

棲み分けと可積分系

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Smoluchowski-Poisson equation – a model in statistical mechanics

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u \nabla v)$$

$$\left. \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0$$

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

canonical ensemble

2D is critical for blowup of the solution to quadratic nonlinearity under the total mass control

self-similar transformation due to the quadratic growth

$$u_\mu(x, t) = \mu^2 u(\mu x, \mu^2 t), \quad \mu > 0$$

$$\|u\|_1 = \|u_\mu\|_1 \equiv \lambda \Leftrightarrow n = 2 \quad \text{critical dimension}$$

1. total mass conservation $\frac{d}{dt} \|u(t)\|_1 = 0$

2. free energy decreasing

$$\mathcal{F}(u) = \int_{\Omega} u(\log u - 1) - \frac{1}{2} \int \int_{\Omega \times \Omega} G(x, x') u \otimes u$$

$$\frac{d}{dt} \mathcal{F}(u) = - \int_{\Omega} u |\nabla(\log u - v)|^2 \leq 0$$

$$\mathcal{F}(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2} \langle \Gamma * u, u \rangle, \quad \Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

$$\mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi} \right) \log \mu + \mathcal{F}(u) \quad \text{critical mass } \lambda = 8\pi$$

$$G(x, x') = G(x', x) \quad \text{Green's function}$$

quantized blowup mechanism with Hamiltonian control

1. stationary 2. finite time 3. infinite time

6. 反応拡散系

ロトカ・ボルテラ系

$$\tau_j \frac{\partial u_j}{\partial t} = d_j \Delta u_j + (-e_j + \sum_k a_{jk} u_k) u_j$$

$$\frac{\partial u_j}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad u_j|_{t=0} = u_{j0}(x) \geq 0$$

$$(Au, u) \leq 0, \quad \forall u \geq 0 \quad A = (a_{jk})$$

$$e = (e_j) \leq 0 \implies T = +\infty \quad \|u(\cdot, t)\|_\infty \leq C$$

Masuda-Takahashi 94 (n=1) S.-Yamada 15 (n=2)

scaling invariance (e=0)

$$u_j^\mu(x, t) = \mu^2 u_j(\mu x, \mu^2 t), \quad \mu > 0$$

rigidness (n=2, quadratic growth by L¹ control) $\|u(\cdot, t)\|_1 \leq \|u_0\|_1$

$$\|u_0\|_1 \ll 1 \implies T = +\infty, \quad \sup_{t \geq 0} \|u(\cdot, t)\|_\infty < +\infty$$



$$f_j(u) = (-e_j + \sum_k a_{jk} u_k) u_j, \quad u = (u_j)$$

$$\implies \sum_{j=1}^N f_j(u) \log u_j \leq C(1 + |u|^2)$$

singularity relaxation

L2 estimate in space and time

$$\implies \sup_{0 \leq t < T} \int_\Omega \Phi(u_j(\cdot, t)) \leq C_T$$

$$\Phi(s) = s(\log s - 1) + 1 \geq 0, \quad s > 0$$

global GN inequality $\implies T = +\infty$ if n=1, 2

$$\int_{-1}^1 \left| \frac{d}{dt} \int_\Omega u_j(\cdot, t + t_k) \varphi \right| dt \leq C_\varphi$$

$$\forall \varphi \in C^2(\bar{\Omega}), \quad \frac{\partial \varphi}{\partial \nu} \Big|_{\partial \Omega} = 0 \quad \text{monotonicity formula}$$

by L2 control in space-time

evokes the measure-valued continuation (very weak solution)

blowup in infinite time ?

L2-L1 estimate

$$\tau_j \frac{\partial u_j}{\partial t} - d_j \Delta u_j = f_j(u)$$

$$\frac{\partial u_j}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad u_j|_{t=0} = u_{j0}(x) \geq 0$$

$$\sum_{j=1}^N f_j(u) \leq 0$$

$$\tau = (\tau_j), \quad d = (d_j) > 0$$

$$\frac{\partial}{\partial t}(\tau \cdot u) - \Delta(d \cdot u) \leq 0, \quad u = (u_j) \geq 0$$

$$\frac{\partial}{\partial \nu}(d \cdot u) \Big|_{\partial \Omega} \leq 0, \quad u|_{t=0} = u_0 = (u_{j0})$$

$$\tau \cdot u(\cdot, t) - \tau \cdot u_0 \leq \int_0^t \Delta(d \cdot u(\cdot, s)) \, ds$$

$$\frac{d}{dt} \int_{\Omega} \tau \cdot u \leq 0 \rightarrow \boxed{\sup_{0 \leq t < T} \|u(\cdot, t)\|_1 \leq C}$$

$$\begin{aligned} \rightarrow (\tau \cdot u(\cdot, t), d \cdot u(\cdot, t)) - (\tau \cdot u_0, d \cdot u(\cdot, t)) &\leq -(\nabla d \cdot u(\cdot, t), \nabla \int_0^t d \cdot u(\cdot, s) \, ds) \\ &= -\frac{1}{2} \frac{d}{dt} \|\nabla \int_0^t d \cdot u(\cdot, s) \, ds\|_2^2 \end{aligned}$$

$$\int_0^T (\tau \cdot u(\cdot, t), d \cdot u(\cdot, t)) \, dt \leq \|\tau \cdot u_0\|_2 \cdot \int_0^T \|d \cdot u(\cdot, t)\|_2 \, dt$$

$$\leq CT^{\frac{1}{2}} \|\tau \cdot u_0\|_2 \cdot \left\{ \int_0^T \|d \cdot u(\cdot, t)\|_2^2 \, dt \right\}^{\frac{1}{2}} \rightarrow \boxed{\|u\|_{L^2(Q_T)} \leq CT^{\frac{1}{2}} \|u_0\|_2}$$

L1 pre-compactness

1. semi-group reduction Baras-Pierre 84

$$\frac{\partial w}{\partial t} - \Delta w = H \in L^1(Q_T)$$

$$\frac{\partial w}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad w|_{t=0} = w_0(x) \in L^1(\Omega)$$

$$w = w(\cdot, t) \in L^\infty(0, T; L^1(\Omega)) \cap L^1_{loc}(0, T; W^{1,1}(\Omega))$$

i.e. as distributions weak solution

$$\frac{d}{dt} \int_{\Omega} w \varphi + \int_{\Omega} \nabla w \cdot \nabla \varphi = \int_{\Omega} H \varphi, \quad \forall \varphi \in W^{1,\infty}(\Omega)$$

$w|_{t=0} = w_0$ in the sense of measures

$$\rightarrow w(\cdot, t) = e^{t\Delta} w_0 + \int_0^t e^{(t-s)\Delta} H(\cdot, s) ds$$

in particular $w \in C([0, T], L^1(\Omega))$

$$\mathcal{F} : (w_0, H) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in C([0, T], L^1(\Omega))$$

continuous

2. compactness

c.f. Baras 78

$$\mathcal{F} : (w_0, H) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in L^1(Q_T)$$

compact

Proof

 $\mathcal{F}^* : L^\infty(Q_T) \rightarrow L^\infty(\Omega) \times L^\infty(Q_T)$

$$\mathcal{F}^*(h) = (\theta|_{t=0}, \theta)$$

$$\frac{\partial \theta}{\partial t} + \Delta \theta = h, \quad \frac{\partial \theta}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad \theta|_{t=T} = 0$$

compact from the parabolic regularity

pre-compactness of the orbit in L1

$$0 \leq u_k(\cdot, t) = u(\cdot, t + t_k) \leftarrow$$

$$\leq \exists w_k(\cdot, t) \in L^2(\Omega \times (-1, 1))$$

comparison theorem

compact

dominated convergence theorem

Pierre-Rolland 15 $0 \leq \exists u = (u_j(\cdot, t)) \in C([0, +\infty), L^1(\Omega)^N)$ global-in-time weak solution

Pierre-S.-Yamada 19 $\mathcal{O} = \{u(\cdot, t)\} \subset L^1(\Omega)^N$ pre-compact

weak solution to $0 \leq u = (u_j(\cdot, t)) \in L_{loc}^\infty([0, T], L^1(\Omega)^N)$

$$\tau_j \frac{\partial u_j}{\partial t} - d_j \Delta u_j = f_j(u), \quad \frac{\partial u_j}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad u_j|_{t=0} = u_{j0}(x) \geq 0$$

\longleftrightarrow (def.)

$$f_j(u) \in L_{loc}^1(\bar{\Omega} \times (0, T))$$

as distributions

$$\frac{d}{dt} \int_{\Omega} u_j \varphi - d_j \int_{\Omega} u_j \Delta \varphi = \int_{\Omega} f_j(u) \varphi, \quad \forall \varphi \in W^{2,\infty}(\Omega), \quad \frac{\partial \varphi}{\partial \nu} \Big|_{\partial \Omega} = 0$$

$$u_j|_{t=0} = u_{j0}(x) \quad \text{in the sense of measures}$$

$$\tau_j \frac{\partial u_j}{\partial t} - d_j \Delta u_j = f_j(u) \text{ in } Q_T$$

$$\frac{\partial u_j}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad u_j|_{t=0} = u_{j0}(x)$$

$\Omega \subset \mathbf{R}^n$ bounded domain, $\partial \Omega$ smooth
 $Q_T = \Omega \times (0, T) \quad 1 \leq j \leq N$

ν outer unit normal
 $\tau = (\tau_j) > 0, \quad d = (d_j) > 0$
 $u_0 = (u_{j0}) \geq 0$ smooth

[local. Lipschitz cont.]

$f_j : \mathbf{R}^N \rightarrow \mathbf{R}, \quad 1 \leq j \leq N$
 loc. Lipschitz cont.



∃! classical solution local-in-time

$T \in (0, +\infty]$ maximal existence time

[quadratic]

$$|\nabla f_j(u)| \leq C(1 + |u|), \quad \forall j$$

[quasi-positive]

$$f_j(u_1, \dots, u_{j-1}, 0, u_{j+1}, \dots, u_n) \geq 0, \quad \forall j$$

$$0 \leq u_0 = (u_{j0}) \in \mathbf{R}^N \quad \longrightarrow$$

$$u = (u_j(\cdot, t)) \geq 0$$

[mass dissipation]

$$\sum_{j=1}^N f_j(u) \leq 0, \quad u = (u_j) \geq 0$$

$$\longrightarrow \frac{\partial}{\partial t} (\tau \cdot u) - \Delta (d \cdot u) \leq 0$$

$$\frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = 0$$

Theorem (Fellner-Morgan-Tang 20, 21)

$$T = +\infty \quad \|u(\cdot, t)\|_\infty \leq C$$

$$\|\tau \cdot u(t)\|_1 \leq \|\tau \cdot u_0\|_1$$

1. Sobolev inequality in space-time
2. Parabolic Giorgi-Nash-Moser regularity
3. Regularity interpolation
4. Souplet's trick by semigroup estimate

7. 可積分な生態系モデル

理論生物学の数理モデル

prey predator system

$$\frac{dx}{dt} = (a - by)x$$

$$\tau \frac{dy}{dt} = (-c + dx)y, \quad x, y > 0$$

$$\xi = \log x \quad \frac{d\xi}{dt} = a - be^\eta$$

$$\eta = \log y \quad \tau \frac{d\eta}{dt} = -c + de^\xi$$

$$H = -a\eta + be^\eta - \tau^{-1}c\xi + \tau^{-1}de^\xi$$

$$\frac{d\xi}{dt} = -H_\eta, \quad \frac{d\eta}{dt} = H_\xi \quad \frac{d}{dt}H(\xi, \eta) = 0$$

Gierer-Meinhardt system

$$\frac{du}{dt} = -u + \frac{u^p}{v^q}$$

$$\tau \frac{dv}{dt} = -v + \frac{u^r}{v^s}, \quad u, v > 0$$

$$u^{-p}(u_t + u) = v^{-q}$$

$$v^s(v_t + \tau^{-1}) = \tau^{-1}u^r$$

$$\xi = \frac{u^{-p+1}}{p-1}$$

$$\eta = \frac{v^{s+1}}{s+1}$$

$$\frac{d\xi}{dt} = (p-1)\xi - \{(s+1)\eta\}^{-\frac{q}{s+1}}$$

$$\frac{d\eta}{dt} = -\tau^{-1}(s+1)\eta + \tau^{-1}\{(p-1)\xi\}^{-\frac{1}{p-1}}$$

$$p-1 = \tau^{-1}(s+1) \equiv a$$

$$H = a\xi\eta + \left(\frac{r}{p-1} - 1\right)^{-1}A(\xi) + \left(\frac{q}{s+1} - 1\right)^{-1}B(\eta)$$

$$A(\xi) = \tau^{-1}(p-1)^{-\frac{r}{p-1}}\xi^{1-\frac{r}{p-1}}$$

$$B(\eta) = (s+1)^{-\frac{q}{s+1}}\eta^{1-\frac{q}{s+1}}$$

$$\rightarrow \frac{d\xi}{dt} = H_\eta, \quad \frac{d\eta}{dt} = -H_\xi$$

three system $x_1, x_2, x_3 > 0$

{Hamilton structures} \subset {Poisson structures} General form?

$$\tau_1 \frac{dx_1}{dt} = (x_2 - x_3)x_2$$

$$\tau_1 \frac{d\xi_1}{dt} = e^{\xi_2} - e^{\xi_3}$$

$$\tau_2 \frac{dx_2}{dt} = (x_3 - x_1)x_2$$

$$\tau_2 \frac{d\xi_2}{dt} = e^{\xi_3} - e^{\xi_1}$$

$$\tau_3 \frac{dx_3}{dt} = (x_1 - x_2)x_3$$

$$\tau_3 \frac{d\xi_3}{dt} = e^{\xi_1} - e^{\xi_2}$$

$$0 = \frac{d}{dt} M\xi \cdot a = \frac{d}{dt} (\tau_1 \xi_1 + \tau_2 \xi_2 + \tau_3 \xi_3)$$

$$0 = \frac{d}{dt} M(\xi) \cdot H(\xi) = \frac{d}{dt} (\tau_1 e^{\xi_1} + \tau_2 e^{\xi_2} + \tau_3 e^{\xi_3})$$

$$\xi_1 = \log x_1$$

$$\xi_2 = \log x_2$$

$$\xi_3 = \log x_3$$

$$M \frac{d\xi}{dt} = H(\xi) \times a$$

$$M = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad H(\xi) = \begin{pmatrix} e^{\xi_1} \\ e^{\xi_2} \\ e^{\xi_3} \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

可積分系と微分形式

$\Lambda^p(\mathbf{R}^n)$ p -forms on \mathbf{R}^n * Hodge

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \quad \xi = \sum_j \xi_j(t) d\xi_j, \quad \frac{d\xi}{dt} = \sum_j \dot{\xi}_j d\xi_j$$

Theorem A $H(\xi) \in \Lambda^0(\mathbf{R}^n)$ $h \in \Lambda^{n-2}(\mathbf{R}^n)$

$$*\frac{d\xi}{dt} = dH \wedge h \quad \longrightarrow \quad \frac{dH}{dt} = 0$$

Proof

$$\begin{aligned} 0 &= dH \wedge *\frac{d\xi}{dt} \\ &= \sum_{jk} \frac{\partial H}{\partial \xi_j} d\xi_j \wedge (-1)^{k+1} \frac{d\xi_j}{dt} d\xi_1 \wedge \cdots \wedge \widehat{d\xi_k} \wedge \cdots \wedge d\xi_n \\ &= \sum_j (-1)^{j+1} \frac{\partial H}{\partial \xi_j} d\xi_j \wedge \frac{d\xi_j}{dt} \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \\ &= \sum_j \frac{\partial H}{\partial \xi_j} \frac{d\xi_j}{dt} d\xi_1 \wedge \cdots \wedge d\xi_n = \frac{dH}{dt} d\xi_1 \wedge \cdots \wedge d\xi_n \end{aligned}$$

Corollary

$$1 \leq r \leq n - 1$$

$$H^1, \dots, H^r \in \Lambda^0(\mathbf{R}^n) \quad h \in \Lambda^{n-r-1}(\mathbf{R}^n)$$

$$*\frac{d\xi}{dt} = dH^1 \wedge \cdots \wedge dH^r \wedge h \quad \longrightarrow \quad \frac{dH^i}{dt} = 0, \quad 1 \leq i \leq r$$

... rank r

Tensor representation of p-form

$$\begin{aligned} \theta &= \sum_{j_1 \cdots j_p} \theta_{j_1 \cdots j_p} d\xi_{j_1} \wedge \cdots \wedge d\xi_{j_p} \in \Lambda^p(\mathbf{R}^n) \\ \theta_{j_1 \cdots j_p} &= \text{sgn} \sigma \cdot \theta_{\sigma(j_1) \cdots \sigma(j_p)}, \quad \sigma \in S_p \end{aligned}$$

Theorem B

rank r \longleftrightarrow

$$\begin{aligned} \frac{d\xi_i}{dt} &= \sum_{i_1 \cdots i_r} \frac{\partial H^1}{\partial \xi_{i_1}} \frac{\partial H^2}{\partial \xi_{i_2}} \cdots \frac{\partial H^r}{\partial \xi_{i_r}} h_{i, i_1 \cdots i_r}, \quad 1 \leq i \leq n \\ h_{\sigma(i) \sigma(i_1) \sigma(i_2) \sigma(i_r)} &= \text{sgn} \sigma \cdot h_{i i_1 i_2 \cdots i_r}, \quad \forall \sigma \in S_{r+1} \end{aligned}$$

alternative group

$$\xi = (\xi_j(t)) \in \mathbf{R}^n, \quad \xi = \sum_j \xi_j(t) d\xi_j, \quad \frac{d\xi}{dt} = \sum_j \dot{\xi}_j(t) d\xi_j$$

$$\frac{d\xi_i}{dt} = \sum_{i_1 \dots i_r} \frac{\partial H^1}{\partial \xi_{i_1}} \frac{\partial H^2}{\partial \xi_{i_2}} \dots \frac{\partial H^r}{\partial \xi_{i_r}} h_{i, i_1 \dots i_r}, \quad 1 \leq i \leq n$$

$$h_{\sigma(i)\sigma(i_1)\sigma(i_2)\sigma(i_r)} = \text{sgn } \sigma \cdot h_{ii_1i_2 \dots i_r}, \quad \forall \sigma \in S_{r+1}$$

... rank r

Corollary rank 1 \longleftrightarrow $\frac{d\xi}{dt} = h \nabla H, \quad h^T + h = 0$

$$\exists H \in \Lambda^0(\mathbf{R}^n) \quad \exists h = (h_{jk}) \in M_n(\mathbf{R})$$

$$\frac{dH}{dt} = 0 \quad \text{generalized Hamilton system}$$

Corollary rank n-1 \longleftrightarrow

$$\frac{d\xi_j}{dt} = \det A_j, \quad 1 \leq j \leq n \quad h_j^i = (-1)^{j+1} \frac{\partial H^i}{\partial \xi_j}$$

$$A_j = \begin{pmatrix} h_1^1 & \dots & \overset{j}{\cdot} & \dots & h_n^1 \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ h_1^{n-1} & \dots & \cdot & \dots & h_n^{n-1} \end{pmatrix} \quad \frac{dH^i}{dt} = 0, \quad 1 \leq i \leq n-1$$

- Aim**
1. matrix representation
 2. formation of periodic orbits

歪対称ロトカ・ボルテラ系

$$x_j > 0, 1 \leq j \leq N$$

$$\tau_j \frac{dx_j}{dt} = (-e_j + \sum_k a_{jk} x_k) x_j$$

$$\tau_j, e_j, a_{jk} \in \mathbf{R}$$

$$\tau = (\tau_j) > 0, e = (e_j) \quad A = (a_{jk})$$

(a1) irreducible $\nexists P$ permutation

$$P^T A P = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

A_{11}, A_{22} square (no sub-system)

(a2) skew symmetric $A + A^T = 0$

$$\longrightarrow \frac{d}{dt} \tau \cdot v = -e \cdot v \quad T = +\infty$$

(a3) sign condition

Each row of A contains both positive and negative components

(P) $E = L \cap \mathbf{R}_+^N, \exists L$ affine space of co-dimension 2

Any non-stationary solution is periodic-in-time with the orbit $\mathcal{O} \cong S^1$ contractible to a stationary solution in $\mathbf{R}_+^N \setminus E$

Any distinct two orbits $\mathcal{O}_1, \mathcal{O}_2 \cong S^1$ do not link in \mathbf{R}_+^N

$E = \{v = (v_j) \in \mathbf{R}_+^N \mid Av = e\}$ set of stationary solutions

Kobayashi-S.-Yamada 2019

Remark

$$e = (e_j) = 0$$

$$\frac{d}{dt} \tau \cdot x = 0 \quad \text{total mass}$$

$$\xi_j = \log x_j$$

$$\tau_j \frac{d\xi_j}{dt} = \sum_k a_{jk} e^{\xi_k}$$

$$b \in \text{Ker } A^T = \text{Ker } A \quad \tilde{b} = (\tau_j b_j) \quad \dim \text{Ker } A = n - 2$$

$$\frac{d}{dt} \tilde{b} \cdot \xi = \sum_{jk} a_{jk} b_j e^{\xi_k} = 0 \quad \text{entropy} \quad \dots \text{rank } n-1$$

Theorem C $(e_j) = 0, N \geq 3$ $A = (a_{jk})$ (a1), (a2), (a3)

$a_{ij}a_{kl} + a_{il}a_{jk} - a_{ik}a_{jl} = 0, \forall i, j, k, l \in \{1, \dots, N\} \rightarrow (P)$
 algebraic condition

Remark

$a_{12} \neq 0$ \updownarrow

(Q) $a_{kl} = \frac{a_{1k}a_{2l} - a_{1l}a_{2k}}{a_{12}}, 3 \leq k < l \leq N$

Remark

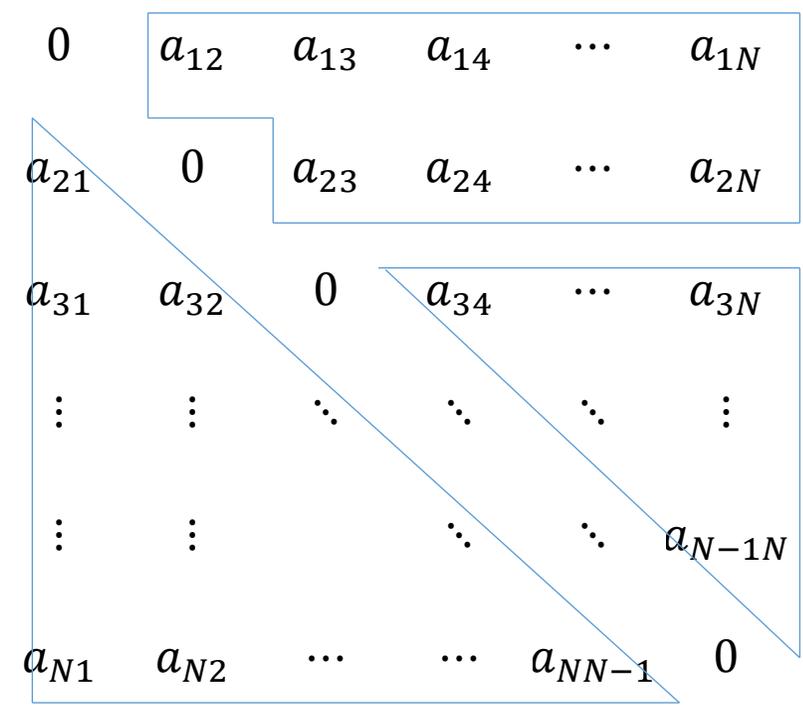
(a1)(a2) ~~(P)~~

\rightarrow extinction or blowup in infinite time

Example. N=3

$$A = \begin{pmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_2 & -c_1 & 0 \end{pmatrix}$$

Skew symmetric matrices
 $N(N-1)/2$ dimension



free
 $2N-3$ dimension

Theorem D

$$A = (a_{jk}) \quad (a1) \ (a2) \ (a3)$$

$$N \geq 3$$

$$(e_j)$$

has both positive and negative components

... sign condition

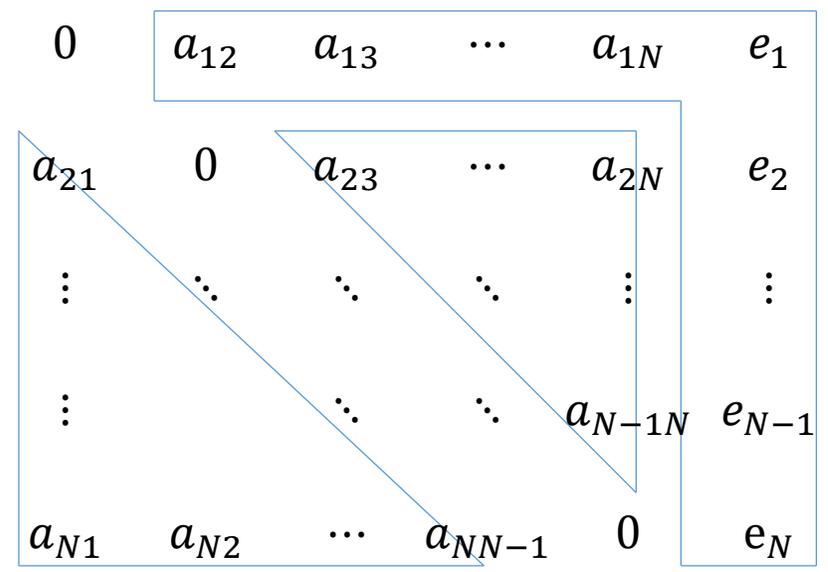
$$a_{jk}e_i - a_{ik}e_j + a_{ij}e_k = 0, \quad \forall i, j, k \in \{1, \dots, N\} \quad \rightarrow \quad (P)$$

Remark

$$e_1 \neq 0 \quad \updownarrow$$

$$a_{jk} = \frac{a_{1k}e_j - a_{1j}e_k}{e_1}, \quad 2 \leq j < k \leq N$$

Skew symmetric matrices + vectors
N(N+1)/2 dimension



← free 2N-1 dimension

Not contain prey predator system

growth rate model

existence time

$$x = (x_i(t)) > 0, \quad -T_- < t < T_+$$

$$\frac{dx_i}{dt} = g_i(x)x_i, \quad x_i|_{t=0} = x_{i0} > 0, \quad 1 \leq i \leq n$$

$$g = (g_i) : \mathbf{R}_+^n \rightarrow \mathbf{R}^n \quad \text{locally Lipschitz conti.}$$

$$\mathbf{R}_+^n = \{(x_i) \in \mathbf{R}^n \mid x_i > 0, \quad 1 \leq i \leq n\}$$

$$\frac{d\xi_i}{dt} = f_i(\xi), \quad \xi_i|_{t=0} = \xi_{i0} \in \mathbf{R}, \quad 1 \leq i \leq n$$

$$\xi_i = \log x_i, \quad f_i(\xi) = g_i(\xi), \quad \xi = (\xi_i), \quad x = (x_i)$$

Theorem 1

Assumption 1-3



$$T_{\pm} = +\infty, \quad \exists T > 0$$

$$\xi(t+T) = \xi(t)$$

Assumption 1 $f_i = (-1)^{i+1} \det B_i, \quad 1 \leq i \leq n$

$$B_i = \begin{pmatrix} \frac{\partial H}{\partial \xi_1} & \cdots & \overbrace{\quad}^i & \cdots & \frac{\partial H}{\partial \xi_n} \\ c_{1,1} & \cdots & \cdot & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n-2,1} & \cdots & \cdot & \cdots & c_{n-2,n} \end{pmatrix}$$

$$H = \varphi(\xi) - \vec{b} \cdot \xi, \quad \vec{b} = (b_i) \in \mathbf{R}^n$$

$$\varphi : \mathbf{R}^n \rightarrow \mathbf{R}, \quad C^2 \quad \text{strictly convex}$$

$$\vec{c}_\ell = (c_{\ell i}) \in \mathbf{R}^n, \quad 1 \leq \ell \leq n-2 \quad \text{linearly independent}$$

Assumption 2

$$\forall \vec{a}, \forall \varepsilon > 0, \exists C_\varepsilon > 0 \quad \vec{a} \cdot \xi \leq \varepsilon \varphi(\xi) + C_\varepsilon, \quad \forall \xi$$

$$\lim_{\xi \rightarrow +\infty} \varphi(\xi) = +\infty, \quad \xi \rightarrow +\infty \Leftrightarrow \exists i, \xi_i \rightarrow +\infty$$

$$\inf_{\xi} \varphi(\xi) > -\infty$$

Assumption 3 $(\vec{b} + X) \cap \mathbf{R}_+^n \neq \emptyset \quad X = \langle \vec{c}_\ell \mid 1 \leq \ell \leq n-2 \rangle$

n system

$$\frac{d\xi_i}{dt} = f_i(\xi), \quad 1 \leq i \leq n \quad f_i = (-1)^{i+1} \det B_i$$

$$B_i = \begin{pmatrix} \frac{\partial H}{\partial \xi_1} & \cdots & \overbrace{\cdot}^i & \cdots & \frac{\partial H}{\partial \xi_n} \\ c_{1,1} & \cdots & \cdot & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n-2,1} & \cdots & \cdot & \cdots & c_{n-2,n} \end{pmatrix}$$

$$B \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\partial H}{\partial \xi_1} & \cdots & \frac{\partial H}{\partial \xi_n} \\ c_{1,1} & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots \\ c_{n-2,1} & \cdots & c_{n-2,n} \end{pmatrix}$$

$n = 2$

$$B = \begin{pmatrix} \frac{\partial H}{\partial \xi_1} & \frac{\partial H}{\partial \xi_2} \end{pmatrix}$$

$$\frac{d\xi_1}{dt} = \frac{\partial H}{\partial \xi_2}, \quad \frac{d\xi_2}{dt} = -\frac{\partial H}{\partial \xi_1} \quad \text{Hamilton system}$$

example

$$H(\xi) = -b_1 \xi_2 + a_1 e^{\xi_2} - b_2 \xi_1 + a_2 e^{\xi_1}$$

$$\frac{d\xi_1}{dt} = -b_1 + a_1 e^{\xi_2}, \quad \frac{d\xi_2}{dt} = b_2 - a_2 e^{\xi_1}$$

$$\frac{dx_1}{dt} = (-b_1 + a_1 x_2) x_1, \quad \frac{dx_2}{dt} = (b_2 - a_2 x_1) x_2$$

$$\vec{a} = (a_i) > 0, \quad \vec{b} = (b_i) > 0 \quad \text{predator-prey system}$$

$$\varphi(\xi) = a_1 e^{\xi_1} + a_2 e^{\xi_2} \quad \text{strictly convex} \quad \text{Assumption 2}$$

A special form

$$\frac{d\xi_i}{dt} = f_i(\xi), \quad 1 \leq i \leq n \quad f_i = (-1)^{i+1} \det B_i$$

$$H = \varphi(\xi) - \vec{b} \cdot \xi, \quad \vec{b} = (b_i) \in \mathbf{R}^n$$

$$\varphi(\xi) = \vec{a} \cdot e^\xi, \quad e^\xi = (e^{\xi_i}), \quad \vec{a} = (a_i) > 0$$

$$\frac{dx_i}{dt} = (-1)^{i+1} (\det \tilde{B}_i) x_i$$

$$\tilde{B}_i = \begin{pmatrix} a_1 x_1 - b_1 & \cdots & \overbrace{\cdot}^i & \cdots & a_n x_n - b_n \\ c_{1,1} & \cdots & \cdot & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n-2,1} & \cdots & \cdot & \cdots & c_{n-2,n} \end{pmatrix}$$

Lotka-Volterra system

$$\tau_i \frac{dx_i}{dt} = \sum_{j=1}^n a_{ij} x_i x_j - e_i x_i, \quad x_i|_{t=0} = x_{i0} > 0, \quad 1 \leq i \leq n$$

Theorem 2 $\exists \vec{f}_i \in \mathbf{R}^{n-2}, \quad 1 \leq i \leq n$

$$\langle \vec{f}_i \mid 1 \leq i \leq n \rangle = \mathbf{R}^{n-2}$$

$$(-1)^{i+j} \tau_i^{-1} \tau_j^{-1} a_{ij}$$

$$= \begin{cases} \det(\vec{f}_1, \dots, \overbrace{\cdot}^i, \dots, \overbrace{\cdot}^j, \dots, \vec{f}_n), & i < j \\ 0, & i = j \\ -\det(\vec{f}_1, \dots, \overbrace{\cdot}^j, \dots, \overbrace{\cdot}^i, \dots, \vec{f}_n), & i > j \end{cases}$$

$$\exists \vec{b} = (b_i), \quad \tau_i^{-1} e_i = \sum_{j=1}^n b_j \tau_i^{-1} \tau_j^{-1} a_{ij}$$

$$\Rightarrow T_{\pm} = +\infty$$

$$(\vec{b} + \text{Ker } \tilde{A}) \cap \mathbf{R}_+^n \neq \emptyset, \quad \tilde{A} = (\tau_i^{-1} \tau_j^{-1} a_{ij})$$

\Rightarrow Any non-stationary solution is periodic-in-time

Comparison

Skew-symmetric Lotka-Volterra system



Theorem 2 $\exists \vec{f}_i \in \mathbf{R}^{n-2}, 1 \leq i \leq n$

$$\langle \vec{f}_i \mid 1 \leq i \leq n \rangle = \mathbf{R}^{n-2}$$

$$\tau_i \frac{dx_i}{dt} = \sum_{j=1}^n a_{ij} x_i x_j - e_i x_i, \quad x_i|_{t=0} = x_{i0} > 0, \quad 1 \leq i \leq n$$

$$A + A^T = 0 \quad A = (a_{ij})$$

Normal form $y_i = \tau_i x_i$

$$\frac{dy_i}{dt} = \sum_{j=1}^n \tilde{a}_{ij} y_i y_j - \tilde{e}_i y_i, \quad 1 \leq i \leq n$$

$$\tilde{a}_{ij} = \tau_i^{-1} \tau_j^{-1} a_{ij}, \quad \tilde{e}_i = \tau_i^{-1} e_i$$

the same algebraic condition (except for the irreducibility)
different sign condition

Any distinct orbits do not link
Any periodic orbit links the set of stationary solutions,
an affine space with codimension 2 in the first orthant in ξ variable

$$(-1)^{i+j} \tau_i^{-1} \tau_j^{-1} a_{ij}$$

$$= \begin{cases} \det(\vec{f}_1, \dots, \overbrace{\cdot}^i, \dots, \overbrace{\cdot}^j, \dots, \vec{f}_n), & i < j \\ 0, & i = j \\ -\det(\vec{f}_1, \dots, \overbrace{\cdot}^j, \dots, \overbrace{\cdot}^i, \dots, \vec{f}_n), & i > j \end{cases}$$

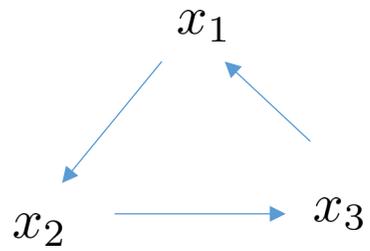
$$\exists \vec{b} = (b_i), \quad \tau_i^{-1} e_i = \sum_{j=1}^n b_j \tau_i^{-1} \tau_j^{-1} a_{ij}$$

$$(\vec{b} + \text{Ker } \tilde{A}) \cap \mathbf{R}_+^n \neq \emptyset, \quad \tilde{A} = (\tau_i^{-1} \tau_j^{-1} a_{ij})$$

skew-symmetric

\Rightarrow Any non-stationary solution is periodic-in-time

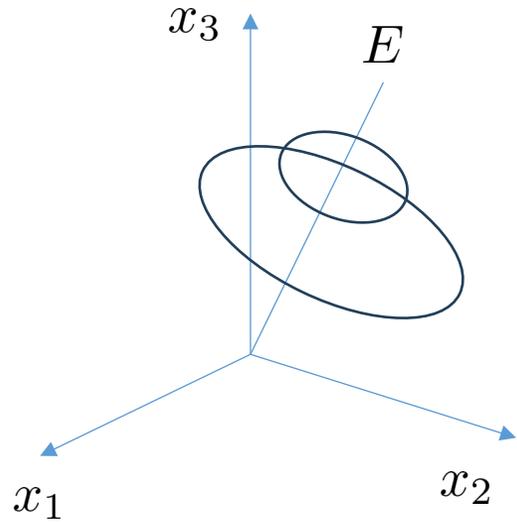
Example 1



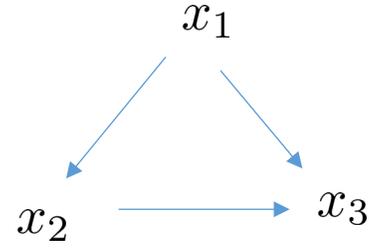
$$\tau_1 \frac{dx_1}{dt} = (x_2 - x_3)x_1$$

$$\tau_2 \frac{dx_2}{dt} = (x_3 - x_1)x_2$$

$$\tau_3 \frac{dx_3}{dt} = (x_1 - x_2)x_3$$



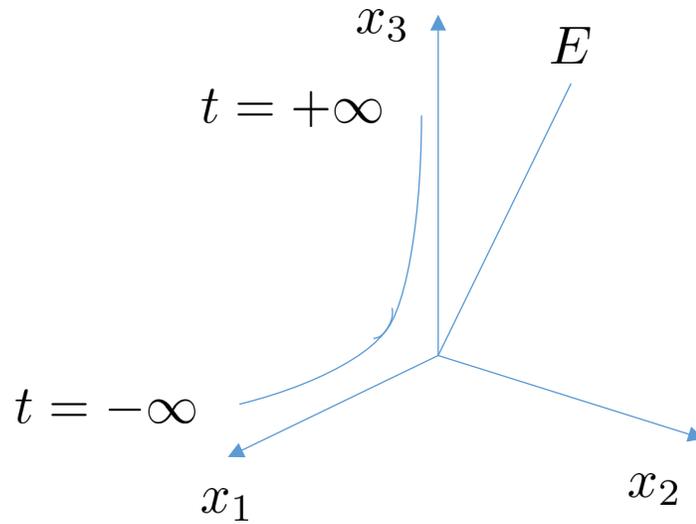
Example 2



$$\tau_1 \frac{dx_1}{dt} = (x_2 + x_3)x_1$$

$$\tau_2 \frac{dx_2}{dt} = (x_3 - x_1)x_2$$

$$\tau_3 \frac{dx_3}{dt} = (-x_1 - x_2)x_3$$



$$\lim_{t \rightarrow +\infty} (x_1, x_2, x_3) = (+\infty, 0, 0)$$

$$\lim_{t \rightarrow -\infty} (x_1, x_2, x_3) = (0, 0, +\infty)$$

Proof of Theorem 1

Lemma 1 $H^i(\xi), 0 \leq i \leq m, m \leq n - 2 \in C^1$

$$h_{i,j}(\xi) = \frac{\partial H^{i-1}}{\partial \xi_j}, 1 \leq i \leq m + 1, 1 \leq j \leq n$$

$$B_i(\xi) = \begin{pmatrix} h_{1,1} & \cdots & \overbrace{\quad}^i & \cdots & h_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{n-1,1} & \cdots & \cdot & \cdots & h_{n-1,n} \end{pmatrix}$$

$$\frac{d\xi_i}{dt} = (-1)^{i+1} \det B_i, 1 \leq i \leq n$$

$$\implies \frac{dH^i}{dt} = 0, 0 \leq i \leq m$$

Proof

$$\hat{B} = \begin{pmatrix} h_{1,1} & \cdots & h_{1,n} \\ h_{1,1} & \cdots & h_{1,n} \\ h_{2,1} & \cdots & h_{2,n} \\ \vdots & \vdots & \vdots \\ c_{n-2,1} & \cdots & c_{n-2,n} \end{pmatrix}$$

$$\begin{aligned} \frac{dH^0}{dt} &= \sum_{j=1}^n h_{1,j} \frac{d\xi_j}{dt} = \sum_{j=1}^n (-1)^{j+1} h_{1,j} \det B_j \\ &= \det \hat{B} = 0 \quad \blacksquare \end{aligned}$$

$$\frac{dH}{dt} = 0, \frac{d}{dt}(\vec{c}_\ell \cdot \xi) = 0, 1 \leq \ell \leq n - 2$$

in Theorem 1

$(n - 1)$ conservation laws

Proof of Theorem 1

$$\mathcal{O} = \{\xi(t) \mid -T_- < t < T_+\} \subset \mathcal{O}_* = \mathcal{M} \cap \mathcal{P} \longleftarrow \text{Assumption 1}$$

$$\mathcal{M} = \{\xi \in \mathbf{R}^n \mid H(\xi) = \beta_0\}, \quad \mathcal{P} = \{\xi \in \mathbf{R}^n \mid \vec{c}_\ell \cdot \xi = \beta_\ell, 1 \leq \ell \leq n-2\}$$

2-d hyper plane

$$\beta_0 = H(\xi_0), \quad \beta_\ell = \vec{c}_\ell \cdot \xi_0, \quad 1 \leq \ell \leq n-2\}$$

$$\mathcal{M} = \partial M \cap \{\eta = \beta_0\}, \quad M = \{(\xi, \eta) \mid \eta \geq H(\xi)\} \subset \mathbf{R}^{n+1}$$

C^2 hyper-surface with positive sectional curvatures strictly convex

$$\mathcal{O}_* \subset \mathcal{P} \quad C^2 \text{ curve} \quad \text{Lemma 2} \implies \mathcal{O}_* = \mathcal{O}$$

$$\frac{d\xi_i}{dt} = (-1)^{i+1} \det B_i, \quad 1 \leq i \leq n$$

stationary solutions

$$E = \{\xi \in \mathbf{R}^n \mid \det B_i = 0, 1 \leq i \leq n\}$$

Lemma 2 $\varphi(\xi)$ strictly convex $\implies \mathcal{O} \cap E = \emptyset$ or $\mathcal{O}_* \subset E, \#\mathcal{O} = 1$

Lemma 3 **Assumption 2-3** $\implies \mathcal{O}_*$ bounded

Proof of Lemma 2 $\xi \in \mathcal{O}_*$ $\nabla H(\xi) = 0 \implies \xi \in \mathbf{R}^n$ attains the global minimum of the convex function $\#\mathcal{M} = 1, \mathcal{M} = \{H(\xi) = \beta_0\}$
 $\implies \xi \in E$

$$B = \begin{pmatrix} \frac{\partial H}{\partial \xi_1} & \cdots & \frac{\partial H}{\partial \xi_n} \\ c_{1,1} & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots \\ c_{n-2,1} & \cdots & c_{n-2,n} \end{pmatrix}$$

$\nabla H(\xi) \neq 0 \implies T_\xi \mathcal{M} = \langle \nabla H(\xi) \rangle^\perp$

$$E = \{\xi \in \mathbf{R}^n \mid \det B_i = 0, 1 \leq i \leq n\}$$

more on the structure of E

$\xi \in E \implies \det B_i = 0, 1 \leq i \leq n \quad \forall \vec{a} = (a_i), \det \hat{B} = 0$

$$B = \begin{pmatrix} a_1, & \cdots & a_n \\ \frac{\partial H}{\partial \xi_1} & \cdots & \frac{\partial H}{\partial \xi_n} \\ c_{1,1} & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots \\ c_{n-2,1} & \cdots & c_{n-2,n} \end{pmatrix}$$

arbitrary linearly independent
 $\vec{a}, \nabla H(\xi), \vec{c}_1, \dots, \vec{c}_{n-2} \implies \nabla H(\xi) \in X = \langle \vec{c}_\ell \mid 1 \leq \ell \leq n-2 \rangle$

linearly dependent

$$X^\perp = \mathcal{P} - \{\xi\}, \mathcal{P} = \{\vec{c}_\ell \cdot \xi = \beta_\ell, 1 \leq \ell \leq n-2\}$$

$$\mathcal{P} - \{\xi\} \subset \langle \nabla H \rangle^\perp = T_\xi \mathcal{M}$$

$$\mathcal{O}_* \subset \mathcal{P} \cap \mathcal{M} \subset (\{\xi\} + T_\xi \mathcal{M}) \cap \mathcal{M} = \{\xi\}$$

$H(\xi)$ strictly convex

Reaction diffusion systems

$\Omega \subset \mathbf{R}^N$ bounded domain $\partial\Omega$ smooth

$$\tau_i \frac{\partial u_i}{\partial t} - d_i \Delta u_i = \sum_{j=1}^n a_{ij} u_j u_i - e_i u_i \text{ in } \Omega \times (0, T)$$

$$\frac{\partial u_i}{\partial \nu} \Big|_{\partial\Omega} = 0, \quad u_i|_{t=0} = u_{i0}(x) > 0, \quad 1 \leq i \leq n$$

$$\tau_i^{-1} \left(\sum_{j=1}^n a_{ij} u_j - e_i \right) = (-1)^{i+1} \det \tilde{B}_i, \quad 1 \leq i \leq n$$

$$\tilde{B}_i = \begin{pmatrix} a_1 u_1 - b_1 & \cdots & \overbrace{\cdot}^i & \cdots & a_n u_n - b_n \\ c_{1,1} & \cdots & \cdot & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n-2,1} & \cdots & \cdot & \cdots & c_{n-2,n} \end{pmatrix}$$

Theorem E

$\vec{a} = (a_i) > 0$, $\vec{c}_\ell = (c_{\ell,i})$, $1 \leq \ell \leq n-2$ linearly independent

$$\vec{b} \cdot \vec{e} = 0, \quad \vec{e} = (\tau_i^{-1} e_i)$$

$n = 2$, $\vec{b} > 0$ or $n \geq 3$, $X \cap \mathbf{R}_+^n \neq \emptyset$, $X = \langle \vec{c}_\ell \mid 1 \leq \ell \leq n-2 \rangle$

predator-prey Alikakos 79, Latos-S.-Yamada 12

$\implies T = +\infty$, $\|u(\cdot, t)\|_\infty \leq C$ for any space dimension

$\exists \mathcal{O}$ spatially homogeneous orbit stationary or periodic-in-time

$$\lim_{t \rightarrow +\infty} \text{dist}_{C^2}(u(\cdot, t), \mathcal{O}) = 0$$

Mass dissipative systems

$$\Omega \subset \mathbf{R}^N \text{ bounded domain } \partial\Omega \text{ smooth} \quad A = (a_{ij}), \vec{e} = (e_i)$$

$$A + A^T \leq 0, \vec{e} \geq 0$$

$$\tau_i \frac{\partial u_i}{\partial t} - d_i \Delta u_i = \sum_{j=1}^n a_{ij} u_j u_i - e_i u_i \text{ in } \Omega \times (0, T)$$

$$\Rightarrow T = +\infty, \|u(\cdot, t)\|_\infty \leq C$$

Fellner-Morgan-Tang 20, 21 $\forall N$

$$\left. \frac{\partial u_i}{\partial \nu} \right|_{\partial\Omega} = 0, u_i|_{t=0} = u_{i0}(x) > 0, 1 \leq i \leq n$$

Total mass control + quadratic growth c.f. Smoluchowski-Poisson equation

compact, connected, invariant

$$\omega(u_0) = \{u^* \in C^2(\bar{\Omega}) \mid \exists t_k \uparrow +\infty, \lim_{k \rightarrow \infty} \|u(\cdot, t_k) - u^*\|_{C^2} = 0\}$$

Theorem F $\forall u^* = (u_i^*) \in \omega(u_0), u_i^* \neq 0, 1 \leq i \leq n \implies \omega(u_0) \subset \mathbf{R}_+^N$ 棲み分けはおこらない

c.f. cross-diffusion model

$$\frac{\partial u}{\partial t} = \Delta(au + a_{11}u^2 + a_{12}uv) + cu - \gamma_{11}u^2 - \gamma_{12}uv$$

$$\frac{\partial v}{\partial t} = \Delta(bv + a_{21}uv + a_{22}v^2) + du - \gamma_{21}uv - \gamma_{22}v^2$$

Shigesada-Kawasaki-Teramoto

three types of diffusion