in\Lambda_k\times\Lambda_{k+1},\ 0\le k\le n-1$$

tangential derivative

$$\theta\in\Lambda_k,\ 1\le k\le n,\ \nu\wedge*\theta|_{\partial\Omega}=0 \Longrightarrow \nu\wedge*\delta\theta|_{\partial\Omega}=0$$

Proof $k=1$ obvious $k\ge 2,\ \alpha\in\Lambda_{k-2}$

$$(-1)^k\int_{\partial\Omega}(\nu\wedge*\delta\theta,*\alpha)\ ds + \int_\Omega(\delta^2\theta,\alpha)\ dx = \int_\Omega(\delta\theta,d\alpha)\ dx$$

$$= \int_\Omega(\theta,d^2\alpha)\ dx - (-1)^{k-1}\int_\Omega(\nu\wedge*\theta,*d\alpha)\ ds$$

$$\Longrightarrow \int_\Omega(\nu\wedge*\delta\theta,*\alpha)\ ds = 0 \qquad \square$$

normal component ↔ codifferentiation

$$n\delta=\delta n$$

alternative proof

H^1 estimate

Gaffney's inequality

$$\|\nabla u\|_2^2 \le C\left(\|du\|_2^2 + \|\delta u\|_2^2 + \|u\|_2^2\right) \quad u \in \breve{\Lambda}_k$$

$$\breve{\Lambda}_k = \{\theta \in \Lambda_k \mid \nu \wedge *\theta|_{\partial\Omega} = 0\}$$

Lemma $u, \varphi \in \Lambda_k$ $\Longrightarrow$ classical Gauss

$$\int_\Omega (\nabla u, \nabla\varphi)\, dx = \int_\Omega (du, d\varphi) + (\delta u, \delta\varphi)\, dx + \int_{\partial\Omega} R\, ds$$

$$R = \left(u, \frac{\partial\varphi}{\partial\nu}\right) - (u \wedge \nu, d\varphi)$$

Lemma $u, \varphi \in \breve{\Lambda}_k$ $\Longrightarrow$

$$\vec{u}^{i_2\cdots i_k} \perp \vec{\nu},\ \vec{\varphi}^{i_2\cdots i_k} \perp \vec{\nu} \ \text{ on } \partial\Omega$$

Proof $\nu \wedge *\omega = *\beta$

$$\beta = \sum_{i_2<\cdots<i_{k+1}} (\vec{\omega}^{i_2\cdots i_{k+1}}, \vec{\nu})dx_{i_2} \wedge \cdots \wedge dx_{i_{k+1}} \in \Lambda_k$$

□

Corollary

Preliminary

tangential derivative

$$\vec{u}^{i_2\cdots i_k} \cdot \nabla(\vec{\varphi}^{j_2\cdots j_k} \cdot \vec{\nu})\big|_{\partial\Omega} = 0 \qquad i_2 < \cdots < i_k$$

vanishing on the boundary

$$j_2 < \cdots < j_k$$

$$\sum_{\ell,m} \tilde{u}^{i_2\ldots i_k\ell}\nu^m \tilde{\varphi}_\ell^{j_2\ldots j_k m} = -\sum_{\ell,m} \tilde{u}^{i_2\ldots i_k\ell}\nu_\ell^m \tilde{\varphi}^{j_2\ldots j_k m}$$

$$\partial/\partial x_\ell$$

---

notation

$$\omega = \sum_{i_1<\cdots<i_{k+1}} \omega^{i_1\ldots i_{k+1}} dx_{i_1} \wedge \cdots \wedge dx_{i_{k+1}}$$

$$\omega \in \Lambda_{k+1}$$

$$\tilde{\omega}^{i_1\ldots i_{k+1}} = \mathrm{sgn}\ \sigma \cdot \omega^{i'_1\ldots i'_{k+1}}$$ extension

$$\sigma : (i_1, \cdots, i_{k+1}) \mapsto (i'_1, \cdots, i'_{k+1})$$

$$\{i'_1, \cdots, i'_{k+1}\} = \{i_1, \cdots, i_{k+1}\} \qquad i'_1 < \cdots < i'_{k+1}$$

$$\hat{\omega}^{i_2\ldots i_{k+1}} = \sum_\ell \tilde{\omega}^{i_2\ldots i_{k+1}\ell} dx_\ell$$ 1-form

$\longleftrightarrow$ $$\vec{\omega}^{i_2\ldots i_{k+1}} = (\tilde{\omega}^{i_2\ldots i_{k+1}\ell})_\ell$$ vector field

H^1 estimate continued

$$\|\nabla u\|_2^2 \le C\left(\|du\|_2^2 + \|\delta u\|_2^2 + \|u\|_2^2\right) \quad u \in \breve{\Lambda}_k$$

$$\breve{\Lambda}_k = \{\theta \in \Lambda_k \mid \nu \wedge *\theta|_{\partial\Omega} = 0\}$$

Lemma $u, \varphi \in \Lambda_k$

classical Gauss

$$\int_\Omega (\nabla u, \nabla \varphi)\, dx = \int_\Omega (du, d\varphi) + (\delta u, \delta\varphi)\, dx + \int_{\partial\Omega} R\, ds$$

$$R = \left(u, \frac{\partial \varphi}{\partial \nu}\right) - (u \wedge \nu, d\varphi)$$

Lemma $i_2 < \cdots < i_k;\ j_2 < \cdots < j_k$

$$\sum_{\ell,m} \tilde{u}^{i_2\cdots i_k \ell} \nu^m \tilde{\varphi}_\ell^{j_2\cdots j_k m} = -\sum_{\ell,m} \tilde{u}^{i_2\cdots i_k \ell} \nu_\ell^m \tilde{\varphi}^{j_2\cdots j_k m}$$

Lemma $$\left(u, \frac{\partial \varphi}{\partial \nu}\right) = \frac{1}{k!} \sum_{i_1,\cdots,i_{k+1}} \nu^{i_1} \tilde{\varphi}_{i_1}^{i_2\cdots i_{k+1}} \tilde{u}^{i_2\cdots i_{k+1}}$$

$$(\nu \wedge u, d\varphi) = \frac{1}{(k+1)!} \sum_{i_1,\cdots,i_{k+1}} (\nu^{i_1} \tilde{u}^{i_2\cdots i_k i_{k+1}} + \cdots + \nu^{i_{k+1}} \tilde{u}^{i_1\cdots i_{k-1} i_k})(\tilde{\varphi}_{i_1}^{i_2\cdots i_k i_{k+1}} + \cdots + \tilde{\varphi}_{i_{k+1}}^{i_1\cdots i_{k-1} i_k})$$

$\Longrightarrow$

$$R = -\frac{1}{(k+1)!} \cdot \sum_{i_1,\cdots,i_k} \sum_{j_1,\cdots,j_k} \sum_{\ell \in \{i_1\cdots,i_k\}} \sum_{m \in \{j_1,\cdots,u_k\}} \tilde{u}^{i_1\cdots i_k} \nu^m \tilde{\varphi}_\ell^{j_1\cdots j_k}$$

$\Longrightarrow$ $|R| \le C|u|_2|\varphi|_2$

$$\|u\|_{L^2(\partial\Omega)} \le \varepsilon\|u\|_{H^1} + C_\varepsilon\|u\|_2$$

Gauss

Lemma $0 \le k \le n-1$

$$\int_\Omega (d\theta, \omega)\ dx = \int_\Omega (\theta, \delta\omega)\ dx + (-1)^k \int_{\partial\Omega} (*\theta, \nu \wedge *\omega)\ ds$$

$(\theta, \omega) \in \Lambda_k \times \Lambda_{k+1}$

Lemma $\Lambda_k \subset W^{1,r}_{k,d}$ (dense), $\Lambda_k \subset W^{1,r}_{k,\delta}$ (dense)

trace form

Proposition $\frac{1}{r'} + \frac{1}{r} = 1$

$\exists J^r_k = J^r_k(\theta, \omega) : W^{1,r'}_{k,d} \times W^{1,r}_{k+1,\delta} \to \mathbf{R},\ 0 \le k \le n-1$ (bounded bilinear)

$$J^r_k(\theta, \omega) = \int_{\partial\Omega} (*\theta, \nu \wedge *\omega)\ ds,\ (\theta, \omega) \in W^{1,r'}_k \times W^{1,r}_{k+1}$$

Lemma

$$\|\nabla u\|_r \le C\left(\|du\|_r + \|\delta u\|_r + \|u\|_r\right) \quad u \in \breve{\Lambda}_k$$

$$\breve{\Lambda}_k = \{\theta \in \Lambda_k \mid \nu \wedge *\theta|_{\partial\Omega} = 0\}$$

Notation

$$W^{1,r}_{k,d} = \{\theta \in L^r_k \mid d\theta \in L^r_{k+1}\},\ 0 \le k \le n-1$$

$$W^{1,r}_{k,\delta} = \{\theta \in L^r_k \mid \delta\theta \in L^r_{k-1}\},\ 1 \le k \le n$$

$$W^{1,r}_{k,d\delta} = W^{1,r}_{k,d} \cap W^{1,r}_{k,\delta}$$

$\omega \in W^{1,r}_{k+1,\delta}$

weak trace

$$\nu \wedge *\omega|_{\partial\Omega} = 0 \overset{\text{def}}{\Longleftrightarrow} J^r_k(\theta, \omega) = 0,\ \forall \theta \in W^{1,r'}_{k,d}$$

$$\breve{W}^{1,r}_{k,d\delta} \overset{\text{def}}{=} \{\theta \in W^{1,r}_{k,d\delta} \mid \nu \wedge *\theta|_{\partial\Omega} = 0\}$$

Duvaut-Lions

Lemma $\breve{\Lambda}_k \subset \breve{W}^{1,r}_{k,d\delta}$ (dense)

$\Longrightarrow$ **Proposition**

$$\breve{W}^{1,r}_{k,d\delta} = \{\theta \in W^{1,r}_{k,d\delta} \mid \nu \wedge *\theta|_{\partial\Omega} = 0\} \subset W^{1,r}_k$$

H^1 regularity $\quad r = 2$

$$u \in \breve{\Lambda}_k \quad \|\nabla u\|_2^2 \le C\left(\|du\|_2^2 + \|\delta u\|_2^2 + \|u\|_2^2\right)$$

Duvaut-Lions

Lemma

$$H^1_{k,d\delta} = W^{1,2}_{k,d\delta}$$

dense

$$\breve{\Lambda}_k \subset \breve{H}^1_{k,d\delta} = \{u \in H^1_{k,d\delta} \mid \nu \wedge *u|_{\partial\Omega} = 0\}$$

weak trace

$$\breve{\Lambda}_k = \{\theta \in \Lambda_k \mid \nu \wedge *\theta|_{\partial\Omega} = 0\}$$

$$\longrightarrow \quad \breve{H}^1_{k,d\delta} \subset H^1_k$$

localization

Remark $\quad (p,\omega) \in \Lambda_0 \times \breve{H}^1_{k,\delta} \longrightarrow p\omega \in \breve{H}^1_{k,\delta}$

Proof $\quad \omega \in H^1_{k,\delta}$

$$\omega \in \breve{H}^1_{k,\delta} \Longleftrightarrow \int_\Omega (d\theta,\omega)\,dx = \int_\Omega (\theta,\delta\omega)\,dx, \ \forall\theta \in \Lambda_{k-1}$$

$$\rightarrow \int_\Omega (d\theta,p\omega)\,dx = \int_\Omega (\theta,\delta(p\omega))\,dx, \ \forall\theta \in \Lambda_{k-1} \quad \square$$

Preliminary

Proof of Lemma

$$\Omega \subset \bigcup_{\ell=0}^{L} \Omega_\ell, \ \overline{\Omega_0} \subset \Omega, \ \Omega_\ell \cap \partial\Omega \neq \emptyset, \ 1 \le \ell \le L$$

$$0 \le \varphi_\ell \in C^\infty(\mathbf{R}^n), \ \text{supp}\ \varphi_\ell \subset \Omega_\ell$$

$$\sum_\ell \varphi_\ell = 1 \text{ on } \partial\Omega \quad \longrightarrow \quad f_\ell = f\cdot\varphi_\ell \in \breve{H}^1_{k,d\delta}$$

$X_\ell : \Omega_\ell \to U_\ell \subset \mathbf{R}^n$ diffeomorphism

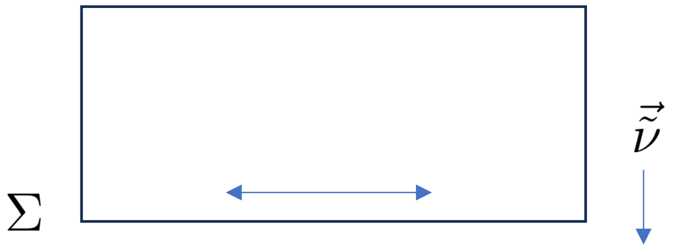
$$X_\ell(\text{supp}\ \varphi_\ell) \subset Q_\ell \cap \overline{\mathbf{R}^n_+} \subset U_\ell$$

$$Q_\ell = \Sigma \times (-\varepsilon_0, \varepsilon_0), \ \Sigma = (0,\varepsilon_0)^{n-1}$$

$$\vec{\tilde{\nu}} = X_\ell(\vec{\nu}) = (0,\cdots,0,-1)^T$$

$$\longrightarrow \vec{\tilde{\nu}} \wedge *g_\ell|_{\Sigma\times\{0\}} = 0, \ g_\ell = Y_\ell^* f_\ell, \ Y_\ell = X_\ell^{-1}$$

tangential regularization Duvaut-Lions



$du \in L^2$

$\longrightarrow$ normal derivative

Elliptic regularity

Preliminary

Proposition $\tilde{H}_k = H_k$ harmonic forms

$$\tilde{H}_k = \{h \in L^2_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge *h|_{\partial\Omega} = 0\}$$

Proof $d\delta$ regularity $h \in H^1_k$

covering partition of unity local chart $(\Omega_\ell, \varphi_\ell, X_\ell)$

$$\partial\Omega \cap \Omega_\ell \neq \emptyset,\ 1 \le \ell \le L \quad \overline{\Omega}_0 \subset \Omega,\ \Omega \subset \bigcup_{\ell=0}^{L} \Omega_\ell$$

$$h_\ell = h \cdot \varphi_\ell$$

difference quotient localization by local chart

$$\Delta_{s,j} h_\ell(x) = \frac{1}{s}\{h_\ell(x + se_j) - h_\ell(x)\}$$ **tangential direction**

$$\|dX^*_\ell \Delta_{s,j} h_\ell\|_2,\ \|\delta X^*_\ell \Delta_{s,j} h_\ell\|_2,\ \|X^*_\ell \Delta_{s,j} h_\ell\|_2 \le C$$

$$X^*_\ell \Delta_{s,j} h_\ell \in \breve{H}^1_{k,d\delta} \xrightarrow{d\delta \text{ regularity}} \|X^*_\ell \Delta_{s,j} h_\ell\|_{H^1_k} \le C$$

Nirenberg

normal derivatives $dh = 0$

$$\longrightarrow \quad h \in H^2_k \qquad \int_\Omega (dh, d\theta) + (\delta h, \delta\theta)\ dx = 0,\ \forall \theta \in \Lambda_k$$

natural boundary condition

$$-\Delta h = 0,\ \nu \wedge *h|_{\partial\Omega} = 0,\ \nu \wedge *dh|_{\partial\Omega} = 0$$

elliptic estimate $\longrightarrow \quad h \in \Lambda_k$ □

Lemma elliptic estimate Agmon-Douglis-Nirenberg

$$\theta \in H^2_k$$

$$-\Delta\theta = \alpha \text{ in } \Omega,\ \nu \wedge *\theta = *\beta,\ \nu \wedge *d\theta = *\gamma \text{ on } \partial\Omega$$

$$\longrightarrow \quad s \ge 0$$

$$\|\theta\|_{H^{s+2}_k} \le C(\|\alpha\|_{H^s_k} + \|\beta\|_{H^{s+3/2}_{k-1}(\partial\Omega)} + \|\gamma\|_{H^{s+1/2}_k(\partial\Omega)} + \|\theta\|_{H^s_k})$$

Does not assure the existence of the strong solution by itself

Rem. 1. $v \in H^1,\ \Delta v = 0,\ v|_{\partial\Omega} = g \in H^{3/2}(\partial\Omega)$

$$\longrightarrow \quad v \in H^2$$

existence of the solution

Rem. 2. $v \in H^1,\ \Delta v = f \in L^2,\ \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$ natural boundary condition **weak trace**

Gilberg-Trudinger Chap.8 topological degree difference quotient L^2 theory

Preliminary

Elliptic estimate

$$\theta \in H^2_k,\ -\Delta\theta = \alpha \text{ in } \Omega,\ \nu \wedge *\theta = *\beta,\ \nu \wedge *d\theta = *\gamma \text{ on } \partial\Omega$$

$$\Longrightarrow \quad \|\theta\|_{H^{s+2}_k} \le C(\|\alpha\|_{H^s_k} + \|\beta\|_{H^{s+3/2}_{k-1}(\partial\Omega)} + \|\gamma\|_{H^{s+1/2}_k(\partial\Omega)} + \|\theta\|_{H^s_k})$$

$$\forall x_0 \in \partial\Omega \qquad \beta(x_0) \in \exists L(x_0),\ \gamma(x_0) \in \exists M(x_0) \quad \dim\, L(x_0) = {}_{n-1}C_{k-1},\ \dim\, M(x_0) = {}_{n-1}C_k$$

complementing condition (Agmon-Douglis-Nirenberg)

algebraic pointwise condition

$${}_{n-1}C_{k-1} + {}_{n-1}C_k = {}_nC_k$$

$$x_0 = 0,\ \vec{\nu} = (1, 0, \cdots, 0)^T \Longrightarrow$$

reduced to the complementing condition to

$$-\Delta\theta^{i_1\dots i_k} = \alpha^{i_1\dots i_k} \text{ in } \Omega, \qquad 1 \le i_1 < \cdots < i_k \le n$$

$$\theta^{1 i_2\dots i_k} = \beta^{i_2\dots i_k} \text{ on } \partial\Omega, \qquad 2 \le i_2 < \cdots < i_k \le n$$

$$\mu^{1 i_2\dots i_{k+1}} = \gamma^{i_2\dots i_{k+1}} \text{ on } \partial\Omega, \qquad 2 \le i_2 < \cdots < i_{k+1} \le n$$

$$\mu^{1 i_2\dots i_{k+1}} = \sum_{j=1}^{k+1} \sum_{i_j < \ell < i_{j+1}} \tilde{\theta}_\ell^{1 i_2\dots i_j i_{j+2}\dots i_{k+1}} \qquad \theta_\ell = \frac{\partial\theta}{\partial x_\ell}$$

$$u = (u^1, u^2, \cdots, u^{n-k+1})^T$$

normal form

$$u^1 = (\theta^{12\dots k}, \theta^{13\dots k+1}, \cdots, \theta^{1,n-k+2,\dots n})^T$$

$$u^2 = (\theta^{23\dots k+1}, \theta^{24\dots k+2}, \cdots, \theta^{2,n-k+2,\dots n})^T$$

$$\vdots$$

$$u^{n-k+1} = (\theta^{n-k+1,\dots n})$$

L^2 theory

Proof

variational method

L^2 —elliptic regularity→ C^∞ —a priori estimate→ L^r

$$H^1_{k,d\delta} = W^{1,2}_{k,d\delta}, \quad \breve{H}^1_{k,d\delta} = \{\theta \in H^1_{k,d\delta} \mid \nu \wedge *\theta|_{\partial\Omega} = 0\} \qquad \breve{H}^1_{k,d\delta} \subset H^1_k$$

Proposition $H_k \oplus \delta W_k \oplus dV_k$ L^2 orthogonal ← Gauss formula

isomorphism

Proposition $h \in H_k \mapsto [h] \in H^k_{DR}(\Omega) = Z^k(\Omega)/B^k(\Omega)$

$$Z^k(\Omega) = \{\omega \in \Lambda_k \mid d\omega = 0\}, \; B^k(\Omega) = d\Lambda_{k-1}$$

$$\dim H_k = b_k$$

Proof

Poincare inequality $\|\theta\|_{L^2_k} \le C\left(\|d\theta\|_{L^2_{k+1}} + \|\delta\theta\|_{L^2_{k-1}}\right),\ \theta \in \breve{\Lambda}_k \cap H_k^{\perp}$ solvability of the variational problem

$$\breve{\Lambda}_k = \{\theta \in \Lambda_k \mid \nu \wedge *\theta|_{\partial\Omega} = 0\} \qquad H_k = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge *h|_{\partial\Omega} = 0\}$$

Proof Assume the contrary $\exists \theta^j \in \breve{\Lambda}_k \cap H_k^{\perp},\ j = 1, 2, \cdots$ $\quad \|\theta^j\|_{L^2_k} = 1,\ \varepsilon_j = \|d\theta^j\|_{L^2_{k+1}} + \|\delta\theta^j\|_{L^2_{k-1}} \to 0$

$\longrightarrow$ H^1 estimate $\quad \|\theta^j\|_{H^1_k} \le C$ $\quad$ subsequence $\quad \theta^j \rightharpoonup \exists\theta \text{ in } H^1_k \quad \theta^j \to \theta \text{ in } L^2_k$

$\theta \in H_k^{\perp},\ \|\theta\|_{L^2_k} = 1,\ \nu \wedge *\theta|_{\partial\Omega} = 0 \quad \|d\theta\|_{L^2_{k+1}} + \|\delta\theta\|_{L^2_{k-1}} \le \liminf_{j\to\infty} \varepsilon_j = 0 \quad \theta \in H_k$ contradiction □

variational method

Proposition $\forall v \in \Lambda_k \cap H_k^{\perp},\ \exists 1\eta \in \breve{H}^1_{k,d\delta} \cap H_k^{\perp} \equiv Z_k$

$$\int_\Omega (d\eta, d\theta) + (\delta\eta, \delta\theta)\ dx = \int_\Omega (v, \theta)\ dx,\ \forall\theta \in Z_k$$

$$-\Delta\theta = \alpha \text{ in } \Omega,\ \nu \wedge *\theta = *\beta,\ \nu \wedge *d\theta = *\gamma \text{ on } \partial\Omega$$

$d\delta$ estimate $\quad$ difference quotient $\longrightarrow$ Nirenberg

elliptic estimate $\longrightarrow$ $\|\theta\|_{H^{s+2}_k} \le C(\|\alpha\|_{H^s_k} + \|\beta\|_{H^{s+3/2}_{k-1}(\partial\Omega)} + \|\gamma\|_{H^{s+1/2}_k(\partial\Omega)} + \|\theta\|_{H^s_k})$ $\quad \eta \in \Lambda_k$ $\longrightarrow$ **Theorem 1**

$\theta \in H^2_k$

natural boundary condition

$$-\Delta\theta = \alpha \text{ in } \Omega,\ \nu \wedge *\theta = 0,\ \nu \wedge *d\theta = 0 \text{ on } \partial\Omega$$

L^r theory

$L^r_k,\ W^{m,r}_k,\ 1<r<\infty,\ m\geq 0$

Proof

Theorem 3

$$\forall u\in L^r_k,\ \exists(h,\eta)\in H_k\times(W^{2,r}_k\cap H_k^{\perp})$$
$$u=h+\delta d\eta+d\delta\eta,\ \ \nu\wedge *\eta|_{\partial\Omega}=0,\ \ \nu\wedge *d\eta|_{\partial\Omega}=0$$

→

Theorem 4

$$L^r_k=H_k\oplus\delta W^r_k\oplus dV^r_k,\ 1\leq k\leq n-1$$

$$W^r_k=\{\omega\in W^{1,r}_{k+1,d\delta}\mid d\omega=0,\ \ \nu\wedge *\omega|_{\partial\Omega}=0\}$$
$$V^r_k=\{p\in W^{1,r}_{k-1,d\delta}\mid \delta p=0,\ \ \nu\wedge *p|_{\partial\Omega}=0\}$$

$$V^r_1=W^{1,r},\ W^{1,r}_{n-1}\cong W^{1,r}_0$$

↑

smooth category

$$\|h\|_{L^r_k}+\|\eta\|_{W^{2,r}_k}\leq C\|u\|_{L^r_k}$$ a priori estimate

$$u=h+\delta d\eta+d\delta\eta\in\Lambda_k$$
$$h,\eta\in\Lambda_k$$

L^r theory

$\forall u \in \Lambda_k,\ \exists (h,\eta) \in H_k \times (\Lambda_k \cap H_k^\perp)$

$u = h + \delta d\eta + d\delta\eta,\ \nu \wedge *\eta|_{\partial\Omega} = 0,\ \nu \wedge *d\eta|_{\partial\Omega} = 0$

OK

a priori estimate

Proposition $\|h\|_{L^r_k} + \|\eta\|_{W^{2,r}_k} \le C\|u\|_{L^r_k}$

$\longrightarrow$ Theorem 3

Lemma $\|h\|_{L^r_k} \le C\|u\|_{L^r_k}$

Proof $\|h\|_2 \le \|u\|_2$ projection $\dim H_k = b_k$

$2 < r < \infty \quad \|h\|_r \approx \|h\|_2 \le \|u\|_2 \le C\|u\|_r$

dual exponent

$1 < r < 2,\ 2 < r' < \infty \quad \|h_2\|_{r'} \le C\|u_2\|_{r'}$

$u_i \in \Lambda_k,\ (h_i, \eta_i) \in H_k \times (\Lambda_k \cap H_k^\perp),\ i = 1, 2$

$u_i = h_i + \delta d\eta_i + d\delta\eta_i$

$\|h_1\|_r = \sup_{\|u_2\|_{r'}=1} (h_1, u_2) = \sup_{\|u_2\|_{r'}=1} (h_1, h_2)$ duality method

$\le C \sup_{\|h_2\|_{r'}=1} (h_1, h_2) = C \sup_{\|h_2\|_{r'}=1} (u_1, h_2) \le C\|u_1\|_r$ □

Proof

Lemma $\|\eta\|_{W^{2,r}} \le C\|v\|_r,\ v = u - h$

Proof $\|\eta\|_{W^{2,r}} \le C(\|v\|_r + \|\eta\|_r)$ elliptic estimate

$\|d\eta\|_2^2 + \|\delta\eta\|_2^2 \le \|v\|_2\|\eta\|_2 \xrightarrow{\text{Poincare}} \|\eta\|_2 \le C\|v\|_2$

variational form $r = 2$

$2 < r < \infty$ interpolation inequality

$$\begin{aligned}\|\eta\|_r &\le \varepsilon\|\eta\|_{W^{2,r}} + C\|\eta\|_2 \le \varepsilon\|\eta\|_{W^{2,r}} + C\|v\|_2\\ &\le \varepsilon\|\eta\|_{W^{2,r}} + C\|v\|_r\end{aligned}$$

duality method

$1 < r < 2 \Longrightarrow \|\delta d\eta\|_r + \|d\delta\eta\|_r \le C\|v\|_r$

Poincare

$\|d\eta\|_r \le C\|\delta d\eta\|_r$

$\|\delta\eta\|_r \le C\|d\delta\eta\|_r$

$\|\eta\|_r \le C(\|d\eta\|_r + \|\delta\eta\|_r)$ □

Lemma

$\eta \in \Lambda_k \cap H_k^\perp,\ \nu \wedge *\eta|_{\partial\Omega} = 0,\ \nu \wedge *d\eta|_{\partial\Omega} = 0$

$\Longrightarrow$ tangential derivative

$p \overset{\text{def}}{=} \delta\eta \in \breve{\Lambda}_{k-1} \cap H_{k-1}^\perp,\ \omega \overset{\text{def}}{=} d\eta \in \breve{\Lambda}_{k+1} \cap H_{k+1}^\perp$

Applications

Application

vector field

Bourguignon-Brezis 74 $k=1$

1. a priori estimate $\|u\|_{H^{s+2}} \le C\left(\|du\|_{H^{s+1}} + \|\delta u\|_{H^{s+1}} + \|u\|_{H^s} + \|\nu\wedge *u\|_{H^{s+3/2}(\partial\Omega)}\right),\ u\in\Lambda_k,\ s\ge 0$

c.f. $\|u\|_{H^1} \le C\left(\|du\|_2 + \|\delta u\|_2 + \|u\|_2\right),\ u\in\breve{\Lambda}_k$

Proof $\alpha = \delta du + d\delta u,\ *\beta = \nu\wedge *u,\ *\gamma = \nu\wedge *du$

$\|\alpha\|_{H^s_k} \le \|du\|_{H^{s+1}_{k+1}} + \|\delta u\|_{H^{s+1}_{k-1}}$

$\|u\|_{H^{s+2}_k} \le C(\|\alpha\|_{H^s_{k+1}} + \|\beta\|_{H^{s+3/2}_{k-1}(\partial\Omega)} + \|\gamma\|_{H^{s+1/2}_k(\partial\Omega)} + \|u\|_{H^s_k})$

$\|\gamma\|_{H^{s+1/2}_k(\partial\Omega)} \le C\|du\|_{H^{s+1}_{k+1}}$ □

elliptic estimate

dual exponent

Tartar 79

Kozono-Yanagisawa 09 $n=3$

2. compensated compactness $r_1 = r,\ r_2 = r'$ $u_i^j \rightharpoonup u_i$ in $L^{r_i}_k,\ j\to\infty,\ i=1,2$

$\|du_1^j\|_{L^{t_1}_{k+1}} + \|\delta u_2^j\|_{L^{t_2}_{k-1}} \le C,\ \exists t_i > \dfrac{nr_i}{n+r_i},\ i=1,2$ $\Longrightarrow$ $(u_1^j, u_2^j) \to (u_1,u_2)$ in $\mathcal{D}'(\Omega),\ j\to\infty$

Proof $\int_\Omega (u,u')\,dx = \int_\Omega (h,h') + (\delta\omega,\delta\omega') + (dp,dp')\,dx,\ (u,u')\in L^r_k\times L^{r'}_k$

$u = h + \delta\omega + dp,\ (h,\omega,p)\in H_k\times W^r_k\times V^r_k$

$u' = h' + \delta\omega' + dp',\ (h',\omega',p')\in H_k\times W^{r'}_k\times V^{r'}_k$

+

$\|\delta\omega\|_{W^{1,t}_k} \le C(\|du\|_{L^r_{k+1}} + \|u\|_{L^r_k})$

$\|dp\|_{W^{1,t}_k} \le C(\|\delta u\|_{L^t_{k-1}} + \|u\|_{L^r_k} + \|\nu\wedge *u\|_{W^{1-1/t,t}_{n-k+1}(\partial\Omega)})$

structure theorem
elliptic estimate

□

3. spectral property $D(A) = \{u\in W^{2,r}_k \mid \nu\wedge *u|_{\partial\Omega} = 0,\ \nu\wedge *du|_{\partial\Omega} = 0\}$

$Au = -\Delta u,\ u\in D(A)$ generator of an analytic semigroup in L^r_k

$W^{1,r}$ estimate of the Poisson equation

Simader-Sohr 92

Application

$$W_k^r = \{\omega \in W^{1,r}_{k+1,d\delta} \mid d\omega = 0,\ \nu \wedge *\omega|_{\partial\Omega} = 0\}$$

$$V_k^r = \{p \in W^{1,r}_{k-1,d\delta} \mid \delta p = 0,\ \nu \wedge *p|_{\partial\Omega} = 0\}$$

$$\inf_{\omega \in W_k^r \setminus \{0\}} \sup_{\theta \in W_k^{r'} \setminus \{0\}} \frac{\int_\Omega (\delta\omega, \delta\theta) dx}{(\|\omega\|_r + \|\delta\omega\|_r)(\|\theta\|_r + \|\delta\theta\|_{r'})} > 0$$

$$\inf_{q \in V_k^r \setminus \{0\}} \sup_{q \in V_k^{r'} \setminus \{0\}} \frac{\int_\Omega (dp, dq)\ dx}{(\|p\|_r + \|dp\|_r)(\|q\|_{r'} + \|dq\|_{r'})} > 0$$

$k = n - 1$ $\Longrightarrow$ $k = n$

$$-\Delta u = \text{div } f \text{ in } \Omega$$

$$\frac{\partial u}{\partial \nu} = \nu \cdot f \text{ on } \partial\Omega, \quad \int_\Omega u\ dx = 0$$

$$-\Delta u = \text{div } f, \quad u|_{\partial\Omega} = 0$$

$$\|u\|_{W^{1,r}} \leq C\|f\|_r$$

Banach-Necas-Babuska's theorem

$Y, Z :$ Banach space $/\mathbf{R}$ $\quad Z$ reflexive $\quad a = a(y,z) : Y \times Z \to \mathbf{R}$ bounded bilinear

$$\forall \ell \in Z',\ \exists 1 y \in Y \quad a(y,z) = \langle z, \ell \rangle,\ \forall z \in Z \quad \Longleftrightarrow \quad \inf_{y \in Y \setminus \{0\}} \sup_{z \in Z \setminus \{0\}} \frac{a(y,z)}{\|y\|_Y \|z\|_z} = b > 0 \quad \Longleftrightarrow \quad \inf_{z \in Z \setminus \{0\}} \sup_{y \in Y \setminus \{0\}} \frac{a(y,z)}{\|y\|_Y \|z\|_Z} = B > 0$$

$$z \in Z,\ a(y,z) = 0,\ \forall y \in Y \Longrightarrow z = 0$$

Babuska-Brezzi-Kikuchi inequality

$$b = B,\ \|y\|_Y \leq C\|\ell\|_{Z'}$$

Remark

$$\Lambda_k = H_k \oplus \delta W_k \oplus dV_k,\ 1 \le k \le n-1$$

Hodge $(H)_k$

$$H_k = \{h \in \Lambda_k \mid dh = 0,\ \delta h = 0,\ \nu \wedge *h|_{\partial\Omega} = 0\}$$
$$W_k = \{\omega \in \Lambda_{k+1} \mid d\omega = 0,\ \nu \wedge *\omega|_{\partial\Omega} = 0\}$$
$$V_k = \{p \in \Lambda_{k-1} \mid \delta p = 0,\ \nu \wedge *p|_{\partial\Omega} = 0\}$$

Helmholtz $(Hd)_k$

$$\exists (v, p) \in \Lambda_k \times \Lambda_{k-1},\ \delta v = 0,\ \nu \wedge *v|_{\partial\Omega} = 0$$
$$u = v + dp$$

$(H\delta)_k$ $\nu \wedge *u|_{\partial\Omega} = 0 \Longrightarrow$

$$\exists (z, \omega) \in \Lambda_k \times \Lambda_{k+1},\ dz = 0,\ \nu \wedge *z|_{\partial\Omega} = 0$$
$$u = z + \delta\omega$$

Theorem $(H)_k \Longrightarrow (Hd)_k,\ (H\delta)_k$

Remark

Babuska-Brezzi-Kikuchi inequality $r = 2$ proven directly $\Longrightarrow$ variational solution

Formulation taken by Kozono-Yanagisawa 09

$$\Longrightarrow \quad \begin{aligned} &\exists \omega \in X_k, \ \int_\Omega (\delta\omega, \delta\theta)\ dx = \int_\Omega (u, \delta\theta)\ dx, \ \forall \theta \in W_k \cap H_{k+1}^\perp \\ &\exists p \in Y_k, \ \int_\Omega (dp, dq)\ dx = \int_\Omega (u, dq), \ \forall q \in V_k \cap H_{k-1}^\perp \end{aligned}$$

$$X_k = [W_k \cap H_{k+1}^\perp]^{H^1_{k+1}}$$

$$Y_k = [V_k \cap H_{k-1}^\perp]^{H^1_{k-1}}$$

$$W_k = \{\omega \in \Lambda_{k+1} \mid d\omega = 0, \ \nu \wedge *\omega|_{\partial\Omega} = 0\}$$

$$V_k = \{p \in \Lambda_{k-1} \mid \delta p = 0, \ \nu \wedge *p|_{\partial\Omega} = 0\}$$

$$(H\delta)_{k+1} \longrightarrow \theta \in \Lambda_{k+1} \longrightarrow d\delta\omega = du, \ d\omega = 0, \ \nu \wedge *\omega|_{\partial\Omega} = 0$$

$$(Hd)_{k-1} \longrightarrow q \in \breve{\Lambda}_{k-1} \longrightarrow \delta dp = \delta u, \ \delta p = 0, \ \nu \wedge *p|_{\partial\Omega} = 0, \ \nu \wedge *dp|_{\partial\Omega} = \nu \wedge *u|_{\partial\Omega}$$

Nirenberg $\Longrightarrow$ $\omega \in W_k$, $p \in V_k$

Theorem $(Hd)_{k-1}, \ (H\delta)_{k+1} \longrightarrow (H)_k$

Poisson

$$(Hd)_0, \ (Hd)_1, \ (H\delta)_{n-1}, \ (H\delta)_n \qquad 2 \le n \le 4 \Longrightarrow (H)_k, \ 1 \le k \le n-1$$

obvious

Conclusion

1. An analytic proof of the Hodge decomposition on bounded domains in Euclidean space is given
2. It is reduced to an elliptic equation in orthogonal space to harmonic forms
3. This equation is solved by L^2 theory; Poincare inequality, H^1-regularity of d-delta system, and elliptic estimate
4. Then the L^r theory follows from a priori estimate
5. It is a natural extension of the recent result on three-dimensional vector fields
6. Applications; Helmholtz decomposition, a priori estimate of d-delta system, $W^{1,r}$-regularity of the Poisson equation, and compensated compactness lemma; are presented
7. A numerical scheme for Helmholtz decomposition is proposed based on this study