微分形式と領域変動

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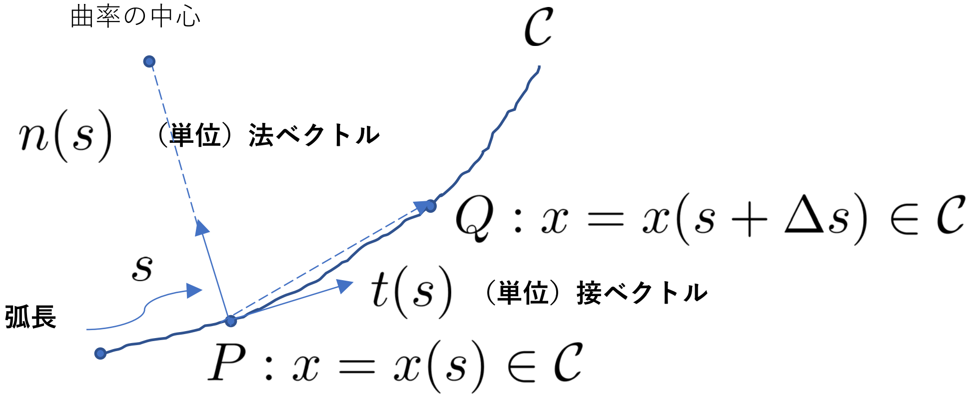
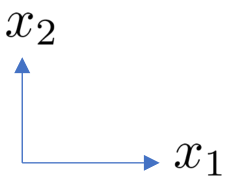
8. 幾何学の言葉

曲線のパラメータ表示





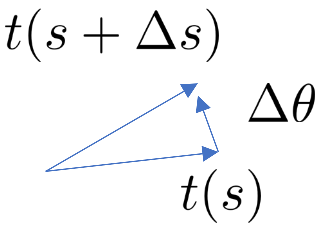
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^2$$

$$t = \lim_{\Delta s \to 0} \frac{1}{\Delta s}(x(s+\Delta s) - x(s)) = \frac{dx}{ds}$$

$$\left|\frac{dx}{ds}\right| = \lim_{\Delta s \to 0} \left|\frac{x(s+\Delta s) - x(s)}{\Delta s}\right| = 1$$

$$|t|^2 = t \cdot t = 1 \quad \Longrightarrow \quad \frac{dt}{ds} \cdot t = 0$$

$$|t(s+\Delta s) - t(s)| = \Delta\theta + o(\Delta\theta)$$

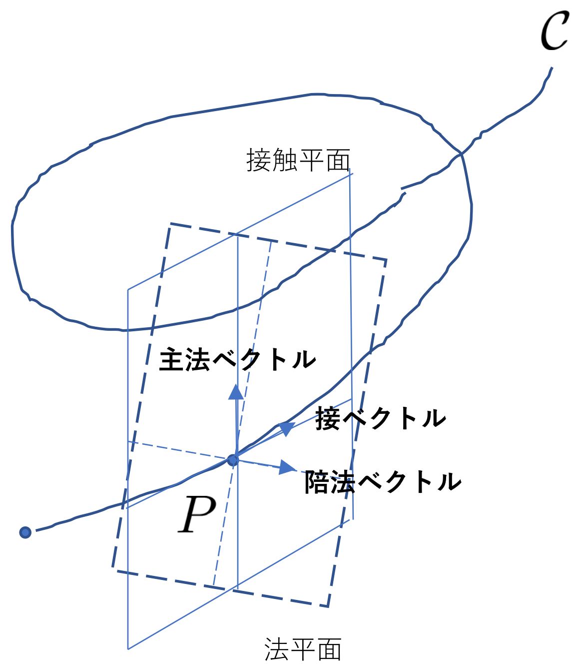


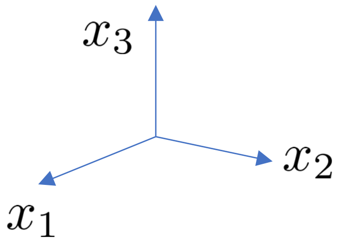
$$\left|\frac{dt}{ds}\right| = \lim_{\Delta \to 0} \left|\frac{t(s+\Delta s) - t(s)}{\Delta s}\right| = \lim_{\Delta s \to 0} \left|\frac{\Delta\theta}{\Delta s}\right| = \left|\frac{1}{\rho}\right|$$

$$\Longrightarrow \quad \frac{dt}{ds} = \frac{1}{\rho} n$$

$$\boxed{\frac{d}{ds}(t \cdot t) = \frac{dt}{ds} \cdot t + t \cdot \frac{dt}{ds} = 2\frac{dt}{ds} \cdot t}$$

空間曲線





$$x = (x_1, x_2, x_3)^T \in \mathbf{R}^3$$

$$P : x = x(s) \in \mathcal{C}$$

弧長パラメータ

$$t = \frac{dx}{ds}$$ 接ベクトル

n 主法ベクトル

b 陪法ベクトル

フレネ・セレの公式

$$\frac{dt}{ds} = \frac{1}{\rho} n$$ 接触平面上の運動

$$\frac{dn}{ds} = -\frac{1}{\rho} t + \tau b$$

$\frac{1}{\rho}$ 曲率

接触平面の変動 $$\frac{db}{ds} = -\tau n$$

τ 捩率

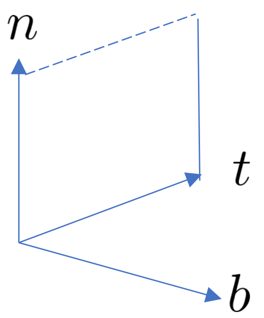
空間曲線を接触面上の円弧で近似すると曲率が表示される
接ベクトル方向に進むと接触平面が変動する

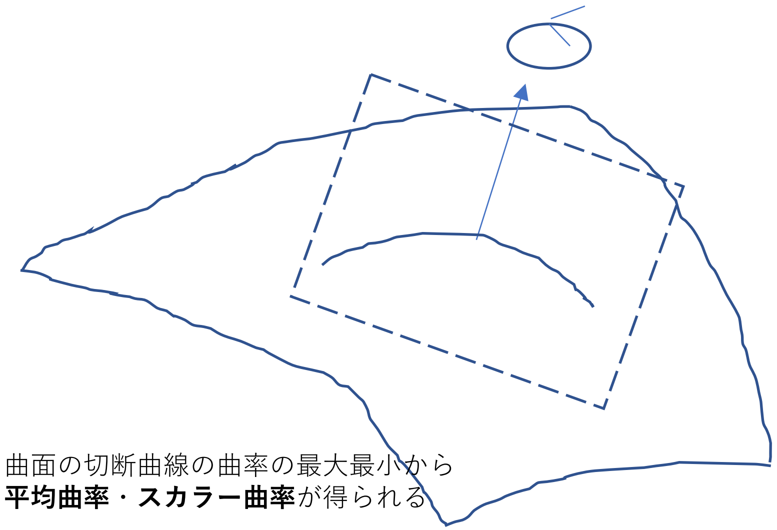
接触平面の変動は捩率で表示される

$$b = t \times n, \ |b|^2 = 1$$

$$\frac{db}{ds} = \frac{dt}{ds} \times n + t \times \frac{dn}{ds} = t \times \frac{dn}{ds}$$

$$\frac{db}{ds} \cdot b = 0$$

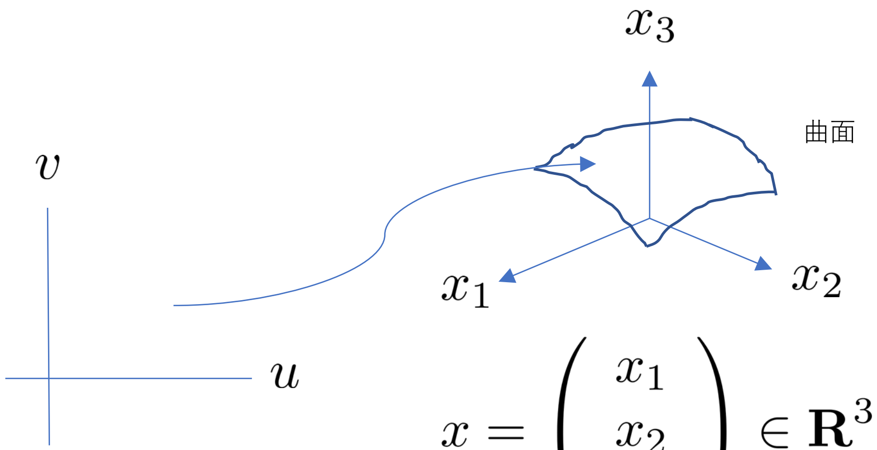




曲面の切断曲線の曲率の最大最小から
平均曲率・スカラー曲率が得られる

第2基本形式

曲面のパラメータ表示



$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3$$

$$x = x(u, v) \qquad dx = x_u du + x_v dv$$

$$x_u = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u} \right)^T$$

$$x_v = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v} \right)^T$$

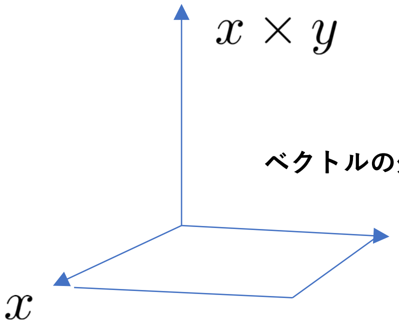
第1基本形式

曲面上2点の無限小距離

$$ds^2 = dx \cdot dx = Edu^2 + 2Fdudv + Gdv^2$$

第1基本量

$$E = |x_u|^2, \ F = 2x_u \cdot x_v, \ G = |x_v|^2$$



ベクトルの外積

ベクトルの成分表示
基底ベクトル間の演算
交換・結合・分配則

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

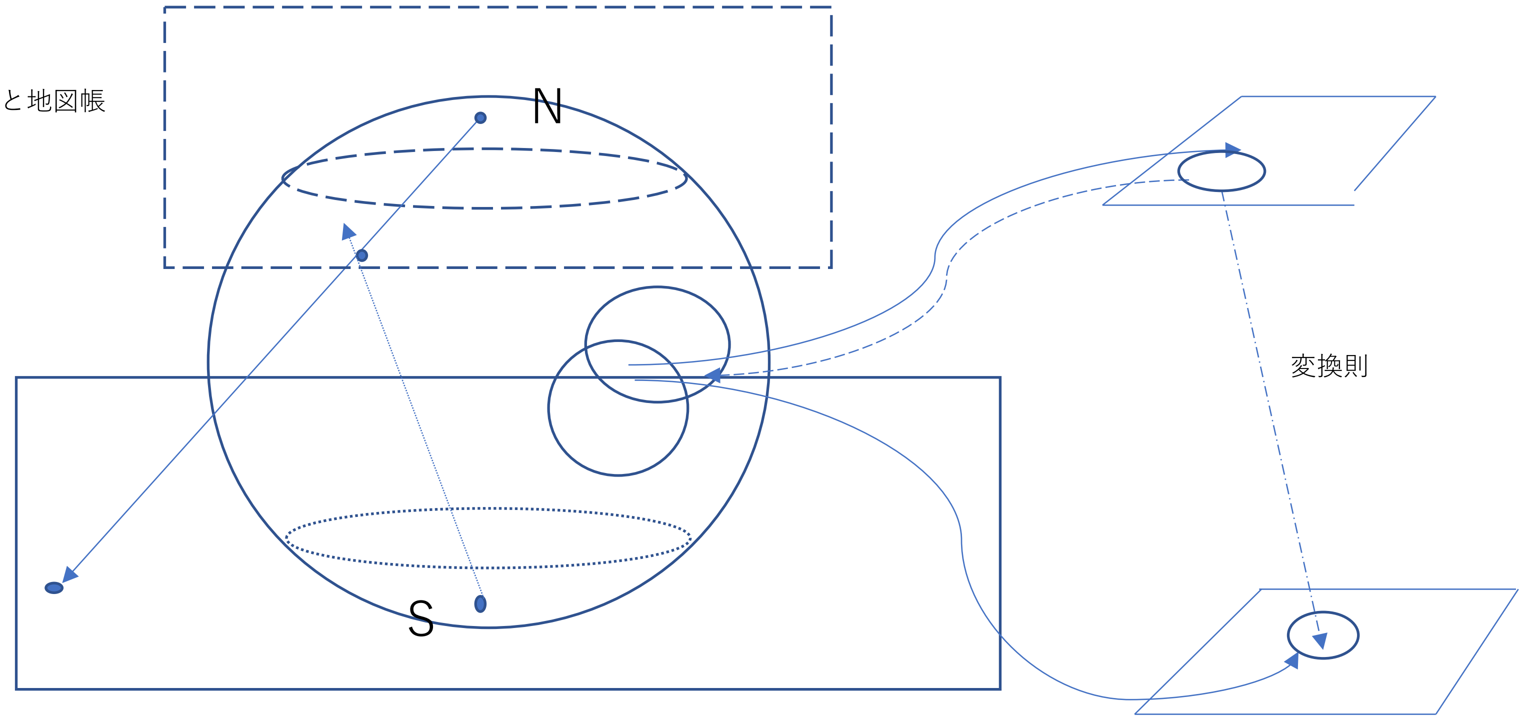
面積要素

$$dS = |x_u \times x_v| dudv = \sqrt{EG - F^2} dudv$$

曲面上平行四辺形の無限小面積

$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

地図と地図帳



多様体

ハウスドルフ空間 $\mathcal{M} = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ 開被覆 アトラス

局所座標 チャート 同相

$$\exists \psi_\alpha : U_\alpha \to \psi_\alpha(U_\alpha) = (x^1, \cdots, x^n) \subset \mathbf{R}^n,\ \alpha \in \mathcal{A}$$

$$U_\alpha \cap U_\beta \neq \emptyset$$

$$\Longrightarrow\ \psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\text{滑らか}} \psi_\beta(U_\alpha \cap U_\beta)$$

$$p \in \mathcal{M}$$

$T_p\mathcal{M}$ 接空間 n 次ベクトル空間 　基底 $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ 　接バンドル $T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p\mathcal{M}$

$T_p^*\mathcal{M}$ 余接空間 　基底 $dx^1, \cdots, dx^n$ 　余接バンドル $T^*\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p^*\mathcal{M}$

$$X : p \in \mathcal{M} \xrightarrow{\text{滑らか}} X_p \in T_p\mathcal{M}$$ ベクトル場 $\mathcal{X}(\mathcal{M})$

$$\omega : p \in \mathcal{M} \xrightarrow{\text{滑らか}} \omega_p \in T_p^*\mathcal{M}$$ 1-form $\mathcal{D}^1(\mathcal{M})$

接続

$$\nabla : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathcal{X}(\mathcal{M})$$
$$(X, Y) \mapsto \nabla_X Y$$

0-form

$$\forall X, Y, Z \in \mathcal{X}(\mathcal{M}),\ \forall f \in C^\infty(\mathcal{M})$$

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$$
$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$
$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$$
$$\nabla_{fX} Y = f\nabla_X Y$$

$(\mathcal{M}, g)$ リーマン多様体

計量

$$g : p \in \mathcal{M} \xrightarrow{\text{滑らか}} g_p$$ $T_p\mathcal{M}$ 上の(0,2)テンソル 対称正定値

局所座標で $g = g_{ij} dx^i \otimes dx^j,\ g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

双対アフィン接続

(∇, ∇^*) 　$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z)$

レビ・チビタ接続 $\nabla^* = \nabla$

情報幾何学 …双対平坦

解析力学

位置エネルギー　ニュートンの運動方程式

$$\frac{dp_i}{dt} = -\frac{\partial U}{\partial x_i}, \quad 1 \le i \le f$$

運動量　運動エネルギー

$$p_i = \frac{\partial K}{\partial \dot{x}_i},\ K = \frac{1}{2}\sum_i m\dot{x}_i^2$$

一般座標

$$x_i = x_i(q_1, \cdots, q_f; t),\ 1 \le i \le f$$

$$\Rightarrow \dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}$$

ラグランジュの運動方程式

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j},\ 1 \le j \le f$$

ラグランジアン

$$L = K(q, \dot{q}, t) - U(q, t)$$

最小作用の原理

ハミルトニアン　ルジャンドル変換

$$H(p, q, t) = \sup_{\dot{q}}(p \cdot \dot{q} - L(q, \dot{q}, t))$$

$$\Rightarrow p = \frac{\partial L}{\partial \dot{q}}$$

正準方程式　保存則

$$\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q} \Rightarrow \frac{d}{dt}H(p(t), q(t)) = 0$$

$$\Longleftrightarrow \dot{F} = \{F, H\},\ \forall F = F(p, q) \Longleftrightarrow \dot{z} = X_H z, \quad z = (q, p)^T$$

ポアソン括弧

$$\{F, G\} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q}$$

歪対称双線形形式

ハミルトンベクトル場

$$X_H = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)^T$$

$$\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0$$ ヤコビの等式

$$\{FG, H\} = F\{G, H\} + \{F, H\}G$$ ライプニッツの等式

シンプレティック多様体

$$\mathcal{D}^1(\mathcal{M}) \xrightarrow{\ d\ } \Lambda^2 T^*\mathcal{M}$$

1-forms　exterior derivative　∧ product　2-forms

$\mathcal{M}$ **シンプレティック多様体** $\overset{\text{def}}{\Longleftrightarrow}$ $\exists\omega \in \Lambda^2 T^*\mathcal{M},\ d\omega = 0$

non-degenerate　closed　3-form

定理（ダルブー） $(\mathcal{M},\omega) \cong (\mathbf{R}^{2n},\omega_0)$ 局所的に

$$\omega_0 = dx^1\wedge dy^1 + \cdots + dx^n \wedge dy^n$$

$$(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$$

シンプレティック多様体はポアソン多様体になる

定理 $(\mathcal{M},\omega)$ シンプレティック多様体　$H \in C^\infty(\mathcal{M})$

$X_H \in \mathcal{X}(\mathcal{M}) \overset{\text{def}}{\Longleftrightarrow} \omega(X_H,\cdot) = dH$ 1-formとして

$\longrightarrow$ $\{F,G\} = \omega(X_F,X_G)$ ポアソン括弧

歪対称双線形形式, ヤコビの等式, ライプニッツの等式

一致

リー微分

$X \in \mathcal{X}(\mathcal{M}),\ f \in C^\infty(\mathcal{M})$

$$\mathcal{L}_X f = \frac{d}{dt}\Phi^*_{-t} f\Big|_{t=0}$$

引き戻し

力学系, 流れ

$$\frac{d}{dt}\Phi_t(x) = X(\Phi_t(x)),\quad \Phi_t(x)|_{t=0} = x$$

ポアソン多様体 $\mathcal{P}$

$\exists\{\ ,\ \} : C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$ ポアソン括弧

$H \in C^\infty(\mathcal{P})$ given

$X_H \in \mathcal{X}(\mathcal{M}) \overset{\text{def}}{\Longleftrightarrow} \mathcal{L}_{X_H} F = \{F,H\},\ \forall F \in C^\infty(\mathcal{P})$

$\dot{x} = X_H(x) \overset{\text{def}}{\Longleftrightarrow} \mathcal{P}$ 上のハミルトン系

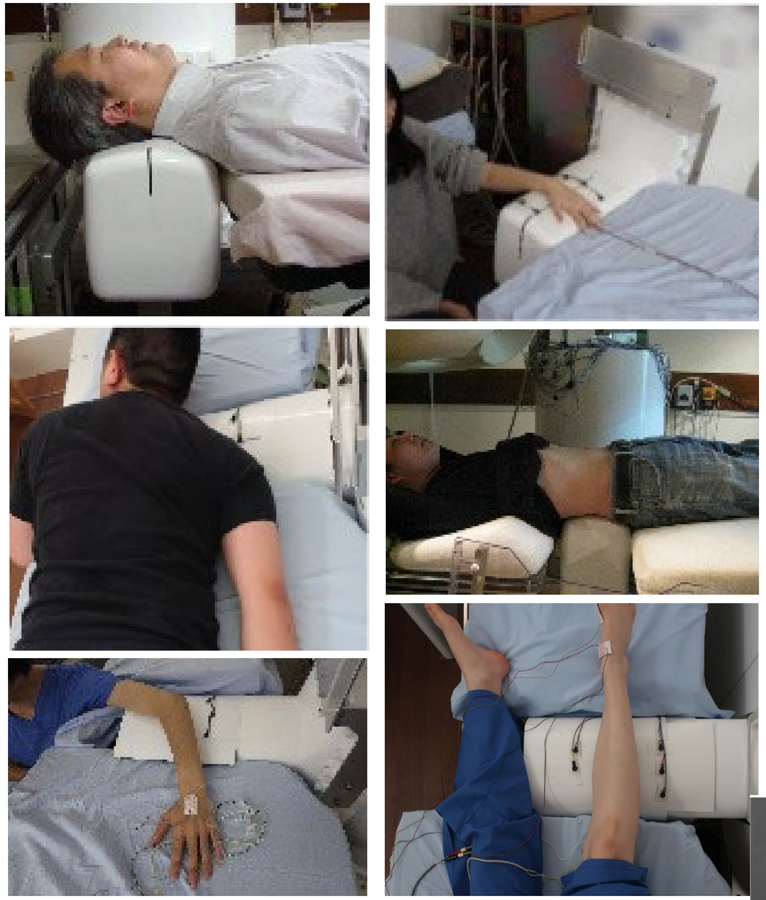
例　$\mathcal{P} = \mathbf{R}^3,\ C \in C^\infty(\mathcal{P})$

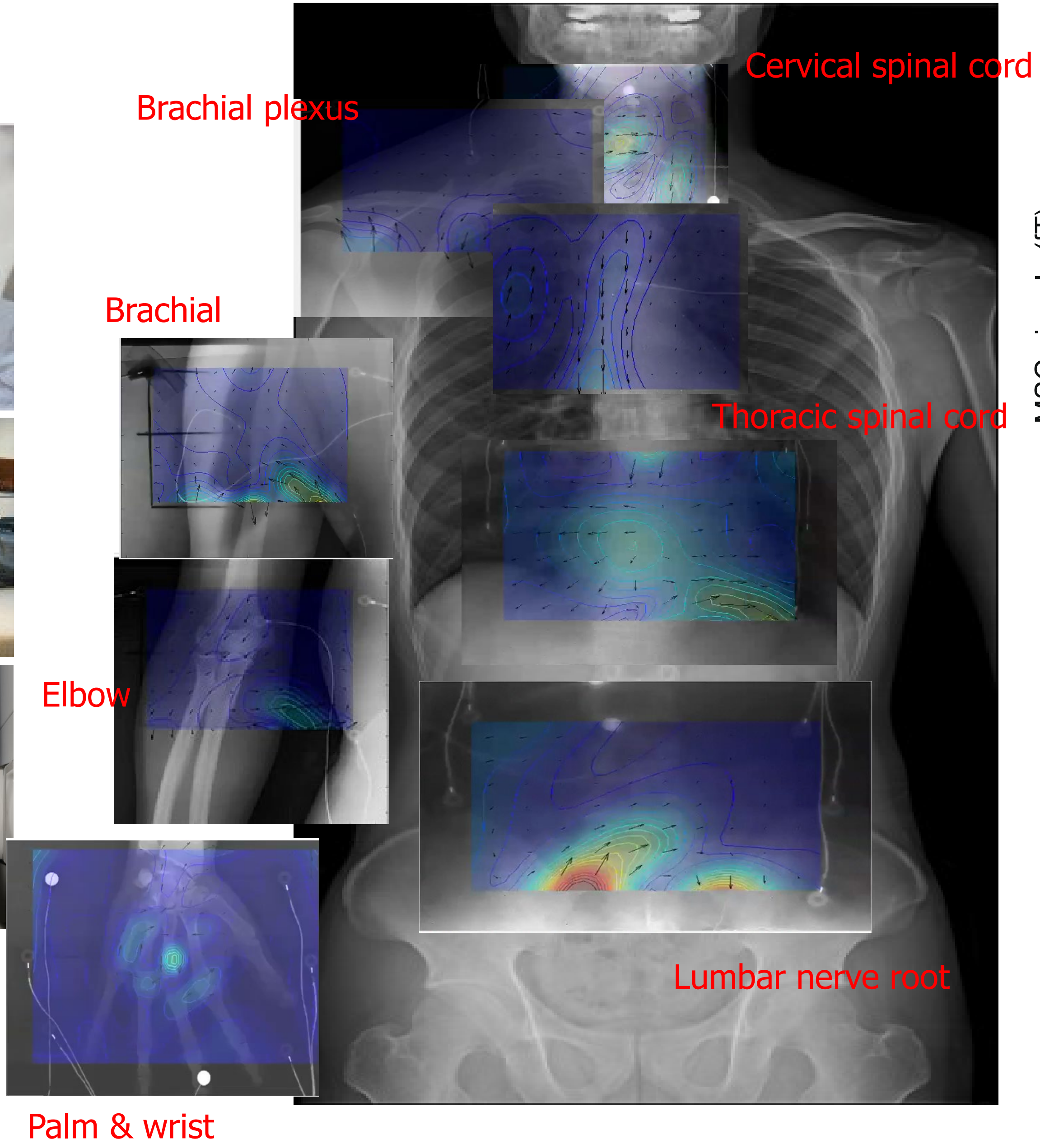
$$\{F,G\} = -\nabla C \cdot \nabla F \times \nabla G$$

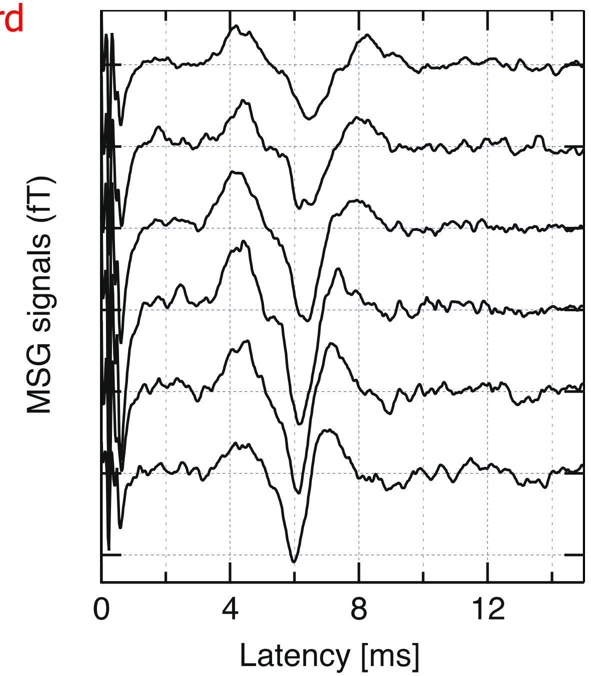
$$\dot{X} = \nabla C \times \nabla H$$

9. 非定常マクスウェル方程式の界面消滅

磁気神経イメージング





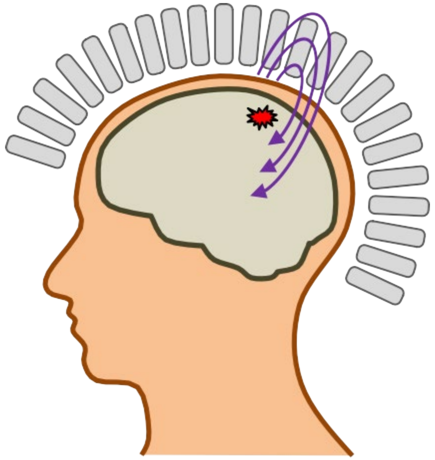


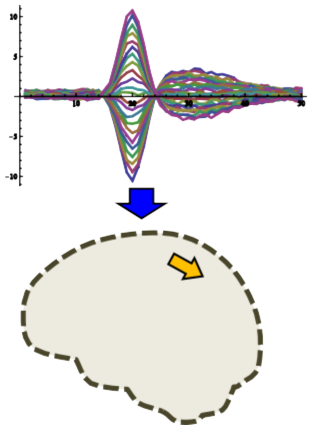
MSG signal waveforms

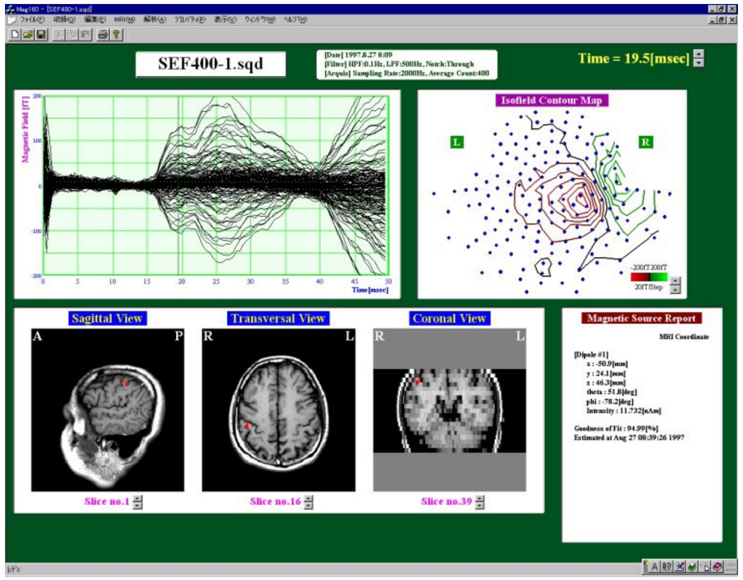
逆問題
神経可塑性

計測
・安静
・リハビリ動画視聴

MEG







Ω volume conductor

$\mu_0 > 0$ permeability

$\nabla \times B = \mu_0 J,\ \nabla \cdot B = 0$

secondary current

$$J = J^p - \sigma(x)\nabla V$$

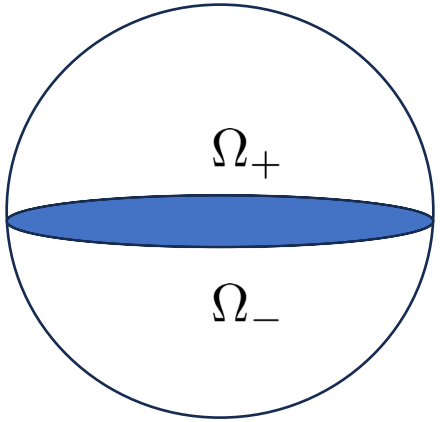
$$\sigma(x) = \begin{cases} \sigma, & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$$

B magnetic field

J total current density

J^p primary current

$\sigma(x)$ conductivity



layer potential Geselowitz 67, 70

$$\frac{\sigma}{2}V(\xi) = -\int_\Omega \nabla \cdot J^p(y)\Gamma(\xi - y)dy - \sigma\int_{\partial\Omega} V(y)\frac{\partial}{\partial \nu_y}\Gamma(\xi - y)dS_y, \quad \xi \in \partial\Omega$$

$$B(x) = -\mu_0\int_\Omega J^p(y)\times\nabla\Gamma(x-y)dy + \mu_0\sigma\int_{\partial\Omega} V(y)\nu_y \times \nabla\Gamma(x-y)dS_y, \quad x \notin \partial\Omega$$

$$\Gamma(x) = \frac{1}{4\pi|x|}$$

Motivation

Interface vanishing

$\Omega \subset \mathbf{R}^n$ bounded domain interface $\mathcal{M} \subset \mathbf{R}^n : C^{2,1}$ hyper-surface $\Gamma \equiv \Omega \cap \mathcal{M} \neq \emptyset$

$$\longrightarrow \quad \Omega = \Omega_+ \cup \Gamma \cup \Omega_-, \quad \Gamma_\pm = \partial\Omega_\pm = \partial\Omega(=\Gamma)$$

$\nu = (\nu^i) \in C^{1,1}(\Omega, \mathbf{R}^n), \ |\nu| = 1$ $\nu|_{\Gamma_-}$ outer unit normal vector $\longleftrightarrow$ $\nu = \sum_i \nu_i dx_i$

Theorem $\omega \in H^1(\Omega)$ $\theta \in L^2(\Omega)$ $d\omega = \theta, \ \delta\omega = 0 \text{ in } \Omega, \ \delta\theta \in L^2(\Omega_\pm) \longrightarrow \Delta(\nu \wedge *\omega) \in L^2(\Omega)$

S.-Watanabe 24 normal component

Maxwell equation

S. 2021

$$(x,t) \in \Omega \subset \mathbf{R}^{3+1}, \ x = (x_1, x_2, x_3), \ \nabla = \nabla_x$$

$$\nabla \times B - \frac{\partial E}{\partial t} = J, \ \nabla \cdot E = \rho, \ \nabla \times E + \frac{\partial B}{\partial t} = 0, \ \nabla \cdot B = 0 \quad \text{in } \Omega$$

Minkowski space $(x,t) \in \Omega \subset \mathbf{R}^{3+1}$ 2form $\omega = \sum_{i=1}^{3} (E^i dx_0 \wedge dx_i + B^i * (dx_0 \wedge dx_i)), \ \theta = -\sum_{i=1}^{3} J^i * dx_i + \rho * dx_0$

S. 21

Theorem $E, B \in H^1(\Omega)^3, \ J \in L^2(\Omega)^3, \ \rho \in L^2(\Omega)$ $\nabla \times J \in L^2(\Omega_\pm)^3, \ \frac{\partial J}{\partial t} + \nabla\rho \in L^2(\Omega_\pm)^3$

$$\nu = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \\ \nu^0 \end{pmatrix}, \ \tilde{\nu} = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix} \Longrightarrow (-\partial_t^2 + \Delta_x)(\nu^0 B + \tilde{\nu} \times E) \in L^2(\Omega)^3, \ (-\partial_t^2 + \Delta_x)(\tilde{\nu} \cdot B) \in L^2(\Omega)$$

Remark $(-\partial_t^2 + \Delta_x)E \in L^2(\Omega_\pm)^3, \ (-\partial_t^2 + \Delta_x)B \in L^2(\Omega_\pm)^3$

微分方程式と微分形式

$D \subset \mathbf{R}^n$ open set

$\Lambda^p = \Lambda^p(D)$ p-forms

$\wedge$ wedge product

$d : \Lambda^p \to \Lambda^{p+1}$ outer derivative

inner product

$\alpha = \sum_\ell \alpha^\ell dx_\ell,\ \beta = \sum_\ell \beta^\ell dx_\ell$ 1-forms $\Rightarrow$ $(\alpha, \beta) = \sum_\ell \alpha^\ell \beta^\ell$

$\lambda = \alpha_1 \wedge \cdots \wedge \alpha_p,\ \mu = \beta_1 \wedge \cdots \wedge \beta_p$ p-forms $\Rightarrow$ $(\lambda, \mu) = \det\ ((\alpha_i, \beta_j))_{i,j}$

$* : \Lambda^p(D) \to \Lambda^{n-p}(D)$ Hodge operator ↔ metric

$$\omega \wedge \tau = (*\omega, \tau)\ dx_1 \wedge \cdots \wedge dx_n$$

$$\omega \in \Lambda^p(D),\ \ \tau \in \Lambda^{n-p}(\Lambda)$$

co-derivative

$$\delta = (-1)^p *^{-1} d* : \Lambda^p(D) \to \Lambda^{p-1}(D)$$

$$B = \sum_i B^i dx_i \ \Rightarrow\ \delta B = -\sum_i B^i_i$$

1 form

$$\omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j \ \Rightarrow\ \delta\omega = -\sum_{i,\ell} \tilde{\omega}^{\ell i}_\ell dx_i$$

2 form

$$\tilde{\omega}^{ij} = \begin{cases} \omega^{ij}, & i < j \\ 0, & i = j \\ -\omega^{ij}, & i > j \end{cases}$$

Laplacian

$$-\Delta = \delta d + d\delta : \Lambda^p \to \Lambda^p$$

$D \subset \mathbf{R}^n$ Lipschitz domain $\quad \exists \gamma : H^1(D) \to H^{1/2}(\partial D)$ trace operator $\quad H^{1/2}(\partial D) \cong H^1(D)/H^1_0(\Omega)$

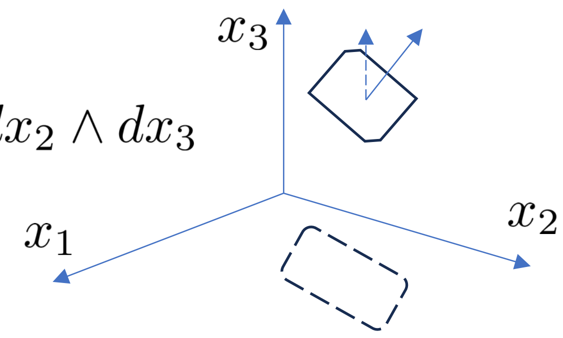
$\longrightarrow$ $C^\infty(\overline{D}) \subset H^1(D)$ dense

write $\quad \varphi|_{\partial D} = \gamma\varphi, \quad \varphi \in H^1(D)$

$n = 3$

integral formulae

$dx = dx_1 \wedge dx_2 \wedge dx_3$



ν outer unit normal vector

$\nu ds = (*dx_1, \cdots, *dx_n)$ vector area element

$\nu^3 ds = dx_1 \wedge dx_2$

Lemma 1 $\quad B \in \Lambda^1(D),\ C \in \Lambda^2(D) \longrightarrow *B = (B, \nu)\ ds,\ \ B \wedge *C = (\nu \wedge B, C)\ ds$

write $\quad \int_D \cdots\ dx_1 \wedge \cdots \wedge dx_n = \int_D, \qquad \int_{\partial D} \cdots\ ds = \int_{\partial D},$

Lemma 2

$\varphi \in H^1(\Lambda^0)$
$B \in H^1(\Lambda^1)$
$J \in H^1(\Lambda^2)$

$\longrightarrow$

$$\int_D (\delta B, \varphi) = \int_D (B, d\varphi) - \int_{\partial D} (B, \nu)\varphi \qquad \text{Gauss}$$

$$\int_D (dB, J) = \int_D (B, \delta J) + \int_{\partial D} (\nu \wedge B, J) \qquad \text{Stokes}$$

$$H^q(\Lambda^p) = H^q(\Lambda^p(\Omega)) = \{p\text{-forms} \mid \text{coefficients are in } H^q\}$$

$$L^2(\Lambda^0) = L^2(D)$$

Definition $\Omega \subset \mathbf{R}^n$ region with interface $\Longleftrightarrow$ $\exists \mathcal{M},\ \Gamma \equiv \Omega \cap \mathcal{M} \neq \emptyset$

smooth non-compact hyper-surface without boundary

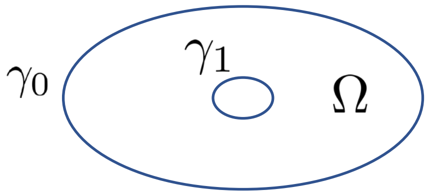
$$\longrightarrow \quad \Omega = \Omega_+ \cup \Gamma \cup \Omega_-, \quad \Gamma_\pm = \partial\Omega_\pm \setminus \partial\Omega (= \Gamma)$$

Theorem 1 $\Omega \subset \mathbf{R}^n$ region with interface ν outer unit normal on Γ_- $\omega \in H^1(\Lambda^2(\Omega))$

$$d\omega = \theta,\ \delta\omega = 0,\ \delta\theta \in L^2(\Lambda^2(\Omega_\pm)) \ \Rightarrow\ \Delta(\nu, \hat{\omega}^i) \in L^2(\Lambda^0(\Omega)),\ 1 \le i \le n$$

$$\hat{\omega}^i = \sum_\ell \tilde{\omega}^{\ell i} dx_i, \quad \tilde{\omega}^{ij} = \begin{cases} \omega^{ij}, & i < j \\ 0, & i = j \\ -\omega^{ij}, & i > j \end{cases}, \quad \omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j$$

10. 固有値の摂動　Hadamard Variation



$\Omega \subset \mathbf{R}^n$ bounded Lipschitz domain

$\gamma_0,\ \gamma_1 \subset \partial\Omega$ relatively open $\overline{\gamma_0} \cup \overline{\gamma_1} = \partial\Omega,\ \gamma_0 \cap \gamma_1 = \emptyset$ $\quad -\Delta u = \lambda u \text{ in } \Omega,\ \ u\big|_{\gamma_0} = 0,\ \left.\frac{\partial u}{\partial \nu}\right|_{\gamma_1} = 0$

$T_t : \Omega \to \Omega_t,\ \ |t| \ll 1$ family of bi-Lipschitz homeomorphisms

$$\gamma_{it} = T_t \gamma_i,\ \ i = 0, 1 \qquad -\Delta u_t = \lambda u_t \text{ in } \Omega_t,\ \ u_t\big|_{\gamma_{0t}} = 0,\ \left.\frac{\partial u_t}{\partial \nu}\right|_{\gamma_{1t}} = 0$$

$$\lambda = \lambda_j(t),\ j = 1, 2, \cdots,$$
$$0 \leq \lambda_1(t) \leq \lambda_2(t) \leq \cdot \to +\infty$$

Garabedian-Shiffer 1952-53 **Harmonic concavity** of the 2D first e.v. under conformal deformation

weak form transformation of variables $u \in V,\quad B_t(u, u) = 1,\quad A_t(u, v) = \lambda B_t(u, v),\ \forall v \in V$

$$B_t(u, v) = \int_\Omega u v a_t\ dx, \quad A_t(u, v) = \int_\Omega Q_t[\nabla u, \nabla v] a_t\ dx \qquad a_t = \det\ DT_t, \quad Q_t = (DT_t)^{-1}((DT_t)^{-1})^T$$

<u>Rayleigh principle</u>

Rayleigh quotient

$$\lambda_j(t) = \min_{L_j \subset V, \dim L_j = j}\ \max_{v \in L_j \setminus \{0\}} R_t[v] = \max_{W_j \subset V, \mathrm{codim} W_j = j-1}\ \min_{v \in W_j \setminus \{0\}} R_t[v]$$

$$R_t[v] = \frac{A_t(v, v)}{B_t(v, v)},\ \ v \in V \setminus \{0\}$$

<u>Proof</u> $L \subset X,\ \dim L = j$ ⟶ $\dim (T_t^{-1})^* L \leq \dim L,\ (T_t^{-1})^* L \subset X_t$

representation formula v.s. differentiability

Abstract Theory $X, \ |\cdot| \quad V, \ \|\cdot\|$ Hilbert spaces / $\mathbf{R}$

$V \overset{\text{dense}}{\hookrightarrow} X$ compact $\quad t \in I = (-\varepsilon_0, \varepsilon_0)$

$A_t : V \times V \to \mathbf{R}, \ B_t : X \times X \to \mathbf{R}$

uniformly bounded, uniformly coercive, bilinear **symmetric** forms

assume $\forall t \in I$

$$\lim_{h\to 0} \sup_{|u|,|v|\le 1} |(B_{t+h} - B_t)(u,v)| = 0$$

$$\lim_{h\to 0} \sup_{\|u\|,\|v\|\le 1} |(A_{t+h} - A_t)(u,v)| = 0 \qquad \boxed{0}$$

---

uniformly bounded

$\exists \dot{B}_t : X \times X \to \mathbf{R} \quad \exists \dot{A}_t : V \times V \to \mathbf{R}$

$$\lim_{h\to 0} \sup_{|u|,|v|\le 1} \frac{1}{h}|(B_{t+h} - B_t - h\dot{B}_t)(u,v)| = 0$$

$$\lim_{h\to 0} \sup_{\|u\|,\|v\|\le 1} \frac{1}{h}|(A_{t+h} - A_t - h\dot{A}_t)(u,v)| = 0$$

$$\lim_{h\to 0} \sup_{|u|,|v|\le 1} |(\dot{B}_{t+h} - \dot{B}_t)(u,v)| = 0$$

$$\lim_{h\to 0} \sup_{\|u\|,\|v\|\le 1} |(\dot{A}_{t+h} - \dot{A}_t)(u,v)| = 0 \qquad \boxed{\text{I}}$$

uniformly bounded

$\exists \dot{B}_t, \ddot{B}_t : X \times X \to \mathbf{R} \qquad \exists \dot{A}_t, \exists \ddot{A}_t : V \times V \to \mathbf{R}$

$$\lim_{h\to 0} \sup_{|u|,|v|\le 1} \frac{1}{h^2}|(B_{t+h} - B_t - h\dot{B}_t - \frac{h^2}{2}\ddot{B}_t)(u,v)| = 0$$

$$\lim_{h\to 0} \sup_{\|u\|,\|v\|\le 1} \frac{1}{h^2}|(A_{t+h} - A_t - h\dot{A}_t - \frac{h^2}{2}\ddot{A})(u,v)| = 0$$

Taylor expansion

$$\lim_{h\to 0} \sup_{|u|,|v|\le 1} |(\ddot{B}_{t+h} - \ddot{B}_t)(u,v)| = 0$$

$$\lim_{h\to 0} \sup_{\|u\|,\|v\|\le 1} |(\ddot{A}_{t+h} - \ddot{A}_t)(u,v)| = 0 \qquad \boxed{\text{II}}$$

---

eigenvalue problem $u \in V, \ A_t(u,v) = \lambda B_t(u,v), \ \forall v \in V$

$\lambda = \lambda_j(t), \ j = 1, 2, \cdots \qquad m_j(t)$ multiplicity

$$0 < \lambda_1(t) \le \lambda_2(t) \le \cdots \to +\infty$$

Theorem 1 $\quad \forall j, \ \lim_{h\to 0} \lambda_j(t+h) = \lambda_j(t)$

$\boxed{5}$

assume $t \in I = (-\varepsilon_0, \varepsilon_0)$ fix $\exists k, m = 1, 2, \cdots$

$\lambda_{k-1}(t) < \lambda \equiv \lambda_k(t) = \cdots = \lambda_{k+m-1}(t) < \lambda_{k+m}(t)$

$\lambda_0(t) = -\infty$ $Y_t,\ \dim Y_t^\lambda = m$ eigenspace ↔ λ

Theorem 2 $k \leq j \leq k+m-1$

$$\exists \dot\lambda_j^\pm(t) \equiv \lim_{h\to\pm 0} \frac{1}{h}(\lambda_j(t+h) - \lambda_j(t))$$

$\nu_j \equiv \dot\lambda_j^+ = \dot\lambda_{2k+m-1-j}^-$

$q = j-k+1$-th eigenvalue of the matrix

independent of the choice of orthonormal basis

$G_t^\lambda = (E_t^\lambda(\tilde\phi_i, \tilde\phi_j))_{1\leq i,j\leq m}$ $E_t^\lambda = \dot A_t - \lambda \dot B_t$

$Y_t = \langle \tilde\phi_j \mid 1 \leq j \leq m \rangle$ $B_t(\tilde\phi_i, \tilde\phi_j) = \delta_{ij}$

Remark $\dot\lambda_k^+(t) \leq \cdots \leq \dot\lambda_{k+m-1}^+(t)$

Theorem 3 $\lim_{h\to\pm 0} \dot\lambda_j^\pm(t+h) = \dot\lambda_j^\pm(t),\ \forall t \in I,\ \forall j$

Theorem 4 $\forall t \in I$

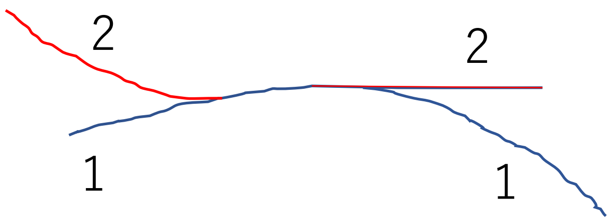
$\lambda_{k-1}(t) < \lambda_k(t) \leq \cdots \leq \lambda_{k+m-1}(t) < \lambda_{k+m}(t)$

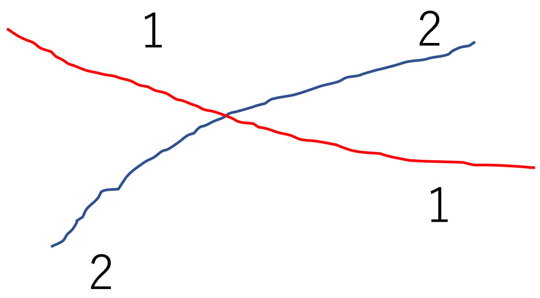
$C_j = \{\lambda_j(t) \mid t \in I\},\ k \leq j \leq k+m-1$

⟹ $\exists \tilde C_j,\ k \leq j \leq k+m-1$

C^1 curves made by the transversal rearrangement of

$C_j,\ k \leq j \leq k+m-1$ at most countably many times





References standard method

Reduction to finite dimension (Lyapunov-Schmidt) Chow-Hale (1982) p.486

F. Rellich (1953) p. 44-52.

T. Kato (1966) p. 122-123

S.N. Chow, J.K. Hale (1982) p. 490

The proof of this theorem is rather complicated.
The original proof due to Rellich is even longer.

It is very difficult and a simpler proof is desirable.

Theorem 5 $t \in I$ fix $k \le \ell < r \le k+m$

$$\lambda_{k-1}(t) < \lambda \equiv \lambda_k(t) = \cdots = \lambda_{k+m-1}(t) < \lambda_{k+m}(t)$$

$$\dot\lambda^+_{\ell-1}(t) < \lambda' \equiv \dot\lambda^+_\ell(t) = \cdots = \dot\lambda^+_{r-1}(t) < \dot\lambda^+_r(t)$$

$$\longrightarrow \quad \exists \lambda''_j = \lim_{h\to 0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda - h\lambda'),\ \ell \le j \le r-1$$

$q = j - \ell + 1$ -th eigenvalue of the matrix

$$H^\lambda_t = \left(F^{\lambda,\lambda'}_t(\tilde\phi_i, \tilde\phi_j)\right)_{\ell \le i,j \le r-1}$$

independent of the choice of orthonormal basis

non-local term

$$F^{\lambda,\lambda'}_t(u,v) = (\ddot A_t - \lambda \ddot B_t - 2\dot\lambda B_t)(u,v) - 2C_t(\gamma(u),\gamma(v))$$

$$C_t = A_t - \lambda B_t, \quad \dot C^{\lambda,\lambda'}_t = \dot A_t - \lambda \dot B_t - \lambda' B_t$$

$$u \in V,\ w = \gamma(u) \in PV$$

$$\overset{\text{def}}{\Longleftrightarrow} \quad C_t(w,v) = -\dot C^{\lambda,\lambda'}_t(u,v),\ \forall v \in PV$$

$P = I - Q$ orthogonal projection w.r.t. $B_t(\cdot,\cdot)$

$$Q : X = L^2(\Omega) \to \langle \tilde\phi_j \mid 1 \le j \le m \rangle$$

Corollary $\forall j, \forall t$

$$\exists \ddot\lambda^\pm_j(t) = \lim_{h\to 0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda_j(t) - h\dot\lambda^\pm_j(t))$$

Theorem 6 $\lim_{h\to\pm 0} \ddot\lambda^\pm_j(t+h) = \ddot\lambda^\pm_j(t)$

Theorem 7

$$\lambda_{k-1}(t) < \lambda_k(t) \le \cdots \le \lambda_{k+m-1}(t) < \lambda_{k+m}(t),\ t \in I$$

$\longrightarrow$ $\tilde C_j,\ k \le j \le k+m-1$ in Theorem 4 are C^2

i.e. $\tilde C_j = \{\tilde\lambda_j(t) \mid t \in I\}$

$$\longrightarrow \quad \exists \tilde\lambda'_j(t) = \lim_{h\to 0} \frac{1}{h}(\lambda_j(t+h) - \lambda_j(t))$$

$$\exists \tilde\lambda''_j(t) = \lim_{h\to 0} \frac{1}{h}(\tilde\lambda'_j(t+h) - \tilde\lambda_j(t))$$

$$\lim_{h\to 0} \tilde\lambda''_j(t+h) = \tilde\lambda''_j(t),\ \forall t \in I$$

Application: harmonic concavity of the first eigenvalue ➡ new isoperimetric inequality

Details

$X, \ |\cdot| \quad V, \ \|\cdot\|$ real Hilbert spaces

$V \hookrightarrow X$ compact $\quad t \in I = (-\varepsilon_0, \varepsilon_0)$

$A_t : V \times V \to \mathbf{R}, \ B_t : X \times X \to \mathbf{R}$ bilinear symmetric forms

$$|A_t(u,v)| \le C\|u\| \cdot \|v\|, \ A_t(u,u) \ge \delta\|u\|^2$$

$$|B_t(u,v)| \le C|u| \cdot |v|, \ B_t(u,u) \ge \delta|u|^2$$

Eigenvalue problem

$$u \in V, \ A_t(u,v) = \lambda B_t(u,v), \ \forall v \in V$$

$$0 < \lambda_1(t) \le \lambda_2(t) \le \cdots \to +\infty$$ eivenvalues

eigenfunctions

$$u_j(t) \in V, \ B_t(u_i(t), u_j(t)) = \delta_{ij}$$

$$A_t(u_j(t), v) = \lambda_j(t) B_t(u_j(t), v), \ \forall v \in V$$

Assume

$t \in I$ fix

$$\lim_{h\to 0} \sup_{|u|,|v|\le 1} |(B_{t+h} - B_t)(u,v)| = 0$$

$$\lim_{h\to 0} \sup_{\|u\|,\|v\|\le 1} |(A_{t+h} - A_t)(u,v)| = 0$$

Theorem 1

$$\forall j, \ \lim_{h\to 0} \lambda_j(t+h) = \lambda_j(t)$$

Proof

$$\alpha(h) = \sup_{\|u\|,\|v\|\le 1} |(A_{t+h} - A_t)(u,v)| = o(1)$$

$$\beta(h) = \sup_{|u|,|v|\le 1} |(B_{t+h} - B_t)(u,v)| = o(1)$$

$$(1 - \delta^{-1}\alpha(h))A_t(v,v) \le A_{t+h}(v,v) \le (1 + \delta^{-1}\alpha(h))A_t(v,v)$$

$$(1 - \delta^{-1}\beta(h))B_t(v,v) \le B_{t+h}(v,v) \le (1 + \delta^{-1}\beta(h))B_t(v,v)$$

$\Longrightarrow$ $(1 - o(1))R_t[v] \le R_{t+h}[v] \le (1 + o(1))R_t[v]$ Rayleigh quotient

$(1 - o(1))\lambda_j(t) \le \lambda_j(t+h) \le (1 + o(1))\lambda_j(t)$ min-max principle

Convergence of eigenvectors

$$\lambda_{k-1}(t) < \lambda \equiv \lambda_k(t) = \cdots = \lambda_{k+m-1}(t) < \lambda_{k+m}(t)$$

$\forall h_\ell \to 0 \ \Longrightarrow$ (subsequence) $u_j(t+h_\ell) \to \phi_j$ in $V, \ k \le j \le k+m-1$

$B_t(\phi_i, \phi_j) = \delta_{ij}$ strong convergence

$A_t(\phi_j, v) = \lambda B_t(\phi_j, v), \ \forall v \in V$

1. boundedness in V
2. weak convergence in V
3. compactness of $V \hookrightarrow X$
4. coerciveness of A_t
5. strong convergence in V

First derivatives $t \in I$ fix $k, m = 1, 2, \cdots$

$$\lambda_{k-1}(t) < \lambda \equiv \lambda_k(t) = \cdots = \lambda_{k+m-1}(t) < \lambda_{k+m}(t)$$

$Y_t^\lambda = \langle u_j(t) \mid k \leq j \leq k+m-1 \rangle$ eigenspace ⟷ λ

$Y_t^\lambda = \langle \tilde{\phi}_j \mid 1 \leq j \leq m \rangle \qquad B_t(\tilde{\phi}_i, \tilde{\phi}_j) = \delta_{ij}$

$$E_t^\lambda = \dot{A}_t - \lambda \dot{B}_t$$

Theorem 2 $k \leq j \leq k+m-1$ unilateral derivatives

$$\exists \dot{\lambda}_j^\pm(t) \equiv \lim_{h \to \pm 0} \frac{1}{h}(\lambda_j(t+h) - \lambda_j(t))$$

$$\nu_j \equiv \dot{\lambda}_j^+ = \dot{\lambda}_{2k+m-1-j}^-, \ k \leq j \leq k+m-1$$

$q = j - k + 1$ -th eigenvalue of the matrix $G_t^\lambda = (E_t^\lambda(\tilde{\phi}_i, \tilde{\phi}_j))_{1 \leq i,j \leq m}$

Remark G_t^λ independent of the choice of orthonormal basis $\{\tilde{\phi}_j\}$ of Y_t^λ

$$\dot{\lambda}_k^+(t) \leq \cdots \leq \dot{\lambda}_{k+m-1}^+(t), \ \dot{\lambda}_k^-(t) \geq \cdots \geq \dot{\lambda}_{k+m-1}^-(t)$$

Each entry of this matrix is reduced to the area integral of the eigenfunctions in the domain perturbation problem

Formal derivation $\lambda_t = \lambda_j(t), \ u_t = u_j(t)$

$$B_t(u_t, u_t) = 1, \ A_t(u_t, v) = \lambda_t B_t(u_t, v), \ v \in V$$

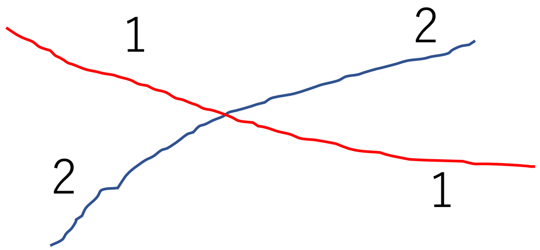
$$\dot{A}_t(u_t, v) + A_t(\dot{u}_t, v) = \dot{\lambda}_t B_t(u_t, v) + \lambda_t \dot{B}_t(u_t, v) + \lambda_t B_t(\dot{u}_t, v)$$

$$\Rightarrow \dot{A}_t(u_t, u_t) + A_t(\dot{u}_t, u_t) = \dot{\lambda}_t + \lambda_t \dot{B}_t(u_t, u_t) + \lambda_t B_t(\dot{u}_t, u_t)$$

$$\dot{\lambda}_t = \dot{A}_t(u_t, u_t) + 2A_t(\dot{u}_t, u_t)$$

$$\dot{B}_t(u_t, u_t) + 2B_t(\dot{u}_t, u_t) = 0$$

$$\Rightarrow \dot{\lambda}_t = E_t^\lambda(u_t, u_t) \qquad E_t^\lambda = \dot{A}_t - \lambda \dot{B}_t$$



Rigorous argument $\lambda_{k-1}(t) < \lambda \equiv \lambda_k(t) = \cdots = \lambda_{k+m-1}(t) < \lambda_{k+m}(t)$

$t \in I$ fix $Y_t^\lambda = \langle u_j(t) \mid k \leq j \leq k+m-1 \rangle$ ↔ λ eigenspace $B_t(u_i(t), u_j(t)) = \delta_{ij}$

Lemma $\forall h_\ell \to 0$ ⟹ (subsequence) $u_j(t+h_\ell) \to \phi_j$ in V, $k \leq j \leq k+m-1$

characterize the first derivatives by a finite dimensional eigenvalue problem

$$\lim_{\ell\to\infty} \frac{1}{h_\ell}(\lambda_j(t+h_\ell)-\lambda) = E_t^\lambda(\phi_j,\phi_j),\ E_t^\lambda(\phi_i,\phi_j) = \delta_{ij},\ B_t(\phi_i,\phi_j) = \delta_{ij},\ k \leq i,j \leq k+m-1$$

Key identity $$\lim_{\ell\to\infty} h_\ell(A_{t+h_\ell} - A_t)\left(\frac{u_j(t+h_\ell)-\phi_j}{h_\ell}, \frac{u_{j'}(t+h_\ell)-\phi_{j'}}{h_\ell}\right) = 0$$

$$h(A_{t+h} - A_t)\left(\frac{u_j(t+h)-\phi_j}{h}, \frac{u_{j'}(t+h)-\phi_{j'}}{h}\right) = \frac{1}{h}(A_{t+h} - A_t)(u_j(t+h), u_{j'}(t+h))$$

$$+\frac{1}{h}(A_{t+h} - A_t)(\phi_j,\phi_{j'}) - \frac{1}{h}(A_{t+h} - A_t)(u_j(t+h),\phi_{j'}) - \frac{1}{h}(A_{t+h} - A_t)(\phi_j, u_{j'}(t+h))$$

+ symmetry, coerciveness

Corollary $$\exists\dot\lambda_j^\pm(t) = \lim_{h\to\pm 0}\frac{1}{h}(\lambda_j(t+h)-\lambda_j(t)),\quad \dot\lambda_j^+(t) = \mu_{j-k+1},\ \dot\lambda_j^-(t) = \mu_{k+m-j},\ k \leq j \leq k+m-1$$

$\mu_q,\ 1 \leq q \leq m$ q -th eigenvalue of $u \in Y_t^\lambda,\ E_t^\lambda(u,v) = \mu B_t(u,v),\ \forall v \in Y_t^\lambda$ ⟹ Theorem 2

Transversal rearrangement $f_j(t) \in C(I),\ 1 \le j \le m;\ f_1(t) \le \cdots \le f_m(t),\ \forall t \in I = (-\varepsilon_0, \varepsilon_0)$

Reduction to the finite dimension via Theorem 1

(P1) $\exists \dot f_j^{\pm}(t) = \lim_{h \to \pm 0} \frac{1}{h}(f_j(t+h) - f_j(t)),\ \forall t, \forall j$ (P2) $\lim_{h \to \pm 0} \dot f_j^{\pm}(t+h) = \dot f_j^{\pm}(t),\ \forall t, \forall j$

(P3) $1 \le n \le m,\ 1 \le k \le m-n+1,\ t \in I;\ f_{k-1}(t) < f_k(t) = \cdots = f_{k+n-1}(t) < f_{k+n}(t)$

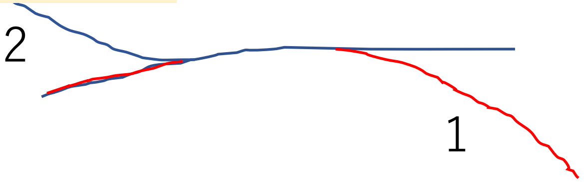
$\Longrightarrow \dot f_j^{+}(t) = \dot f_{2k+n-j-1}^{-}(t),\ k \le \forall j \le k+n-1$ $K = K_{k,n}(t) = \{k, \cdots, k+n-1\}$

cluster at t with entry k and length n

Definition $p(K) = \max\{j \mid k \le j \le k + [\frac{n}{2}] - 1,\ \dot f_j^{+}(t) < \dot f_{2k+n-1-j}^{+}(t)\} - k + 1$ degree of transverse

Lemma $I_1 \overset{\text{def}}{=} \{t \in I \mid p(K) \ge 1,\ \exists K \text{ cluster at } t\}$ is at most countable

unilateral continuity of unilateral derivatives



Definition $\tilde C_j,\ 1 \le j \le m$ **transversal rearrangement** of $C_j = \{f_j(t) \mid t \in I\},\ 1 \le j \le m$

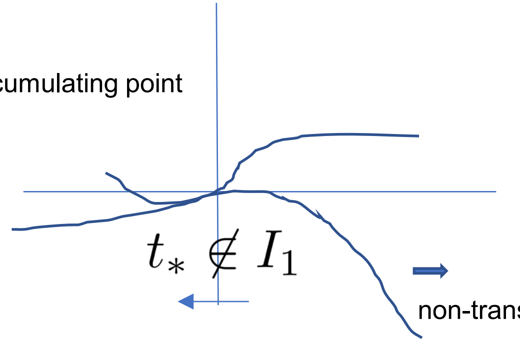
$\overset{\text{def}}{\Longleftrightarrow}$ $t \in I_1,\ 1 \le j \le m,\ j \in K = K_{k,n}(t),\ p(K) \ge 1 \Longrightarrow$

$C_j,\ k \le j \le k+p-1,\ k-n-p \le j \le k-n-1,\ p = p(K)$

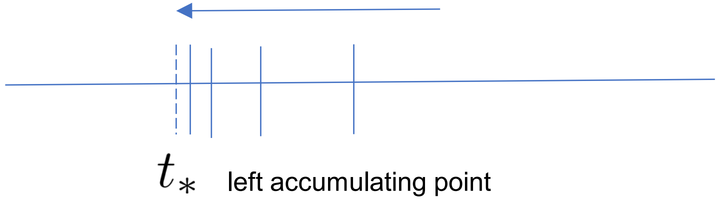
on the right is connected to $C_{2k+n-j-1}$ on the left

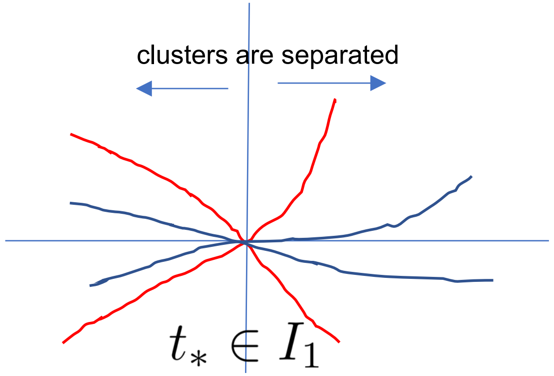
No operation otherwise

not right accumulating point

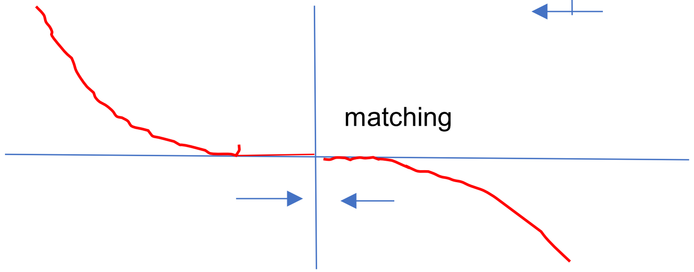


Theorem 4 $\tilde C_j,\ 1 \le j \le m$ are C^1





right accumulating point



Second derivatives $A_t(u_t, v) = \lambda_t B_t(u_t, v), \ v \in V$ $\dot{\lambda}_t = \dot{A}_t(u_t, u_t) - \lambda_t \dot{B}_t(u_t, u_t)$

$$\dot{A}_t(u_t, v) + A_t(\dot{u}_t, v) = \dot{\lambda}_t B_t(u_t, v) + \lambda_t \dot{B}_t(u_t, v) + \lambda_t B_t(\dot{u}_t, v), \ v = \dot{u}_t$$

$$\ddot{\lambda}_t = \ddot{A}_t(u_t, u_t) + 2\dot{A}_t(\dot{u}_t, u_t) - \dot{\lambda}_t \dot{B}_t(u_t, u_t) - \lambda_t \ddot{B}_t(\dot{u}_t, u_t) - 2\lambda_t \dot{B}_t(\dot{u}_t, u_t)$$

$$\ddot{\lambda}_t = -2(A_t - \lambda_t B_t)(z_*, z_*) + \ddot{A}_t(u_t, u_t) - 2\dot{\lambda}_t \dot{B}_t(u_t, u_t) - \lambda_t \ddot{B}_t(u_t, u_t), \ z_* = \dot{u}_t$$

defined up to the ambiguity of the eigenspace

Remark Direct differentiation of $\dot{\lambda}_j^{\pm}(t)$ is difficult because of the multiplicity

$t \in I$ fix $\lambda_{k-1}(t) < \lambda = \lambda_k(t) = \cdots = \lambda_{k+m-1}(t) < \lambda_{k+m}(t)$

Assume $\exists \ddot{B}_t : X \times X \to \mathbf{R}$ $\exists \ddot{A}_t : V \times V \to \mathbf{R}$ bounded, bilinear

Taylor expansion

$$\lim_{h\to 0} \sup_{|u|,|v|\leq 1} \frac{1}{h^2} |(B_{t+h} - B_t - h\dot{B}_t - \frac{h^2}{2}\ddot{B}_t)(u,v)| = 0 \quad \lim_{h\to 0} \sup_{\|u\|,\|v\|\leq 1} \frac{1}{h^2} |(A_{t+h} - A_t - h\dot{A}_t - \frac{h^2}{2}\ddot{A}_t)(u,v)| = 0$$

$$\ell + \ell_- = r + r_- = 2k + m - 1$$

Theorem 5 $\dot{\lambda}^+_{\ell-1} < \lambda' \equiv \dot{\lambda}^+_{\ell} = \cdots = \dot{\lambda}^+_{r-1} < \dot{\lambda}^+_r \Leftrightarrow \dot{\lambda}^-_{\ell_-+1}(t) > \lambda' = \dot{\lambda}^-_{\ell_-}(t) = \cdots = \dot{\lambda}^-_{r_--1}(t) > \dot{\lambda}^-_{r_-}(t)$

$$k \leq \ell < r \leq k+m \longrightarrow \exists \lambda''_j = \lim_{h\to 0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda - h\lambda'), \ \ell \leq j \leq r-1$$

$q = j - \ell + 1$ -th eigenvalue of the matrix

$$H_t^{\lambda} = \left(F_t^{\lambda,\lambda'}(\tilde{\phi}_i, \tilde{\phi}_j)\right)_{\ell\leq i,j\leq r-1}$$

independent of the choice of orthonormal basis

non-local term

$$F_t^{\lambda,\lambda'}(u,v) = (\ddot{A}_t - \lambda\ddot{B}_t - 2\dot{\lambda}B_t)(u,v) - 2C_t^{\lambda}(\gamma(u), \gamma(v))$$

$$u \in V, \ w = \gamma(u) \in PV \longleftrightarrow C_t^{\lambda}(w, v) = -\dot{C}_t^{\lambda,\lambda'}(u,v), \ \forall v \in PV \qquad \dot{C}_t^{\lambda,\lambda'} = \dot{A}_t - \lambda\dot{B}_t - \lambda' B_t$$

$C_t^{\lambda} = A_t - \lambda B_t$ $P = I - Q, \ Q : X = L^2(\Omega) \to Y_t^{\lambda} = \langle \tilde{\phi}_j \mid 1 \leq j \leq m \rangle$ orthogonal projection w.r.t. $B_t(\cdot, \cdot)$

Corollary $\exists \ddot{\lambda}_j^{\pm}(t) = \lim_{h\to 0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda_j(t) - h\dot{\lambda}_j^{\pm}(t)),\ \forall j, \forall t$ bilateral derivatives

Remark $h^2 > 0$

Impossibility of C3 rearrangement

Proof of Theorem 5 recall

$\forall h_\ell \to 0$ subsequence

$\exists \dot{\lambda}_j^{*} = \lim_{\ell\to\infty} \frac{1}{h_\ell}(\lambda_j(t+h_\ell) - \lambda_j(t)),\ k \le j \le k+m-1$

$u_j(t+h_\ell) \to \phi_j \text{ in } V$

$z_\ell^j \overset{\text{def}}{=} \frac{1}{h_\ell}(u_j(t+h_\ell) - \phi_j)$ $\qquad \dot{C}_t^{*j} \overset{\text{def}}{=} \dot{A}_t - \dot{\lambda}_j^{*} B_t - \lambda \dot{B}_t$

$\longrightarrow \lim_{\ell\to\infty} C_t^\lambda(z_\ell^j, v) = -\dot{C}_t^{*j}(\phi_j, v),\ \forall v \in V$ $\qquad C_t^\lambda = A_t - \lambda B_t$

1. boundedness in V
2. weak convergence in V
3. compactness of
4. coerciveness of
5. strong convergence in V

A_t symmetry, coerciveness

$V \hookrightarrow X$ compactness

Key Lemma $\exists \lim_{\ell\to\infty} P z_\ell^j = z_*^j \text{ in } V$ $\qquad z_*^j \overset{\text{def}}{=} \gamma_{\dot{\lambda}_j^*}(\phi_j) \in PV,\ k \le j \le k+m-1$

Remark Convergence of $\{z_\ell^j\}$ is **not expected** $\quad P \overset{\text{recall}}{=} I - Q,\ Q : X = L^2(\Omega) \to Y_t^\lambda = \langle \tilde{\phi}_j \mid 1 \le j \le m \rangle$ orthogonal projection w.r.t. $B_t(\cdot,\cdot)$

Lemma $\exists \ddot{\lambda}_j^{*} \overset{\text{def}}{=} \lim_{\ell\to\infty} \frac{2}{h_\ell^2}(\lambda_j(t+h_\ell) - \lambda_j(t) - h_\ell \dot{\lambda}_j^{*}) = (\ddot{A}_t - \lambda \ddot{B}_t - 2\dot{\lambda}_j^{*} \dot{B}_t)(\phi_j, \phi_j) - 2C_t^\lambda(z_*^j, z_*^j)$

Lemma $k \le j \ne j' \le k+m-1,\ \dot{\lambda}_* \equiv \dot{\lambda}_j^{*} = \dot{\lambda}_{j'}^{*}$ $\longrightarrow$ $(\ddot{A}_t - \lambda \ddot{B}_t - 2\dot{\lambda}_* \dot{B}_t)(\phi_j, \phi_{j'}) - 2C_t^\lambda(z_*^j, z_*^{j'}) = 0$

$\longrightarrow$ characterization of the second unilateral derivatives **Corollary**

Theorem 7 $\tilde{C}_j = \{\tilde{\lambda}_j(t)\},\ k \le j \le k+m-1$ are C^2

Proof $\tilde{C}_j$ is C^1 $\exists \tilde{\lambda}_j''(t) = \lim_{h\to 0} \frac{2}{h^2}(\tilde{\lambda}_j(t+h) - \tilde{\lambda}_j(t) - h\tilde{\lambda}_j'(t))$ **loc. unif.** $\tilde{\lambda}_j''(t+0) = \tilde{\lambda}_j''(t-0)$

$$\tilde{\lambda}_j(t+h) = \tilde{\lambda}_j(t) + h\tilde{\lambda}_j'(t) + \frac{h^2}{2}\tilde{\lambda}_j''(t) + o(h^2)$$

$$\tilde{\lambda}_j(t) = \tilde{\lambda}_j(t+h) - h\tilde{\lambda}_j'(t+h) + \frac{h^2}{2}\tilde{\lambda}_j''(t+h) + o(h^2)$$

⟹

$$\lim_{h\to 0} \frac{1}{h}(\tilde{\lambda}_j'(t+h) - \tilde{\lambda}_j'(t))$$

$$= \lim_{h\to 0} \frac{1}{2}(\tilde{\lambda}_j''(t+h) + \tilde{\lambda}_j''(t)) = \tilde{\lambda}_j''(t)$$

Conclusion

transversal rearrangement of eigenvalues ⟶ C^2 curves

characterization of their derivatives ⟵ finite dimensional eigenvalue problems

Processes

1. Reduction to abstract setting
2. Continuity
3. Unilateral derivatives and characterizations
4. Transversal rearrangement
5. Taylor expansion of the second order
6. Second order derivatives and characterizations
7. The rearrangement is C2

Symmetry of bilinear forms is essential

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