# 場と粒子の双対性

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鈴木　貴 (大阪大学)

# 3. 関連する話題

**Results** (Smoluchowski-Poisson equation)

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$\left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0$$

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

**Theorem B** $T < +\infty$ ⟶

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$m(x_0) \in 8\pi\mathbf{N}$ collapse mass quantization possibly with sub-collapse collision

blowup set; exclusion of boundary blowup

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0,\ t_k \uparrow T,\ u(x_k, t_k) \to +\infty\} \subset \Omega$$

$\sharp\mathcal{S} < +\infty$ finiteness of blowup points

$0 < f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$ measure theoretic regular part

**Theorem C**

$$T = +\infty, \quad \limsup_{t \uparrow +\infty} \|u(\cdot, t)\|_\infty = +\infty$$

⟶ $\lambda \equiv \|u_0\|_1 = 8\pi\ell, \ \exists \ell \in \mathbf{N}$ initial mass quantization

$\exists x_* \in \Omega^\ell \setminus D, \ \nabla H_\ell(x_*) = 0$ recursive hierarchy

point vortex Hamiltonian; Robin function; Green function

$$H_\ell(x_1, \cdots, x_\ell) = \frac{1}{2}\sum_j R(x_j) + \sum_{i<j} G(x_i, x_j)$$

**Corollary 1** $T < +\infty$ if

$\lambda \notin 8\pi\mathbf{N}$, $\nexists$ stationary solution or $\mathcal{F}(u_0) \ll -1$

$\lambda \in 8\pi\ell$, $\ell \in \mathbf{N}$, $\nexists$ critical point of $H_\ell$

**Corollary 2** $\Omega$ convex $\lambda \neq 8\pi$

$\Rightarrow T < +\infty$ or $T = +\infty$ pre-compact orbit

$\exists$ stationary solution

c.f. Grossi-F. Takahashi (2010)

## Bounded free energy and simplicity

$$\mathcal{F}(u)=\int_\Omega u(\log u-1)dx-\frac{1}{2}\iint_{\Omega\times\Omega}G(x,x')u\otimes u\;dxdx'$$

**free energy of Helmholtz**

- (entropy) temperature    inner energy (self-attractive)

$$\frac{d\mathcal{F}}{dt}=-\int_\Omega u|\nabla(\log u-v)|^2\leq 0,\;\; v=(-\Delta)^{-1}u$$

### Simplicity of the blowup points

$$T<+\infty,\;\lim_{t\uparrow T}\mathcal{F}(u(t))>-\infty \longrightarrow m(x_0)=8\pi \quad \forall x_0\in\mathcal{S}$$

simple ⟶

### Emergence

$$\lim_{t\uparrow T}\mathcal{F}_{x_0,b(T-t)^{1/2}}(u(\cdot,t))=+\infty,\;\;\forall b>0$$

$$\mathcal{F}_{x_0,R}(u)=\int_{\Omega\cap B(x_0,R)}u(\log u-1)dx$$
$$-\frac{1}{2}\iint_{\Omega\cap B(x_0,R)\times(\Omega\cap B(x_0,R))}G(x,x')u\otimes u\;dxdx'$$

**Type II Blowup rate** ↓

$$\lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,b(T-t)^{1/2})}=+\infty,\;\;\forall b>0$$

$$u_t=\Delta u-\nabla v\cdot\nabla u+u^2 \longrightarrow u_t=u^2,\;\;u(t)=(T-t)^{-1}$$

$O((T-t)^{-1})$ Type I blowup rate

### Remark

**Rate of blowup is always type II**

$$\exists b>0,\forall x_0\in\mathcal{S},\;\lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,b(T-t)^{1/2})}=+\infty$$

Systems on the whole space $\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$

$$u_t = \Delta u - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^2\times(0,T)$$
$$u|_{t=0} = u_0(x) \in L^\infty\cap L^1(\mathbf{R}^2)$$

$$\int_{\mathbf{R}^2}|x|^2 u_0\ dx < +\infty \ \longrightarrow\ \nexists$$ dichotomy

$$0<\lambda<8\pi \ \longrightarrow\ T=+\infty,\ \|u(\cdot,t)\|_\infty \le C$$
$$\lambda>8\pi \ \longrightarrow\ T<+\infty$$
$$\lambda=8\pi \ \longrightarrow\ T=+\infty$$

vanishing
$$\lim_{t\uparrow+\infty}\|u(\cdot,t)\|_\infty = 0$$

compact (concentration)
$$\lim_{t\uparrow+\infty}\|u(\cdot,t)\|_\infty = +\infty$$

concentration compactness principle

Chemotaxis system in biology

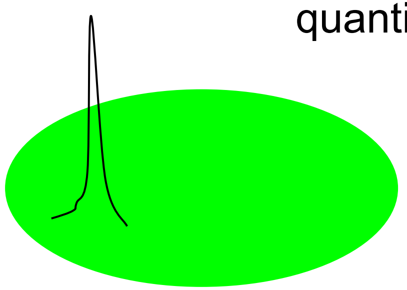
$$u_t = \nabla\cdot(\nabla u - u\nabla v),\ -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$
$$\left(\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu}, \frac{\partial v}{\partial\nu}\right)\Big|_{\partial\Omega} = 0,\ \int_\Omega v = 0$$

$$n=2,\ T=T_{\max}<+\infty$$
$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

existence of the boundary blowup

$$m(x_0)\in m_*(x_0)\mathbf{N},\ m_*(x_0) = \begin{cases} 8\pi, & x_0\in\Omega \\ 4\pi, & x_0\in\partial\Omega \end{cases}$$
$$0<f=f(x)\in L^1(\Omega)\cap C(\overline{\Omega}\setminus\mathcal{S})$$

quantized blowup mechanism



also in infinite time

# Spectral mechanics

Smoluchowski-Poisson equation

$$u_t = \Delta u - \nabla \cdot (u \nabla v), -\Delta v = u \quad \text{in } \Omega \times (0,T)$$

$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = v = 0 \quad \text{on } \partial\Omega \times (0,T)$$

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

$$\rightarrow \quad -\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \quad v|_{\partial\Omega} = 0$$

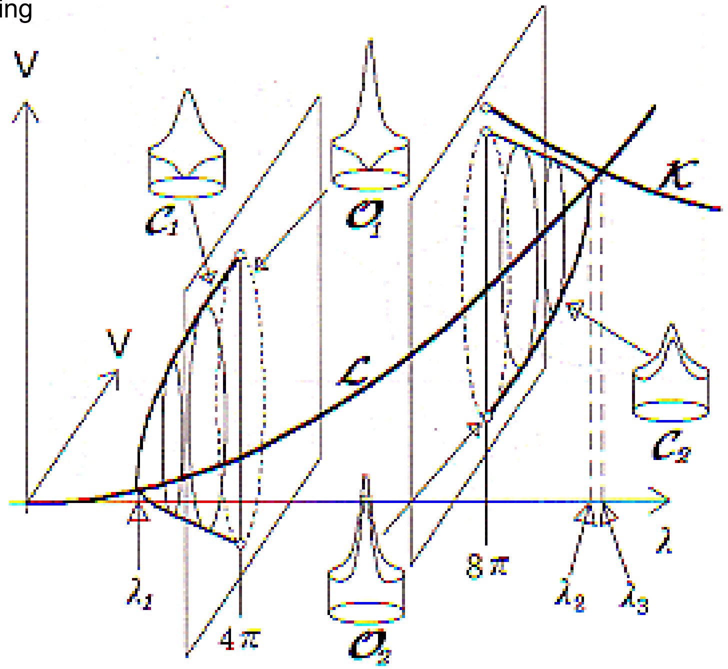
Boltzmann-Poisson equation

$$u \geq 0, \ \frac{d}{dt}\|u(\cdot,t)\|_1 = 0$$

mass conservation
free energy decreasing

$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 dx$$

**Potentials of self-organization**




Senba-S. 00

simplified system of chemotaxis

$$-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right), \ \int_\Omega v = 0$$

$$\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

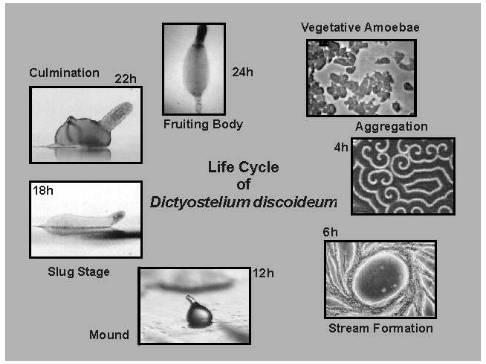
stationary state →

$$\log u - v = \text{constant}, \ \|u\|_1 = \lambda$$

$$\rightarrow \quad u = \frac{\lambda e^v}{\int_\Omega e^v}$$

$$\lambda = 8\pi, \ 4\pi$$

interior boundary




Biler-Hilhorst-Nadieja 94, Nagai 95, Nagai-Senba-Yoshida 97, Gajewski-Zacharius 98, Biler 98

## Systems with relaxation time

full system of chemotaxis

$$\varepsilon u_t = \nabla\cdot(\nabla u - u\nabla v)$$
$$\tau v_t = \Delta v + u \text{ in } \Omega\times(0,T)$$
$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, v\right)\Big|_{\partial\Omega} = 0$$

$\tau = 0$ Smoluchowski-Poisson

quantized blowup mechanism

$\varepsilon = 0$ $$v_t = \Delta v + \frac{\lambda e^v}{\int_\Omega e^v}, \quad v|_{\partial\Omega} = 0$$

non-local parabolic equation

$$\Omega = B(0,1)\subset \mathbf{R}^2, \ v = v(|x|,t), \ \lambda \geq 8\pi$$

Wolansky 97

$$\longrightarrow \quad \frac{\lambda e^v}{\int_\Omega e^v}dx \rightharpoonup \lambda\delta_0, \ t\uparrow T = T_{\max}\in(0,+\infty]$$

Kavallaris-S. 07

$$\lambda > 8\pi \ \Rightarrow \ T = T_{\max} < +\infty$$

**dis-quantized blowup mechanism**

## Duality between field and particles

Lagrangian in Toland duality

$$L(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}|\nabla v|^2 - vu\ dx$$

$\longrightarrow$ unfolding-minimality

$$\inf\{L(u,v) \mid u\geq 0, \|u\|_1 = 8\pi, \ v\in H_0^1(\Omega)\} > -\infty$$

**Model C equation**

$$\varepsilon u_t = \nabla\cdot(u\nabla L_u(u,v)), \ \tau v_t = -L_v(u,v)$$

## Models in non-equilibrium thermo-dynamics

**semi-unfolding-minimality**

infinitesimal stability $\longrightarrow$ dynamical stabilitiy

local minimum of the analytic field functional $\longrightarrow$ infinitesimal stable

S.-Tasaki 10

## Higher-dimensional quantization

$n > 2,\ m = \frac{n}{n-2}$

weak solution

$$u_t = \frac{m-1}{m}\Delta u^m - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^n\times(0,T)$$

$$u|_{t=0} = u_0(x) \geq 0 \in L^\infty\cap L^1(\mathbf{R}^n) \quad \int_{\mathbf{R}^n}|x|^2u_0dx < +\infty$$

$$\Gamma(x) = \frac{|x|^{2-n}}{|\partial B|},\ B = B(0,1) \quad \longrightarrow \quad \frac{d}{dt}\int_{\mathbf{R}^n} u\ dx = 0$$

$$\frac{d}{dt}\mathcal{F}(u) = -\int_{\mathbf{R}^n} u|\nabla u^{m-1} - \Gamma * u|^2 dx \leq 0$$

$$\mathcal{F}(u) = \int_{\mathbf{R}^n}\frac{u^m}{m} - \frac{1}{2}\langle\Gamma * u, u\rangle$$

Tsallis entropy

**Theorem D** $T < +\infty \longrightarrow \mathcal{S}\subset\mathbf{R}^n \quad \sharp\mathcal{S}_{II} < +\infty$

$$\mathcal{S} = \{x_0\in\mathbf{R}^n\cup\{\infty\} \mid \exists x_k\to x_0,\ t_k\uparrow T,\ \lim_{k\to\infty}u(x_k,t_k) = +\infty\}$$

$$S_{II} = \{x_0\in\mathcal{S} \mid \lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(B(x_0,r_0))} = +\infty,\ \forall r_0 > 0$$

S.-R. Takahashi 09a, 09b, 20, 12

## Toward the theory of elliptic uniformization

$$-\Delta w = w_+^m \text{ in } \Omega,\ w = c\in\mathbf{R} \text{ on } \partial\Omega$$

solution sequence

$$\int_\Omega w_+^m = \lambda \quad (w_k, c_k, \lambda_k) \quad \lambda_k\to\lambda_0$$

$\longrightarrow$ Subsequence alternatives

(a) $\|w_k\|_\infty \leq C$

(b) $\sup_\Omega w_k \to -\infty$

(c) $\lambda_0 = m_*\ell,\ \ell\in\mathbf{N}$

$$\mathcal{S} = \{x_1^*, \cdots, x_\ell^*\}\in\Omega$$

$$\nabla_{x_j}H_\ell(x)\big|_{x=x_*} = 0,\ 1\leq j\leq\ell$$

$$x = (x_1,\cdots,x_\ell),\ x_* = (x_1^*,\cdots,x_\ell^*)$$

$$H_\ell(x_1,\cdots,x_\ell) = \frac{1}{2}\sum_j R(x_j) + \sum_{i<j}G(x_i,x_j)$$

# 4. 無限時間爆発

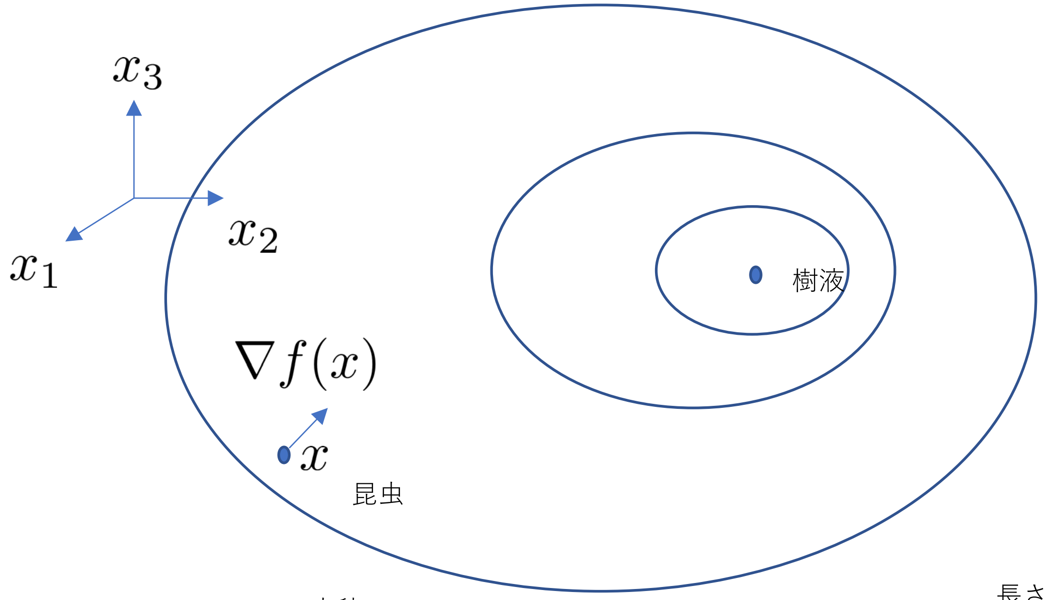
内積は成分ごとの積の和

gradient

$$f(x+se)=f(x_1+se_1,x_2+se_2,x_3+se_3)$$

合成関数の微分　　　内積

$$\left.\frac{d}{ds}f(x+se)\right|_{s=0}=\frac{\partial f}{\partial x_1}(x_1,x_2,x_3)e_1+\frac{\partial f}{\partial x_2}(x_1,x_2,x_3)e_2+\frac{\partial f}{\partial x_3}(x_1,x_2,x_3)e_3=\nabla f(x)\cdot e$$




grad, ナブラ

勾配作用素

$$\nabla f=\begin{pmatrix}\frac{\partial f}{\partial x_1}\\ \frac{\partial f}{\partial x_2}\\ \frac{\partial f}{\partial x_3}\end{pmatrix}$$ **勾配**

単位ベクトル

$$e=\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}$$ 「方向」

$\nabla f(x)=0$ となる $x$

⇕ def

$f$ の**臨界点**

同じ方向を向いたときに最大になる

内積　　　長さ

$\nabla f(x)\cdot e$ を最大にする　$e\in\mathbf{R}^3,\ |e|=1$ ⟶ $e=\dfrac{\nabla f(x)}{|\nabla f(x)|}$

長さ 1

そのときの $\nabla f(x)\cdot e=\dfrac{\nabla f(x)\cdot\nabla f(x)}{|\nabla f(x)|}=|\nabla f(x)|$

**走化性**

昆虫には $e=\dfrac{\nabla f(x)}{|\nabla f(x)|}$ 方向に大きさ $|\nabla f(x)|$ に比例する力が働く

c.f. 万有引力の法則

勾配ベクトル

$$|\nabla f(x)|e=\nabla f(x)$$

等高面に垂直

flux

質量変化 $$\frac{d}{dt}\int_{\Omega}\rho = -\int_{\partial\Omega}\nu\cdot j = -\int_{\Omega}\nabla\cdot j$$

（ガウス；$\rho$：密度；$j$：流束（フラックス））

$$v = v(x) \in \mathbf{R}^3,\ x \in \mathbf{R}^3$$ 速度場

$$\frac{d}{dt}T_t x = v(T_t x),\ \ T_t x|_{t=0} = x$$ $\{T_t\}$ 流れ（力学系）

勾配作用素 $$\nabla = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix}$$

$$x = (x_1, x_2, x_3)$$

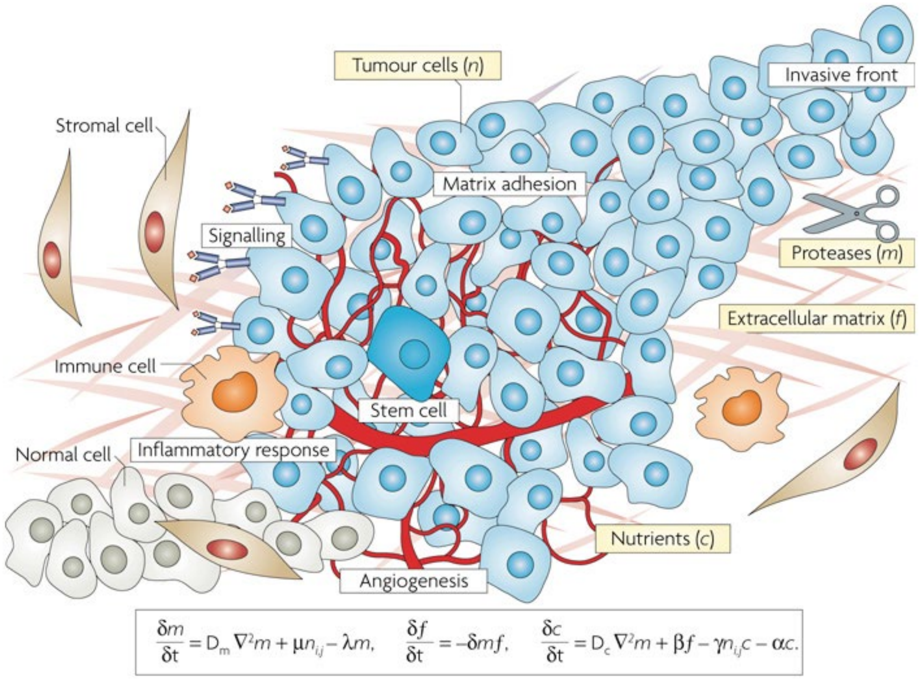
・内積　$\nabla\cdot$ 発散

質量保存 $$\rho_t = -\nabla\cdot j$$

ヤコビアン $$\frac{d}{dt}\det DT_t\Big|_{t=0} = \nabla\cdot v$$

$$\Omega_t = T_t\Omega$$

リュービル $$\frac{d}{dt}\int_{\Omega_t}\rho\Big|_{t=0} = \int_{\Omega}\rho_t + v\cdot\nabla\rho + \rho\nabla\cdot v\ dx\Big|_{t=0} = \int_{\Omega}\rho_t + \nabla\cdot\rho v\ dx\Big|_{t=0} = 0$$




Nature Reviews | Cancer

→ $$j = \rho v$$ 流束 =質量×速度（運動量）

$$j = -d(u)\nabla u$$ 拡散

$$j = d(u, f)u\nabla f$$ 走化性

濃度勾配

速度

## 気体分子の動力学

$v = v(x,t)$ velocity

$\{T(t,s)\}$ propagator

$x(t) = T(t,s)\xi$

$\Leftrightarrow$

$$\frac{dx}{dt} = v(x,t), \ x|_{t=s} = \xi$$

## Flux

$$\frac{d}{dt}\int_\omega \rho dx = -\int_{\partial\omega} \nu \cdot j \ ds$$

$j = v\rho$

$\rho_t + \nabla \cdot v\rho = 0$, mass conservation

## State Equation

$p = A\rho^\gamma, \ A > 0, \ 1 < \gamma < 2$

## Liouville's Theorem

$x = x(\xi, t) \ \leftrightarrow \ x = T(t,s)\xi$

$$J_t = \det\left(\frac{\partial x_i}{\partial \xi_j}\right)$$

$\Rightarrow$

$$J_t(\xi) = 1 + (t-s)\nabla \cdot v(\xi, s) + o(t-s)$$

## Accelation

$$\left.\frac{d^2x}{dt^2}\right|_{t=s} = \left.\frac{d}{ds}v(x(t),t)\right|_{t=s} = \left.\frac{Dv}{Dt}\right|_{t=s}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla, \text{ material derivative}$$

$\rho$ density, $p$ pressure

$\Rightarrow$

$$\rho\frac{Dv}{Dt} = -\nabla p, \text{ equation of motion}$$

## Momentum Balance

$$\left.\frac{d}{dt}\int_{T(t,s)\omega} \rho(x,t)v(x,t)dx\right|_{t=s}$$

$$= \int_\omega \frac{D}{Dt}(\rho v) + \rho v \nabla \cdot v \ dx$$

$$= \int_\omega (\rho v)_t + \nabla \cdot (\rho v \otimes v) \ dx$$

$$= -\int_\omega \nabla p \ dx$$

## Mass Balance

$$0 = \left.\frac{d}{dt}\int_{T(t,s)\omega} \rho(x,t)dx\right|_{t=s}$$

$$= \left.\frac{d}{dt}\int_\omega \rho(x(\xi,t),t)|J_t(\xi)|d\xi\right|_{t=s}$$

$$= \int_\omega \frac{D\rho}{Dt} + \rho\nabla \cdot v \ d\xi$$

$$= \int_\omega \rho_t + \nabla \cdot v\rho \ d\xi$$

圧縮性オイラー方程式

$$\rho_t + \nabla\cdot v\rho = 0$$
$$\rho(v_t + v\cdot\nabla v) + \nabla p = 0$$
$$p = \rho^\gamma \text{ in } \mathbf{R}^n \times (0,T)$$
$$\gamma > 1$$

## Total Mass Conservation

$$\frac{d}{dt}\int \rho = -\int \nabla\cdot v\rho = 0$$
$$\rho \geq 0$$
$$\|\rho\|_1 = \|\rho_0\|_1 = M$$

## Total Energy Conservation

$$\left(\frac{1}{2}\rho|v|^2 + \frac{p}{\gamma-1}\right)_t$$
$$+\nabla\cdot\left(\frac{1}{2}\rho|v|^2 + \frac{\gamma p}{\gamma-1}\right)v = 0$$
$$\frac{d}{dt}\int \frac{1}{2}\rho|v|^2 + \frac{p}{\gamma-1}\ dx = 0$$

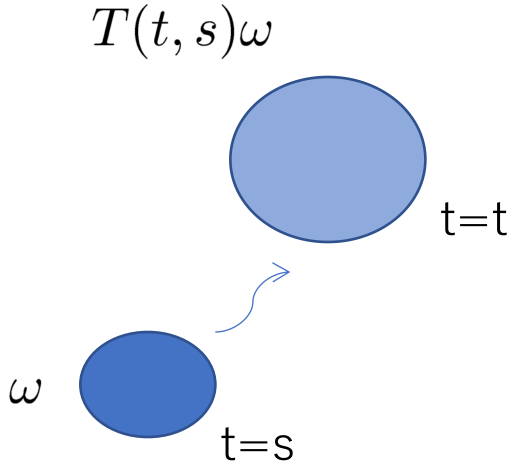
## Total Energy Conservation

$$(\rho v)_t + \nabla\cdot(\rho v\otimes v) + \nabla p = 0$$

$$\int [\nabla\cdot(\rho v\otimes v)]\cdot v = -\int \rho v^j v^i \partial_j v^j$$
$$= -\frac{1}{2}\int \rho v^j \partial_j |v|^2 = \frac{1}{2}\int |v|^2 \nabla\cdot(\rho v)$$
$$= -\frac{1}{2}\int |v|^2 \rho_t$$

$$\int (\rho v)_t\cdot v = \int \rho_t|v|^2 + \frac{1}{2}\rho\partial_t|v|^2$$

$$\int [(\rho v)_t + \nabla\cdot(\rho v\otimes v)]\cdot v = \frac{1}{2}\frac{d}{dt}\int |v|^2\rho$$




$$\int \nabla p\cdot v = \frac{\gamma}{\gamma-1}\int \nabla\rho^{\gamma-1}\cdot\rho v$$
$$= \frac{\gamma}{\gamma-1}\int \rho^{\gamma-1}\rho_t = \frac{d}{dt}\int \frac{p}{\gamma-1}$$

$$\frac{d}{dt}\int \frac{1}{2}\rho|v|^2 + \frac{p}{\gamma-1}\ dx = 0$$

## 無限時間爆発の量子化

**定理 B (blowup in infinite time)**

$$T=+\infty,\ \limsup_{t\uparrow+\infty}\|u(\cdot,t)\|_\infty=+\infty$$

$\Longrightarrow$

$$\lambda\equiv\|u_0\|_1=8\pi\ell,\ \exists\ell\in\mathbf{N}$$ initial mass quantization

$$\exists x_*\in\Omega^\ell\setminus D,\ \nabla H_\ell(x_*)=0$$ recursive hierarchy

point vortex Hamiltonian, Robin function, Green function

$$H_\ell(x_1,\cdots,x_\ell)=\frac{1}{2}\sum_j R(x_j)+\sum_{i<j}G(x_i,x_j)$$

**系 1** $T<+\infty$ if

(1) $\lambda\notin 8\pi\mathbf{N},\ \nexists$ stationary solution or $\mathcal{F}(u_0)\ll -1$

(2) $\lambda\in 8\pi\ell,\ \ell\in\mathbf{N},\ \nexists$ critical point of $H_\ell$

**系 2** $\Omega$ convex $\lambda\neq 8\pi$

$\Rightarrow\ T<+\infty$ or $T=+\infty$ compact orbit

c.f. Grossi-F. Takahashi $\exists$ stationary solution weak limit

assume $T=+\infty,\ t_k\uparrow+\infty,\ \lim_{k\to\infty}\|u(\cdot,t_k)\|_\infty=+\infty$

subsequence $u(\cdot,t+t_k)dx \rightharpoonup \mu(dx,t)\in C_*(-\infty,+\infty;\mathcal{M}(\overline{\Omega}))$ weak solution

$$\mu(dx,t)=\sum_{x_0\in\mathcal{S}_t} m(x_0)\delta_{x_0}(dx)+f(x,t)dx$$

improved regularity
formation of collapse in infinite time

$$m(x_0)\geq\varepsilon_0,\ 0\leq f=f(\cdot,t)\in L^1(\Omega)$$

blowup set

exclusion of boundary blowup

$$\mathcal{S}_t=\{x_0\in\overline{\Omega}\mid \exists x_k\to x_0,\ \lim_k u(x_k,t+t_k)=+\infty\}\subset\Omega$$

dilation $x_0=0\in\mathcal{S}_0,\ \beta>0$

$$\mu_\beta(dx',t')=\beta^2\mu(dx,t),\ x'=\beta x,\ t'=\beta^2 t$$

$\beta_k\downarrow 0$ subsequence

$\mu_{\beta_k}(dx,t)\rightharpoonup\tilde{\mu}(dx,t)\in C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$ scaling limit

$m(x_0)=\tilde{\mu}(\mathbf{R}^2,0)=8\pi\geq\varepsilon_0$ full orbit of weak solutions on the whole space

Liouville property collapse mass quantization

**local second moment traces the collapse dynamics**

$$\sharp\,\mathcal{S}_t\equiv\ell,\ \mu^s(dx,t)=\sum_{i=1}^{\ell}8\pi\delta_{x_i(t)}(dx) \qquad |\dot{x}_j(t)|\leq C \qquad \{x(t)\}\subset\subset\Omega^\ell\setminus D,\ x(t)=(x_j(t))$$

**residual vanishing** $x_i = x_i(t),\ u_k(x,t) = u(x,t+t_k),\ v_k(x,t) = v(x,t+t_k),\ 0 < r \ll 1$

$$\frac{d}{dt}\int_{B(x_i,r)} |x-x_i|^2 u_k = \int_{B(x_i,r)} \frac{\partial}{\partial t}(|x-x_i|^2 u_k) + \dot{x}_i \cdot \nabla(|x-x_i|^2 u_k)\ dx$$

リュービルの第1体積公式
$$= \int_{B(x_i,r)} |x-x_i|^2 u_{kt} + \dot{x}_i \cdot |x-x_i|^2 \nabla u_k\ dx$$

$$\int_{B(x_i,r)} |x-x_i|^2 u_{kt}$$
$$= \int_{B(x_i,r)} |x-x_i|^2 \nabla\cdot(\nabla u_k - u_k \nabla v_k)\ dx$$
$$\le r^2 \int_{\partial B(x_i,r)} \frac{\partial u_k}{\partial \nu} - u_k \frac{\partial v_k}{\partial \nu}\ dS$$
$$+ \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k \nabla v_k\ dx$$
$$= \int_{B(x_i,r)} \cancel{r^2 u_{kt}} + 4u_k + 2(x-x_i)\cdot u_k \nabla v_k\ dx$$

$$\int_{B(x_i,r)} \dot{x}_i \cdot |x-x_i|^2 \nabla u_k$$
$$= \int_{\partial B(x_i,r)} (\dot{x}_i\cdot\nu)|x-x_i|^2 u_k\ dS$$
$$- \int_{B(x_i,r)} 2(x-x_i)\cdot \dot{x}_i u_k$$
$$= \int_{B(x_i,r)} \cancel{r^2 \dot{x}_i \cdot \nabla u_k}$$
$$-2(x-x_i)\cdot \dot{x}_i u_k\ dx$$

$$\frac{d}{dt}\int_{B(x_i,r)} u_k = \int_{B(x_i,r)} \cancel{u_{kt}}$$
$$\cancel{+\dot{x}_i \cdot \nabla u_k}\ dx$$

defect moment

$$\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2) u_k$$
$$\le \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k \nabla v_k - 2(x-x_i)\cdot \dot{x}_i u_k\ dx$$

$$x_i = x_i(t)$$
$$u_k(x,t) = u(x,t+t_k)$$
$$v_k(x,t) = v(x,t+t_k)$$
$$0 < r \ll 1$$

$$v_k(x,t) = \sum_{i=0}^{3} v_k^i(x,t)$$

$$v_k^0(x,t) = \int_{B(x_i,r)} \Gamma(x-x')u_k(x',t)dx'$$

$$v_k^1(x,t) = \int_{B(x_i,r)} K(x,x')u_k(x',t)\ dx'$$

$$v_k^2(x,t) = \int_{\Omega\setminus\mathcal{S}_t^{2r}} G(x,x')u_k(x',t)dx'$$

$$v_k^3(x,t) = \int_{\mathcal{S}_t^{2r}\setminus B(x_i,r)} G(x,x')u_k(x',t)dx'$$

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$
$$G(x,x') = \Gamma(x-x') + K(x,x')$$

$\longrightarrow$

$$\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2)u_k$$
$$\leq \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k\nabla v_k$$
$$-2(x-x_i)\cdot \dot{x}_i u_k\ dx$$

$$2\int_{B(x_i,r)} (x-x_i)\cdot u_k\nabla v_k^0\ dx$$
$$= -\frac{1}{2\pi}\left(\int_{B(x_i,r)} u_k\ dx\right)^2$$

$$\|u_k(\cdot,t)\|_1 = \lambda,\ K(x,x') \in C^1(\Omega\times\Omega)$$
$$\sup_x \int_\Omega |\nabla_x G(x,x')|\ dx' \leq C$$

$\longrightarrow$

$$\|\nabla v^i(\cdot,t)\|_{L^\infty(B(x_i,r))} \leq C,\ 1 \leq i \leq 3$$

$$\frac{d}{dt}\int_{B(x_i,r)}(|x-x_i|^2-r^2)u_k$$

$$\leq 4\int_{B(x_i,r)} u_k - \frac{1}{2\pi}\left(\int_{B(x_i,r)} u_k\right)^2$$

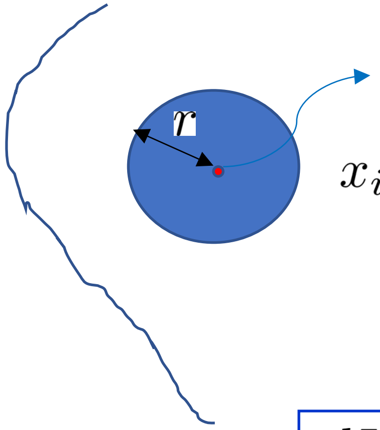
$$+C\int_{B(x_i,r)}|x-x_i|u_k$$

$$r^2\int_{B(x_i,r)} u_k \to 8\pi r^2 + \int_{B(x_i,r)} f$$

$k \to \infty$

as distributions in time

defect moment




$$x_i = x_i(t)$$

$$\frac{d}{dt}\int_{B(x_i,r)}(|x-x_i|^2-r^2)f$$

$$\leq 4\left(8\pi + \int_{B(x_i,r)} f\right)$$

$$-\frac{1}{2\pi}\left(8\pi + \int_{B(x_i,r)} f\right)^2$$

$$+C\int_{B(x_i,r)}|x-x_i|f$$

$0 < r \ll 1$

$$\frac{dI}{dt} \leq \int_{B(x_i,r)} -4f + C|x-x_i|f \ dx \leq \frac{2I}{r^2}$$

$$I(t) \equiv \int_{B(x_i,r)}(|x-x_i|^2-r^2)f \leq 0$$

$$I(t) \equiv 0$$

$$f = 0 \text{ in } B(x_i,r)$$

$$f \equiv 0$$

## コラプスの運動

$$\mu(dx,t)=8\pi\sum_{i=1}^{\ell}\delta_{x_i(t)}(dx)$$

$$\frac{dx_i}{dt}=8\pi\nabla_{x_i}H_\ell(x_1,\cdots,x_\ell),\ 1\leq i\leq\ell$$

a blowup criterion excludes the collapse collision in infinite time

$$x(t)=(x_i(t))\in\Omega^\ell\setminus D$$ pre-compact

$$D=\{(x_i)\mid\exists i\neq j,\ x_i=x_j\}$$

⟶

$$\exists x^*\in\Omega^\ell\setminus D,\ \nabla_{x_i}H_\ell(x^*)=0,\ 1\leq i\leq\ell$$

## 自由エネルギーの有界性

**定理 3 (Senba-S. 02)**

$$T=+\infty,\ \ \lim_{t\uparrow+\infty}\|u(\cdot,t)\|_\infty=+\infty$$

$$\lim_{t\uparrow+\infty}\mathcal{F}(u(\cdot,t))>-\infty$$

⟶

$$\lambda=\|u_0\|_1=8\pi\ell,\ \ell\in\mathcal{N},\ \exists(x_1^*,\cdots,x_\ell^*)\in\Omega^\ell\setminus D$$

$$\nabla_{x_i}H_\ell(x_1^*,\cdots,x_\ell^*)=0,\ 1\leq i\leq\ell$$

$$\mu(dx,t)=8\pi\sum_{i=1}^{\ell}\delta_{x_i^*}(dx)$$

## 勾配不等式

クリニック軌道

$$\exists x_\pm^*\in\Omega^\ell\setminus D,\ \lim_{t\to\pm\infty}x(t)=x_\pm^*,\ \nabla_{x_i}H_\ell(x_\pm^*)=0,\ 1\leq i\leq\ell$$

## 5. 双対変分原理

### 統計集団と非平衡熱力学

S. Mean Field Theories and Dual Variation, 2nd edition, Atlantis Press, 2015

| system | consistency | dynamics | ensemble |
|---|---|---|---|
| isolated | energy | entropy | micro-canonical |
| closed | temperature | Helmholtz free energy | canonical |
| open | pressure | Gibbs free energy | grand-canonical |

### 場と粒子の双対性

**particle density** — duality ⟷ — **field potential**

Smoluchowski

$$u_t = \nabla \cdot (\nabla u - u \nabla v)$$

$$\left. \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right|_{\partial \Omega} = 0$$

Poisson

$$v = (-\Delta)^{-1} u = \int_\Omega G(\cdot, x') u(x') dx'$$

symmetry

$$-\Delta v = u \quad v|_{\partial\Omega} = 0$$

**Helmholtz fee energy**

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2} \langle (-\Delta)^{-1} u, u \rangle$$

$$\delta \mathcal{F}(u) = \log u - (-\Delta)^{-1} u$$

**Model (B) equation**

$$u_t = \nabla u \cdot \nabla \delta \mathcal{F}(u), \quad \left. \frac{\partial}{\partial \nu} \delta \mathcal{F}(u) \right|_{\partial\Omega} = 0$$

total mass conservation, free energy decreaseing

$$\Rightarrow \quad \frac{d}{dt} \int_\Omega u = 0, \quad \frac{d\mathcal{F}}{dt} = -\int_\Omega u |\nabla \delta \mathcal{F}(u)|^2 \leq 0$$

# 双対変分原理

$X/\mathbf{R}$ 実バナッハ空間

$F: X \to (-\infty, +\infty]$, prop. c'x, l.s.c. 適切, 凸, 下半連続

**ルジャンドル変換** $F^*(p) = \sup_{x \in X} \{\langle x, p \rangle - F(x)\}$

**Fenchel-Moreau 双対** $F^{**} = F, \ F^{**}(x) = \sup_{p \in X^*} \{\langle x, p \rangle - F^*(p)\}$

**Toland 双対**

$$F, G: X \to (-\infty, +\infty], \ \text{prop. c'x l.s.c.}$$

$$J(x) = G(x) - F(x), \ J^*(p) = F^*(p) - G^*(p)$$

鈴木・田崎理論

$$L(x,p) = F^*(p) + G(x) - \langle x, p \rangle$$

ラグランジュ関数

$$\inf_{X \times X^*} L = \inf_X J = \inf_{X^*} J^*$$

ミニマリティ

$$L|_{p \in \partial G(x)} = J(x)$$

$$L|_{x \in \partial F^*(p)} = J^*(p)$$

アンフォールディング

$$\frac{d}{dt} L(x(t), p(t)) \le 0 \longrightarrow$$

無限小安定（線形化安定より弱い）定常解は力学安定
非自明解析的非線形性のもとで極小臨界点は無限小安定

非平衡熱力学モデルの多くはセミ・アンフォールディング・ミニマリティをもつ

Nonlinearity 2010

相転移・相分離・記憶形状

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$$\inf\{\mathcal{F}(u) \mid u \ge 0, \ \|u\|_1 = 8\pi\} > -\infty \qquad n = 2 \qquad \inf\left\{J_{8\pi}(v) \mid v \in H_0^1(\Omega)\right\} > -\infty$$

双対

自由エネルギー ⟷ 場の汎関数

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle (-\Delta)^{-1} u, u \rangle$$

$u \ge 0, \ \|u\|_1 = \lambda$ 粒子密度

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v + \lambda(\log \lambda - 1)$$

$v \in H_0^1(\Omega)$ ポテンシャル分布

## フォークモデル

マルテンシティック相転移（NiTi金属）

変位 $u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x$ $(u_x, u_{xxx}, \theta_x)|_{x=0,1} = 0$ or $(u, u_{xx}, \theta_x)|_{x=0,1} = 0$

温度 $\theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt} \quad \text{in } 0 < x < 1,\ t > 0$ $(u, u_t, \theta)|_{t=0} = (u_0(x), u_1(x), \theta_0(x))$

臨界温度

$$f_i = F_i',\ i = 1, 2, \quad F_1(\varepsilon) = \alpha_1 \varepsilon^2, \quad F_2(\varepsilon) = \alpha_3 \varepsilon^6 - \alpha_2 \varepsilon^4 - \alpha_1 \theta_c \varepsilon^2$$

粘性項なし ⟹ 分散系

## 線形部分

ブシネスク

$$(\partial_t + \imath \partial_x^2)(\partial_t - \imath \partial_x^2) = \partial_t^2 + \partial_x^4$$

ストリッカー評価

↔

熱

$$\partial_t - \partial_x^2$$

極大正則性

$$\left\| e^{\pm \imath t \partial_x^2} g \right\|_{L^4(0,T);L^4)} \leq C \|g\|_2$$

$$\left\| \int_0^t e^{\pm \imath (t-1) \partial_x^2} f(\cdot, s) ds \right\|_{L^4(0,T;L^4)} \leq C \|f\|_{L^{4/3}(0,T;L^{4/3})}$$

$$\left\| \int_0^t e^{-\imath (t-s) \partial_x^2} f(\cdot, s) ds \right\|_{L^\infty(0,T;L^2)} \leq C \|f\|_{L^{4/3}(0,T;L^{4/3})}$$

$$\theta_t - \theta_{xx} = f(x,t),\ \ \theta_x|_{x=0,1} = 0,\ \ \theta|_{t=0} = 0$$

⟹

$$\int_0^T \|\theta_t\|_p^p + \|\theta_{xx}\|_p^p \, dt \leq C(p,T) \int_0^T \|f(\cdot,t)\|_p^p \, dt$$

⟹ 局所適切性と大域存在 Yoshikawa 05

## 熱力学的構造

$$u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x, \ \theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt} \qquad (u_x, u_{xxx}, \theta_x)|_{x=0,1} = 0 \qquad f_1(0) = f_2(0) = 0$$

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\|u_{xx}\|_2^2 = -\int_0^1 [f_1(u_x)\theta + f_2(u_x)]u_{xt} \ dx$$

$$= -\int_0^1 \theta_t - \theta_{xx} \ dx - \frac{d}{dt}\int_0^1 F_2(u_x) \ dx = -\frac{d}{dt}\int_0^t \theta + F_2(u_x) \ dx$$

$\Longrightarrow$

エネルギー保存

$$\frac{dE}{dt} = 0$$

$$E = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^t F_2(u_x) + \theta \ dx$$

$$\theta|_{t=0} = \theta_0 > 0 \quad \Longrightarrow \quad \theta(\cdot, t) > 0 \qquad \frac{1}{\theta}(\theta_t - \theta_{xx}) = f_1(u_x)u_{xt} = F_1(u_x)_t$$

エントロピー増大

$$\frac{dW}{dt} = -\int_0^1 \frac{\theta_{xx}}{\theta} \ dx = -\int_0^1 \left(\frac{\theta_x}{\theta}\right)^2 \ dx \leq 0$$

$$W = \int_0^t F_1(u_x) - \log\theta \ dx$$

定常状態 $\quad u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x,\ \theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt} \qquad (u_x, u_{xxx}, \theta_x)|_{x=0,1} = 0$

$u_t = \theta_t = 0 \quad \Longrightarrow \quad \theta = \overline{\theta} > 0$ constant associated with $u$ by $\quad E(u, 0, \overline{\theta}) = b \equiv E(u_0, u_1, \theta_0)$

$u_{xxxx} = (\overline{\theta} f_1(u_x) + f_2(u_x))_x,\ (u_x, u_{xxx})|_{x=0,1} = 0 \qquad E(u, u_t, \theta) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^t F_2(u_x) + \theta\ dx$

変分汎関数 $\quad J_1(u_x) = \int_0^1 F_1(u_x)dx \qquad \Longrightarrow \quad \overline{\theta} + \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^1 F_2(u_x)dx = b$

$$J_2(u_x) = \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^1 F_2(u_x)dx$$

$\Longrightarrow \quad \delta J_2(u_x) = \overline{\theta}\delta J_1(u_x),\ \overline{\theta} = b - J_2(u_x) \quad \Longrightarrow \quad \delta J_2(u_x) = -(b - J_2(u_x))\delta J_1(u_x)$

$\Longrightarrow \quad \delta J_b(u_x) = 0, \quad J_b(u_x) = J_1(u_x) - \log(b - J_2(u_x))$

$v = u_x \quad \Longrightarrow \quad J_b = J_b(v),\ v \in V_b, \quad V_b = \{v \in H_0^1 \mid J_b(v) < b\}$

## セミ・ミニマリティ

$$b = E(u, u_t, \theta) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^1 F_2(u_x) + \theta\ dx \geq \frac{1}{2}\|u_{xx}\|_2^2 + \int_0^1 F_2(u_x) + \theta\ dx$$

$$W(u_x, \theta) = \int_0^1 F_1(u_x) - \log\theta\ dx \geq \int_0^1 F_1(u_x)dx - \log\left(\int_0^1 \theta dx\right)$$

$$\Longrightarrow \quad \geq \int_0^1 F_1(u_x)dx - \log\left(b - \frac{1}{2}\|u_{xx}\|_2^2 - \int_0^1 F_2(u_x)dx\right) = J_b(u_x)$$

## セミ・アンフォールディング

$$\overline{\theta} = b - J_2(u_x) > 0 \Longrightarrow W(u_x, \overline{\theta}) = J_b(u_x)$$

$$\Longrightarrow \boxed{W(u_x, \theta) \geq W(u_x, \overline{\theta}) = J_b(u_x)}$$

$$J_b(u_x) - J_b(\overline{v}) \leq W(u_{0x}, \theta_0) - J_b(\overline{v}) = W(u_{0x}, \theta_0) - W(\overline{v}, \overline{\theta})$$

$$W(u_{0x}, \theta_0) - W(\overline{v}, \overline{\theta}) < \delta \Rightarrow J_b(u_x(\cdot, t)) - J_b(\overline{v}) < \delta$$

無限小安定

$$\exists\varepsilon_0 > 0 \text{ s.t. } \forall\varepsilon \in (0, \frac{\varepsilon_0}{4}],\ \exists\delta > 0 \text{ s.t. } \|(v - \overline{v})_x\|_2 < \varepsilon_0,\ J_b(v) - J_b(\overline{v}) < \delta \Rightarrow \|(v - \overline{v})_x\|_2 < \varepsilon$$

$u_x \in C([0, +\infty); H_0^1) \Longrightarrow (\overline{v}, \overline{\theta})$ は力学系安定　S.-Yoshikawa 07

実解析性

$f_1(v) \neq 0, \quad v \neq 0$

$F_i,\ i = 1, 2$: real analytic

鈴木・田崎理論 $\Longrightarrow$ すべての極小は無限小安定

分岐解析

$\mathcal{C} = \{(b, v)\}$: total set of solutions

1. $v = 0$...the trivial solution, $\forall b$

2. $b_k = \theta_c - \frac{k^2\pi^2}{2\alpha_1}$, $k = 1, 2, \cdots$
...bifurcation points
(for normalized physical constants)

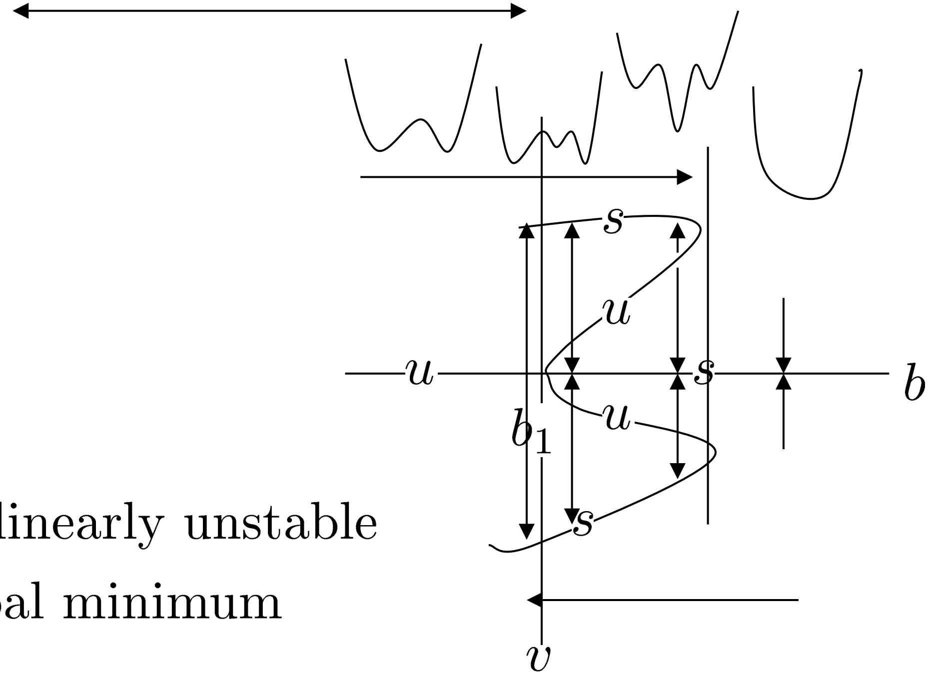
3. ${}^\forall v \neq 0$, stationary state, generates a one-dimensional manifold $\subset \mathcal{C}$ (branch)

$(b(s), v(s))$, $|s| \ll 1$
the bifurcated branch
from the trivial solution at $b = b_1$
$\Rightarrow$

$$\ddot{b}(0) = -\alpha_1\theta_c + \frac{\pi^2}{2} + \frac{3\alpha_2}{\alpha_1}$$

$\ddot{b}(0) > 0$
$\Rightarrow$ oscillation of $b$
$\Rightarrow$ hysteresis
$\Rightarrow$ hetero-clinic orbits

non-trivial global minimizer




$b < b_1 \Rightarrow v = 0$: linearly unstable
${}^\exists$ non-trivial global minimum