循環するハミルトニアン

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1. 点渦系の統計力学

Model Non-equilibrium thermo-dynamics

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Euler-Smoluchowski Part

vortex term, diffusion, gravitational mass, velocity

$$u_t + \beta \nabla \cdot (u \nabla^\perp v) = \nabla \cdot (\nabla u - u \nabla v) \text{ in } \Omega \times (0,T)$$

$$\left. \frac{\partial u}{\partial \nu} - u\left(\frac{\partial v}{\partial \nu} + \beta \frac{\partial v}{\partial \tau}\right) \right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0$$

flux zero

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

Green's function

$$G(x,x') = G(x',x)$$

action at a distance (long range potential)
symmetry (action-reaction)

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}, \quad \nabla^\perp = \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}$$

$$x = (x_1, x_2), \quad x^\perp = (-x_2, x_1)$$

outer unit normal vector

$$\nu = \begin{pmatrix} \nu^1 \\ \nu^2 \end{pmatrix}, \quad \tau = \begin{pmatrix} \nu^2 \\ -\nu^1 \end{pmatrix}$$

$\beta = 0$ Smoluchowski-Poisson equation

total mass conservation, free energy decreasing ⟹ (stationary) Boltzmann-Poisson equation

scaling invariance ⟹ critical dimension, critical mass

weak form ⟹ quantized blowup mechanism, recursive hierarchy
both in blowup in finite and infinite time

The same structure with different sub-collapse dynamics

Keller-Segel system

1971

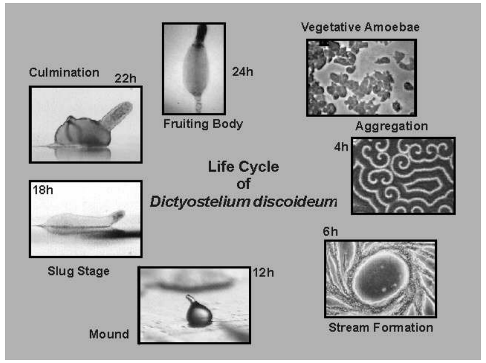
$$u_t = \nabla \cdot (d_1(u,v)\nabla u) - \nabla \cdot (d_2(u,v)\nabla v)$$
$$v_t = d_v \Delta v - k_1 vw + k_{-1}p + f(v)u$$
$$w_t = d_w \Delta w - k_1 vw + (k_{-1}+k_2)p + g(v,w)u$$
$$p_t = d_p \Delta p + k_1 vw - (k_{-1}+k_2)p$$

$u = u(x,t)$ cellular slime molds

$v = v(x,t)$ attractant

$w = w(x,t)$ enzyme

$p = p(x,t)$ complex



1. transport, gradient
   - (a) diffusion u, v, w, p
   - (b) chemotaxis $v \to u$

2. production $u \to (v, w)$

3. chemical reaction v, w, p

$$V + W \underset{\leftarrow (k_{-1})}{\overset{(k_1)\to}{}} P \overset{(k_2)}{\to} W + A$$

$$v_t = -k_1 vw + k_{-1}p$$
$$w_t = -k_1 vw + (k_{-1}+k_2)p$$
$$p_t = k_1 vw - (k_{-1}+k_2)p$$

Reductions

Nanjundiah 73

Michaelis-Menten

$$k_1 vw - (k_{-1}+k_2)p = 0 \quad \text{quasi-static}$$
$$w + p = c \quad \text{total mass conservation}$$

$$u_t = \nabla \cdot (d_1(u,v)\nabla u) - \nabla \cdot (d_2(u,v)\nabla v)$$
$$v_t = d_v \Delta v - k(v)v + f(v)u$$
$$k(v) = \frac{ck_1k_2}{(k_{-1}+k_2)+k_1 v}$$

Childress-Percus 81

$d_1(u,v),\ k(v),\ f(v)$ constant

$d_2(u,v) = u\chi'(v)$ mass×velocity＝flux（momentum）

sensitivity

Jager-Luckhaus 92 short range approximation

no vorticity term Smoluchowski

$$u_t = \nabla \cdot (\nabla u - u\nabla v), \quad \left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

Poisson

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \quad \int_\Omega v = 0, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

Statistic ensemble and non-equilibrium thermodynamics

S. Mean Field Theories and Dual Variation, 2nd edition, Atlantis Press, 2015

system	consistency	dynamics	ensemble
isolated	energy	entropy	micro-canonical
closed	temperature	Helmholtz free energy	canonical
open	pressure	Gibbs free energy	grand-canonical

Duality between Field Distribution and Particle Density

particle density ⟷ **duality** ⟷ **field potential**

Smoluchowski ⟷ Poisson

$$v=(-\Delta)^{-1}u=\int_\Omega G(\cdot,x')u(x')dx'$$

symmetry

$$u_t=\nabla\cdot(\nabla u-u\nabla v)$$

$$\left.\frac{\partial u}{\partial\nu}-u\frac{\partial v}{\partial\nu}\right|_{\partial\Omega}=0$$

$$-\Delta v=u \quad v|_{\partial\Omega}=0$$

Helmholtz fee energy

$$\mathcal{F}(u)=\int_\Omega u(\log u-1)-\frac{1}{2}\langle(-\Delta)^{-1}u,u\rangle$$

$$\delta\mathcal{F}(u)=\log u-(-\Delta)^{-1}u$$

Model (B) equation

$$u_t=\nabla u\cdot\nabla\delta\mathcal{F}(u),\ \left.\frac{\partial}{\partial\nu}\delta\mathcal{F}(u)\right|_{\partial\Omega}=0$$

total mass conservation, free energy decreaseing

$$\Rightarrow\ \frac{d}{dt}\int_\Omega u=0,\ \frac{d\mathcal{F}}{dt}=-\int_\Omega u|\nabla\delta\mathcal{F}(u)|^2\le 0$$

Thermo-dynamical structure

thermally closed system (canonical ensemble)

independent of the vorticity term

$$u(\cdot,t)>0$$

total mass conservation

$$\frac{d}{dt}\int_\Omega u=\int_\Omega \nabla\cdot(\nabla u-u\nabla v-\beta u\nabla^\perp v)=\int_{\partial\Omega}\frac{\partial u}{\partial\nu}-u\frac{\partial v}{\partial\nu}-\beta u\frac{\partial v}{\partial\tau}\ ds=0$$

free energy decreasing

$$u_t=\nabla\cdot u(\nabla(\log u-v)-\beta\nabla^\perp v),\ v(x,t)=\int_\Omega G(x,x')u(x',t)\ dx'$$

-(entropy) inner energy

$$\mathcal{F}(u)=\int_\Omega u(\log u-1)-\frac{1}{2}\int\int_{\Omega\times\Omega}G(x,x')u\otimes u$$

$$\int_\Omega u_t(\log u-v)=\frac{d\mathcal{F}}{dt}$$

method of symmetrization

$$=\int_\Omega[\nabla\cdot(u\nabla(\log u-v)-\beta\nabla^\perp v)](\log u-v)$$

$$=\int_\Omega -u|\nabla(\log u-v)|^2+\beta u\nabla^\perp v\cdot\nabla(\log u-v)\ dx$$

Helmholtz's free energy $A=U-TS$

$$\nabla^\perp v\cdot\nabla v=0$$

$$\int_\Omega u\nabla^\perp v\cdot\nabla\log u=\int_\Omega\nabla^\perp v\cdot\nabla u=\int_{\partial\Omega}\frac{\partial v}{\partial\tau}u-\int_\Omega((\nabla\cdot\nabla^\perp)v)u=0$$

$$\lambda=\int_\Omega u$$

$$\frac{d}{dt}\mathcal{F}(u)=-\int_\Omega u|\nabla(\log u-v)|^2\le 0$$

stationary ⟹

$$u=\frac{\lambda e^v}{\int_\Omega e^v dx}$$

Boltzmann equation

Boltzmann Poisson equation

Euler's equation of motion

$$v_t + (v\cdot\nabla)v = -\nabla p,\ \ \nabla\cdot v = 0,\ \ \nu\cdot v|_{\partial\Omega} = 0$$

2D $\omega = \nabla^\perp \cdot v$ ⟹ $\omega_t + \nabla\cdot(v\omega) = 0,\ \ \nabla\cdot v = 0$

$v = \nabla^\perp \psi$ stream function

$$\omega_t + \nabla\cdot(\omega\nabla^\perp\psi) = 0,\ \ -\Delta\psi = \omega$$ vorticity equation

boundary condition ⟹ $\psi|_{\partial\Omega} = 0$ simply-connected

$$\psi(\cdot,t) = \int_\Omega G(\cdot,x')\omega(x',t)dx'$$ Green function

$$G(x,x') = G(x',x)$$ action reaction law

$$\omega(dx,t) = \sum_{i=1}^{\ell} \alpha_i \delta_{x_i(t)}(dx)$$ point vortex system

local second moment weak form p.v. ⟹ **Kirchhoff equation** $\frac{dx_i}{dt} = \nabla_{x_i}^\perp H_\ell$

point vortex Hamiltonian

$$H_\ell(x_1,\ldots,x_\ell) = \sum_i \frac{\alpha_i^2}{2}R(x_j) + \sum_{i<j}\alpha_i\alpha_j G(x_i,x_j)$$

Robin function $R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x-x'|\right]_{x'=x}$

Thermal equilibrium

Gibbs theory of statistical mechanics

micro-canonical statistics

total energy

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \ 1 \leq i \leq \ell$$

$$\mathbf{R}^{6\ell}/\{H = E\}$$

co-area formula

$$x = (q_1, \ldots, q_\ell, p_1, \ldots, p_\ell), \ dx = dE \cdot \frac{d\Sigma(E)}{|\nabla H|}$$

$$d\Sigma(E) \ \leftrightarrow \ \{x \in \mathbf{R}^{6\ell} \mid H(x) = E\}$$

thermal equilibrium

micro-canonical measure

weight factor

$$d\mu^{E,\ell} = \frac{1}{W(E)} \cdot \frac{d\Sigma(E)}{|\nabla H|}, \ W(E) = \int_{\{H=E\}} \frac{d\Sigma(E)}{|\nabla H|}$$

$$H_\ell(x_1, \ldots, x_\ell) = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

principle of equal probability

$$\alpha_i = \hat{\alpha}, \ \hat{\alpha}\ell = 1, \ \hat{H}_\ell = H, \ \hat{\alpha}^2 \ell \hat{\beta} = \beta \qquad \ell \uparrow +\infty$$

one-point PDF
factorization

canonical statistics

Boltzmann constant

$$\mathbf{R}^{6\ell}/\{T\}, \ \beta = 1/(kT)$$

inverse temperature

$$d\mu^{\beta,\ell} = \frac{e^{-\beta H} dx}{Z(\beta, N)}, \ Z(\beta, \ell) = \int_{\mathbf{R}^{6N}} e^{-\beta H} dx$$

canonical measure

weight factor

thermo-dynamical relation

$$\beta = \frac{\partial}{\partial E} \log W(E)$$

propagation of chaos

Pointin-Ludgren1973 etc

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}$$

vorticity

$$\psi = \int_\Omega G(\cdot, x')\rho(x')dx'$$

stream function

Boltzmann-Poisson equation

duality

$$-\Delta v = u, \ v|_{\partial\Omega} = 0$$

$$u = \frac{\lambda e^v}{\int_\Omega e^v dx}, \ \lambda = \|u\|_1$$

Hamiltonian?

Hamiltonian is recursive

$$u = \frac{\lambda e^v}{\int_\Omega e^v dx} \qquad -\Delta v = u, \ \ v|_{\partial\Omega} = 0$$

Boltzmann　　　　Poisson

$$\rightarrow \quad -\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \ \ v|_{\partial\Omega} = 0$$

Nonlinear eigenvalue problem with nonlocal term

point vortices ~ negative inverse temperature

L. Onsager 49

order structure in negative temperature

$$G(x, x') = G(x', x)$$ action-reaction law

$$R(x) = \left[G(x, x') + \frac{1}{2\pi} \log |x - x'| \right]_{x'=x}$$

Theorem A (Nagasaki-S. 90)

$$\{(\lambda_k, v_k)\}, \ \ \lambda_k \to \lambda_0 \in (0, \infty), \ \ \|v_k\|_\infty \to \infty$$

$$\rightarrow \quad \lambda_0 = 8\pi\ell, \ \ \ell \in \mathbf{N}, \ \ \exists \mathcal{S} \subset \Omega, \ \ \sharp\mathcal{S} = \ell$$ stationary quantization

(sub-sequence) $v_k \to v_0$ loc. unif. in $\overline{\Omega} \setminus \mathcal{S}$

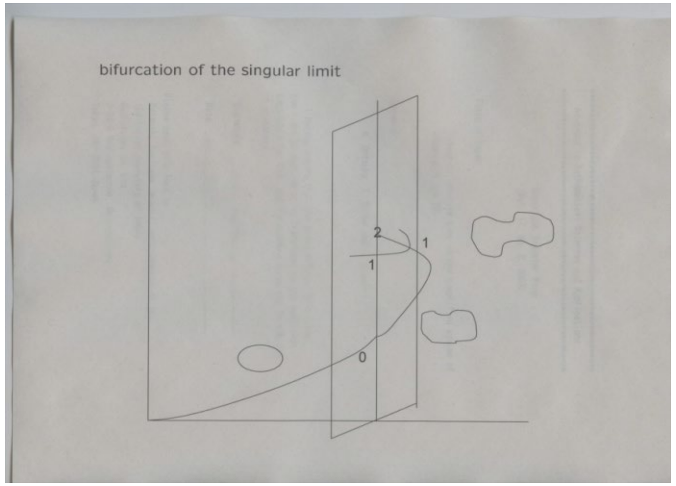
$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0), \ \ \mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$$

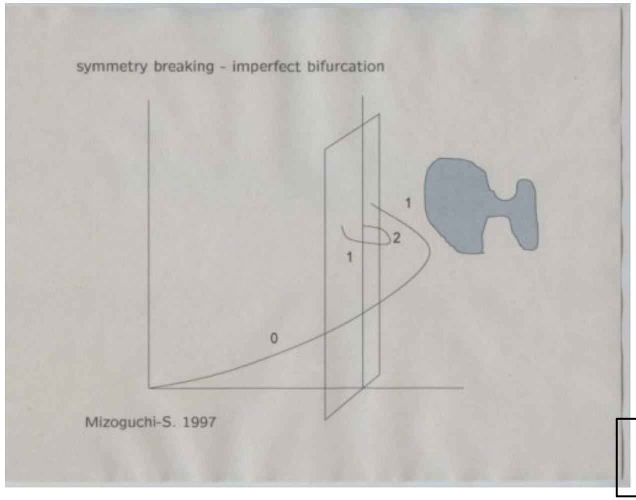
singular limit　　　blowup set

$$\nabla H_\ell|_{(x_1, \cdots, x_\ell) = (x_1^*, \cdots, x_\ell^*)} = 0$$

$$H_\ell(x_1, \ldots, x_\ell) = \frac{1}{2} \sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$$

kinetics?





From quasi-stationary to stationary

$$u_t + \beta\nabla\cdot(u\nabla^\perp v) = \nabla\cdot(\nabla u - u\nabla v) \text{ in } \Omega\times(0,T)$$

Euler-Smoluchowski-Poisson equation

$$\frac{\partial u}{\partial \nu} - u\left(\frac{\partial v}{\partial \nu} + \beta\frac{\partial v}{\partial \tau}\right)\Big|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0 \qquad -\Delta v = u, \quad v|_{\partial\Omega} = 0$$

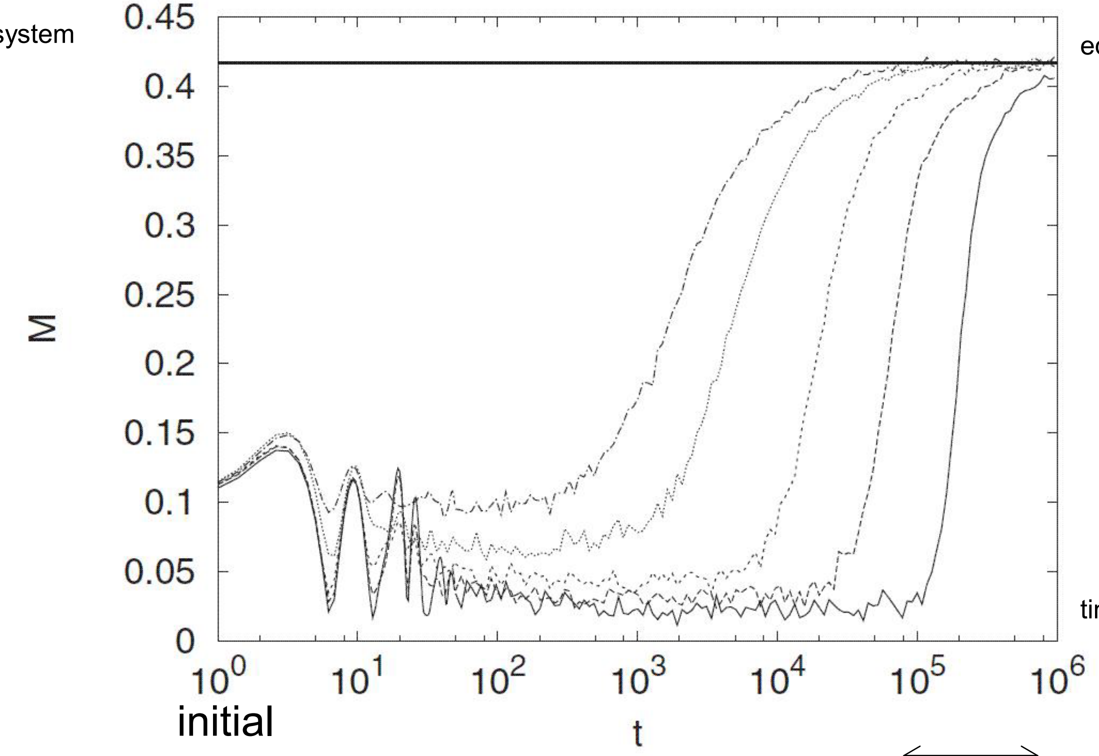
factorization (propagation of chaos)

$$P_N(x_1, x_2, \cdots, x_N, t) = \prod_{i=1}^N P_1(x_i, t)$$

Hamilton system of many particles with inner interaction of long range

Staniscia-Chavanis-Ninno-Fanelli 09

state of the system



equilibrium (stationary)

time

initial

relaxation

quasi-equilibrium (stationary)

relaxation

high energy limit

$$\hat\beta N\alpha^2 = \beta, \ \alpha N = 1, \ \omega = P_1$$

$$\frac{\partial\omega}{\partial t} + \nabla^\perp\psi\cdot\nabla\omega = \nu\nabla\cdot(\nabla\omega + \beta\alpha\omega\nabla\psi)$$

$$-\Delta\psi = \omega, \ \psi|_{\partial\Omega} = 0 \qquad \beta = -\lambda$$

negative inverse temperature

recursive hierarchy?

Kyoto 2011. 8. 28-31

Chavanis 08 relaxation to the equilibrium in the point vortices, kinetic equation + maximum entropy production

Sire-Chavanis 02 motion of the mean field of many self-gravitating Brownian particles, BBGKY hierarchy + factorization

Scaling $u_\mu(x,t)=\mu^2 u(\mu x,\mu^2 t),\ \mu>0$

1. total mass conservation $\frac{d}{dt}\|u(t)\|_1=0$

2. free energy decreasing

$$\mathcal{F}(u)=\int_\Omega u(\log u-1)-\frac{1}{2}\int\int_{\Omega\times\Omega}G(x,x')u\otimes u$$

$$\frac{d}{dt}\mathcal{F}(u)=-\int_\Omega u|\nabla(\log u-v)|^2\le 0$$

$$u_\mu(x)=\mu^2 u(\mu x),\ \ \mu>0$$

$$\|u\|_1=\|u_\mu\|_1\equiv\lambda\ \Leftrightarrow\ n=2$$ critical dimension

$$\mathcal{F}(u)=\int_{\mathbf{R}^2}u(\log u-1)-\frac{1}{2}\langle\Gamma*u,u\rangle,\ \ \Gamma(x)=\frac{1}{2\pi}\log\frac{1}{|x|}$$

$$\mathcal{F}(u_\mu)=\left(2\lambda-\frac{\lambda^2}{4\pi}\right)\log\mu+\mathcal{F}(u)$$ critical mass $\lambda=8\pi$

Weak form $\varphi\in C^2(\overline{\Omega}),\ \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega}=0$

$$\rho_\varphi(x,x')=\nabla\varphi(x)\cdot\nabla_x G(x,x')+\nabla\varphi(x')\cdot\nabla_{x'}G(x,x')$$

$$\rho_\varphi^\perp(x,x')=\nabla\varphi(x)\cdot\nabla_x^\perp G(x,x')+\nabla\varphi(x')\cdot\nabla_{x'}^\perp G(x,x')$$

$$\frac{d}{dt}\int_\Omega\varphi u\ dx=\int_\Omega u\Delta\varphi\ dx+\frac{1}{2}\int\int_{\Omega\times\Omega}(\rho_\varphi(x,x')+\beta\rho_\varphi^\perp(x,x'))u\otimes u\ dxdx'$$ method of symmetrization

$$\in L^\infty(\Omega\times\Omega)\setminus C(\overline{\Omega}\times\overline{\Omega})$$

$$\Gamma(x)=\frac{1}{2\pi}\log\frac{1}{|x|}$$

Green function $G(x,x')=\Gamma(x-x')+K(x,x'),\ K=K(x,x')\in C^{1+\theta,\theta}(\Omega\times\overline{\Omega})\cap C^{\theta,1+\theta}(\overline{\Omega}\times\Omega)$

$$x_0\in\partial\Omega\Longrightarrow G(x,x')=E(X,X')+K(x,x'),\ K=K(x,x')\in(C^{1+\theta,\theta}\cap C^{\theta,1+\theta})(\overline{\Omega\cap B(x_0,R)}\times\overline{\Omega\cap B(x_0,R)})$$

$$X:\overline{\Omega\cap B(x_0,2R)}\to\overline{\mathbf{R}_+}^2=\{(X_1,X_2)\mid X_2\ge 0\}\qquad X(\partial\Omega\cap B(x_0,2R))\subset\partial\mathbf{R}_+^2$$

conformal diffeomorphism

$$E(X,X')=\Gamma(X-X')-\Gamma(X-X'_*)\qquad X=\begin{pmatrix}X_1\\X_2\end{pmatrix}\mapsto X_*=\begin{pmatrix}X_1\\-X_2\end{pmatrix}$$

Results

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

$$u_t + \beta\nabla\cdot(u\nabla^{\perp}v) = \nabla\cdot(\nabla u - u\nabla v) \text{ in } \Omega\times(0,T)$$

$$\left.\frac{\partial u}{\partial \nu} - u\left(\frac{\partial v}{\partial \nu} + \beta\frac{\partial v}{\partial \tau}\right)\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0$$

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

Hamiltonian control of sub-collapse dynamics

Theorem B

$T < +\infty \longrightarrow$

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$m(x_0) \in 8\pi\mathbf{N}$ collapse mass quantization possibly with sub-collapse collision

blowup set — exclusion of boundary blowup

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0,\ t_k \uparrow T,\ u(x_k,t_k) \to +\infty\} \subset \Omega$$

$\sharp\mathcal{S} < +\infty$ finiteness of blowup points

$0 < f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega}\setminus\mathcal{S})$ measure theoretic regular part

Theorem C

$$T = +\infty, \quad \limsup_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = +\infty$$

$\longrightarrow$ $\lambda \equiv \|u_0\|_1 = 8\pi\ell,\ \exists\ell\in\mathbf{N}$ initial mass quantization

$\exists x_* \in \Omega^\ell \setminus D,\ \nabla H_\ell(x_*) = 0$ recursive hierarchy

point vortex Hamiltonian (Robin function, Green function)

$$H_\ell(x_1,\cdots,x_\ell) = \frac{1}{2}\sum_j R(x_j) + \sum_{i<j} G(x_i,x_j)$$

Corollary 1

$T < +\infty$ if

$\lambda \notin 8\pi\mathbf{N}$, $\nexists$ stationary solution or $\mathcal{F}(u_0) \ll -1$

$\lambda \in 8\pi\ell,\ \ell\in\mathbf{N}$, $\nexists$ critical point of H_ℓ

Corollary 2

Ω convex $\lambda \neq 8\pi$

$\Rightarrow T < +\infty$ or $T = +\infty$ pre-compact orbit

$\exists$ stationary solution

c.f. Grossi-F. Takahashi (2010)

Proof of Theorem B

ε regularity

$$\frac{1}{p+1}\frac{d}{dt}\|u\|_{p+1}^{p+1} \leq -\frac{4p}{(p+1)^2}\|\nabla u^{\frac{p+1}{2}}\|_2^2 + \frac{p}{p+1}\|u\|_{p+2}^{p+2}$$

$$\|z\|_{p+1}^{p+1} \leq C_p\|z\|_1\|z\|_{H^1}^p$$

$$n = 2$$

Gagliardo-Nirenberg inequality

$$\exists\varepsilon_0 > 0,\ \|u_0\|_1 < \varepsilon_0 \ \Rightarrow\ T = +\infty,\ \|u(\cdot,t)\|_\infty \leq C$$

elliptic regularity

semi-group estimate

Moser's iteration scheme
maximal regularity

localization

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists\varepsilon_0 \Rightarrow x_0 \notin \mathcal{S}$$

nice cut-off function

$$x_0 \in \overline{\Omega},\ 0 < R \ll 1,\ \varphi = \varphi_{x_0,R} \in \mathcal{Y}$$

$$0 \leq \varphi \leq 1,\ \varphi = \begin{cases} 1, & x \in B(x_0, \frac{R}{2}) \\ 0, & x \in \mathbf{R}^2 \setminus B(x_0,R) \end{cases}$$

$$|\nabla\varphi| \leq CR^{-1}\varphi^{\frac{5}{6}}$$

$$|\nabla^2\varphi| \leq CR^{-2}\varphi^{\frac{2}{3}}$$

$$\mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0\}$$

Formation of collapses

symmetry of the Green function → weak form (symmetrization)

$$\varphi \in C^2(\overline{\Omega}), \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\frac{d}{dt}\int_\Omega \varphi u(\cdot,t) = \int_\Omega \Delta\varphi \cdot u(\cdot,t) + \frac{1}{2}\iint_{\Omega\times\Omega} (\rho_\varphi + \beta\rho_\phi^\perp)(x,x')u\otimes u$$

$$\|\rho_\varphi\|_\infty, \ \|\rho_\varphi^\perp\|_\infty \leq C\|\nabla\varphi\|_{C^1}$$

boundary behavior of the Green function
singularity cancellation by the symmetry

monotonicity formula $\lambda = \|u(\cdot,t)\|_1$

$$\left|\frac{d}{dt}\int_\Omega u\varphi\right| \leq C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1}$$

→ **weak continuation**

$$0 \leq \exists \mu(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$$

$$u(x,t)dx = \mu(dx,t), \ 0 \leq t < T$$

ε -regularity

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists\varepsilon_0 \Rightarrow x_0 \notin \mathcal{S}$$

→

$$x_0 \in \mathcal{S} \Rightarrow \lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0$$

↔ monotonicity formula

$$\lim_{R\downarrow 0}\liminf_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0$$

→ $\sharp\mathcal{S} < +\infty$

formation of collapse

$$\mu(dx,T) = \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0} + f(x)dx, \ m(x_0) \geq \varepsilon_0, \ 0 \leq f = f(x) \in L^1(\Omega)$$

singular part regular part

Generation of the weak solution

$0 \leq \mu = \mu(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$ weak solution

$\overset{\text{def}}{\Longleftrightarrow}$ $0 \leq \exists \mathcal{N} = \mathcal{N}(\cdot,t) \in L^\infty_*([0,T], \mathcal{X}')$ multiplicated operator

1. $t \in [0,T] \mapsto \langle \varphi, \mu(dx,t) \rangle, \ \varphi \in \mathcal{Y}$ a.c.
2. $\frac{d}{dt}\langle \varphi, \mu \rangle = \langle \Delta\varphi, \mu \rangle + \frac{1}{2}\langle \rho_\varphi + \beta\rho_\varphi^\perp, \mathcal{N}(\cdot,t) \rangle$ a.e. $t \in [0,T]$
3. $\mathcal{N}|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu \otimes \mu$

separable

$$\mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0\} \quad \mathcal{X} = [\mathcal{X}_0]^{L^\infty(\Omega\times\Omega)}$$

$$\mathcal{X}_0 = \{\rho_\varphi + \beta\rho_\varphi^\perp + \psi \mid \varphi \in \mathcal{Y}, \ \psi \in C(\overline{\Omega}\times\overline{\Omega})\}$$

$\Rightarrow$

$$\mu(\overline{\Omega},t) = \mu(\overline{\Omega},0) \equiv \lambda, \ 0 \leq t \leq T$$

$$\left|\frac{d}{dt}\langle \varphi, \mu(dx,t) \rangle\right| \leq C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1}$$

Theorem 1 $\mu_k(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$

$\mathcal{N}_k \in L^\infty_*([0,T], \mathcal{X}')$ weak solutions

$0 \leq \mu_k(\overline{\Omega},t) \leq C$

$\|\mathcal{N}_k(\cdot,t)\|_{\mathcal{X}'} \leq C$ $\Rightarrow$ sub-sequence

$\mu_k(dx,t) \rightharpoonup \mu(dx,t)$ in $C_*([0,T], \mathcal{M}(\overline{\Omega}))$

$\mathcal{N}_k(\cdot,t) \rightharpoonup \mathcal{N}(\cdot,t)$ in $L^\infty_*([0,T], \mathcal{X}')$ weal solution

$u = u(x,t)$ classical solution

$\Rightarrow$ $\mathcal{N}(\cdot,t) = u(x,t) \otimes u(x',t) \ dxdx'$

$$\|\mathcal{N}(\cdot,t)\|_{\mathcal{X}'} = \lambda^2, \ \lambda = \|u_0\|_1$$

Weak solution is not unique!

Backward self-similar transformation $x_0 \in \mathcal{S}$

$$y=(x-x_0)/(T-t)^{1/2}, \ s=-\log(T-t)$$
$$z(y,s)=(T-t)u(x,t)$$

weak limit $s_k \uparrow +\infty$ subsequence

$$z(y,s+s_k)dy \rightharpoonup \exists \zeta(dy,s) \ \text{in} \ C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$

Put $z(y,s)=0$ where it is not defined.

$$\mathcal{M}(\mathbf{R}^2)=C_\infty(\mathbf{R}^2)'$$
$$C_\infty(\mathbf{R}^2)=\{z\in C(\mathbf{R}^2\cup\{\infty\}), \ z(\infty)=0\}$$

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx)+f(x)dx$$

First parabolic envelope

$$\left|\frac{d}{dt}\int_\Omega u(\cdot,t)\varphi_{x_0,R}\right| \leq C_\lambda R^{-2}, \ 0<R\ll 1$$

$$|\langle \varphi_{x_0,R}, u(\cdot,t)dx\rangle - \langle \varphi_{x_0,R}, \mu(dx,T)\rangle| \leq C_\lambda (T-t)/R^2$$

$$s_k+s=-\log(T-t), \ R=b(T-t)^{1/2}$$

$\longrightarrow$

$$|\langle \varphi_{0,b}, z(\cdot,s+s_k)dy\rangle - \langle \varphi_{x_0,be^{-(s+s_k)/2}}, \mu(dx,T)\rangle| \leq C_\lambda/b^2$$

$$\mu(dx,T)=\sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx)+f(x)dx$$

$k\to\infty, \ b\uparrow+\infty$ $\longrightarrow$ $m(x_0)=\zeta(\mathbf{R}^2,s)$

Second parabolic envelope $\langle |y|^2, \zeta(dy,s)\rangle \leq C$

Limit equation $x_0 \in \mathcal{S}$

$$y = (x - x_0)/(T-t)^{1/2}, \ s = -\log(T-t)$$
$$z(y,s) = (T-t)u(x,t)$$

$$y \in (T-t)^{-1/2}(\Omega - \{x_0\}) = \Omega_s$$
$$-\log T \leq s < +\infty, \ \|z(\cdot,s)\|_1 = \lambda$$

$$z_s = \nabla\cdot(\nabla z - z\nabla(w + |y|^2/4))$$
$$\frac{\partial z}{\partial \nu} - z\left(\frac{\partial}{\partial \nu}(w + |y|^2/4) + \beta\frac{\partial w}{\partial \tau}\right)\Big|_{\partial\Omega_s} = 0$$

$$w(\cdot,s) = \int_{\Omega_s} G_s(\cdot,y')z(y',s)dy'$$
$$G_s(y,y') = G(x,x')$$

Theorem 2 $x_0 \in \Omega$ $\Longrightarrow$ $s_k \uparrow +\infty$ subsequence

$$z(y,s+s_k)dy \rightharpoonup \exists\zeta(dy,s) \text{ in } C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$

$$\zeta_s + \beta\nabla\cdot(\zeta\nabla^\perp\Gamma*\zeta) = \nabla\cdot(\nabla\zeta - \zeta\nabla(\Gamma*\zeta + |y|^2/4)) \ \text{ in } \mathbf{R}^2\times(-\infty,+\infty)$$

Proof $\varphi \in C_0^2(\mathbf{R}^2), \ s \gg 1$

$$\frac{d}{ds}\int_{\mathcal{O}_s} z\varphi = \int_{\mathcal{O}_s}(\partial_s\varphi + y\cdot\nabla\varphi + \Delta\varphi)z$$
$$+\frac{1}{2}\int_{\mathcal{O}_s\times\mathcal{O}_s}\rho_\varphi^s(y,y')z\otimes z$$

$$\mathcal{O}_s = \Omega_s \times \{s\}$$
$$\rho_\varphi^s(y,y') = \nabla\varphi(y)\cdot\nabla_y G_s(y,y')$$
$$+\nabla\varphi(y')\cdot\nabla_{y'}G_s(y,y')$$

$$G(x,x') = \Gamma(x-x') + K(x,x')$$
$$(x,x') \in (\overline{\Omega}\times\Omega)\cup(\Omega\times\overline{\Omega})$$
$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$
$$G_s(y,y') = \Gamma(y-y') - \frac{s}{4\pi}$$
$$+K(e^{-s}y + x_0, e^{-s}y' + x_0)$$

$\square$

Exclusion of boundary blowup

$x_0 \in \partial\Omega$

$\varphi = |y|^2\psi_R, \ \psi_R(y) = \psi(y/R)$

$\psi = \varphi_{0,2}(|y|)$

$$0 \le \zeta(dy,s), \ \zeta(\mathbf{R}^2, s) \le \lambda \equiv \|u_0\|_1, \ \mathrm{supp}\ \zeta(dy,s) \subset \overline{\mathbf{R}^2_+}$$

$$\zeta_s + \beta\nabla\cdot(\zeta\nabla^\perp E*\zeta) = \nabla\cdot(\nabla\zeta - \zeta\nabla(E*\zeta + |y|^2/4)) \text{ in } \mathbf{R}^2_+ \times (-\infty, +\infty)$$

$$\left.\frac{\partial\zeta}{\partial\nu} - \zeta\left(\frac{\partial}{\partial\nu}\left(E*\zeta + \frac{|y|^2}{4}\right) + \beta\frac{\partial}{\partial\tau}E*\zeta\right)\right|_{\partial\mathbf{R}^2_+} = 0$$

$$E(y,y') = \Gamma(y-y') - \Gamma(y-y'_*)$$

$$\Delta\varphi = 4\psi_R + 4\frac{y}{R}\cdot\nabla\psi\left(\frac{y}{R}\right) + \frac{|y|^2}{R^2}\Delta\psi\left(\frac{y}{R}\right)$$

$$y\cdot\nabla\varphi = 2|y|^2\psi_R + |y|^2\frac{y}{R}\cdot\nabla\psi\left(\frac{y}{R}\right)$$

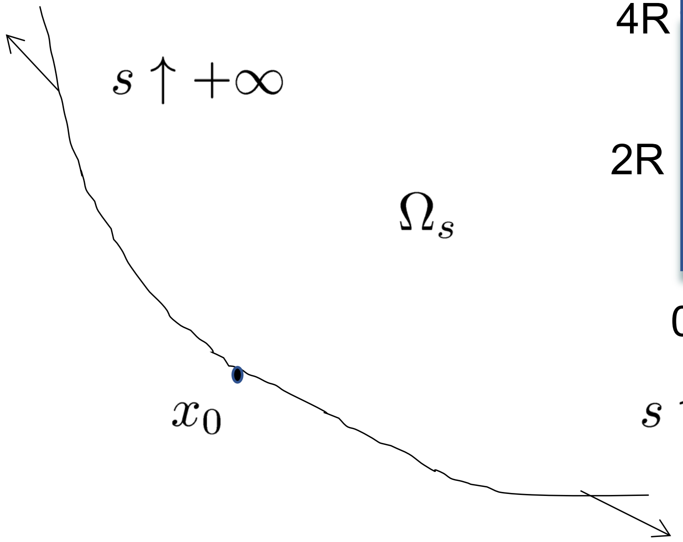
$$\left\langle 1+|y|^2, \zeta(dy,s)\right\rangle \le C$$

$\Rightarrow$ (dominated convergence theorem)

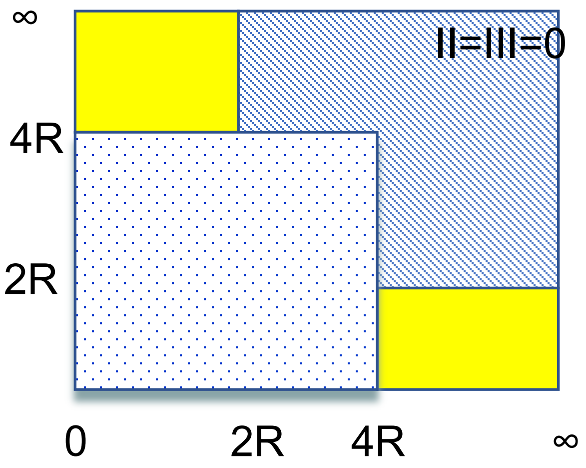
$$\lim_{R\uparrow+\infty}\int_{s_1}^{s_2}\left\langle \Delta\varphi + \frac{y}{2}\cdot\nabla\varphi, \zeta(dy,s)\right\rangle$$

$$= 4(s_2-s_1)m(x_0) + \int_{s_1}^{s_2} I(s)ds$$

$$I(s) = \left\langle |y|^2, \zeta(dy,s)\right\rangle$$



action in long distance



$s \uparrow +\infty$

contradiction

$$\frac{dI}{ds} = 4m(x_0) + I(s) \text{ a.e. } s \Rightarrow \lim_{R\uparrow+\infty} I(s) = +\infty \quad \square$$

Proof of Theorem B (continued) $x_0 \in \mathcal{S}$

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

backward self-similar transformation

$$y = (x - x_0)/(T-t)^{1/2}, \ s = -\log(T-t)$$
$$z(y,s) = (T-t)u(x,t)$$

parabolic envelope

$$m(x_0) = \zeta(\mathbf{R}^2, s) \qquad \langle |y|^2, \zeta(dy,s)\rangle \leq C$$

weak limit $s_k \uparrow +\infty$ subsequence

$$z(y, s+s_k)dy \rightharpoonup \exists \zeta(dy,s) \text{ in } C_*(-\infty,+\infty; \mathcal{M}(\mathbf{R}^2))$$

scaling back

$$\zeta(dy,s) = e^{-s}A(dy',s'), \ y' = e^{-s/2}y, \ s' = -e^{-s}$$

limit equation exclusion of boundary blowup $x_0 \in \Omega$

$$\zeta_s + \beta\nabla\cdot(\zeta\nabla^\perp\Gamma * \zeta) = \nabla\cdot(\nabla\zeta - \zeta\nabla(\Gamma*\zeta + |y|^2/4)) \ \text{ in } \mathbf{R}^2 \times (-\infty,+\infty)$$

$$A_s + \beta\nabla\cdot(A\nabla^\perp\Gamma * A) = \nabla\cdot(\nabla A - A\nabla\Gamma * A) \text{ in } \mathbf{R}^2 \times (-\infty, 0)$$
$$A = A(dy',s') \geq 0, \ A(\mathbf{R}^2, s') = m(x_0)$$

semi-orbit on the whole space

singular part, sub-collapse

$$\zeta^s(dy,s) = \sum_{j=1}^{m(s)} \tilde{m}_j(s)\delta_{y_j(s)}(dy)$$

ϵ – regularity

$$m(s) \leq m(x_0)/\varepsilon_0, \ |y_j(s)| \leq C, \ \tilde{m}_j(s) \geq \varepsilon_0$$

second parabolic envelope

singular part

$$A^s(dy',s') = \sum_{j=1}^{m(s')} \tilde{m}_j(s')\delta_{y'_j(s')}(dy')$$

$$A_s + \beta\nabla\cdot(\nabla^\perp\Gamma * A) = \nabla\cdot(\nabla A - A\nabla\Gamma * A) \text{ in } \mathbf{R}^2\times(-\infty,0)$$

$$A = A(dy,s)\geq 0,\ \ A(\mathbf{R}^2,s) = m(x_0)$$

singular part

$$A^s(dy',s') = \sum_{j=1}^{m(s')} \tilde{m}_j(s')\delta_{y_j'(s')}(dy')$$

$$\nabla^\perp\Gamma(y-y')\cdot\nabla|y|^2 + \nabla^\perp\Gamma(y'-y_*)\cdot\nabla|y'|^2 = 0$$

↓

scaling limit $s_0'<0,\ 1\leq j\leq m(s_0')$ fix

$$\tilde{A}_\beta(dy',s') = \beta^2 A(dy,s)$$

$$y = \beta y' + y_j'(s_0'),\ \ s = \beta^2 s' + s_0'$$

Theorem 3 (weak Liouville property) weak solution (measure valued)

$$a_s + \beta\nabla\cdot(a\nabla^\perp\Gamma * a) = \nabla\cdot(\nabla a - a\nabla\Gamma * a) \text{ in } \mathbf{R}^2\times(-\infty,+\infty)$$

$$\Rightarrow\ \ a(\mathbf{R}^2,s) = 0 \text{ or } 8\pi$$

Kurokiba-Ogawa03

$\beta_k\downarrow 0$ subsequence

$$\tilde{A}_{\beta_k}(dy',s') \rightharpoonup \tilde{A}(dy',s') \in C_*(-\infty,s_0';\mathcal{M}(\mathbf{R}^2))$$

$$\longrightarrow\ \ \tilde{m}_j(s_0') = \tilde{A}(\mathbf{R}^2,0) = 8\pi$$

$$\tilde{A}(dy',s') = m_j'(s_0')\delta_0(dy')$$ weak solution

residual vanishing

if $\boxed{A^{ac}(dy',s') = 0}$ $\longrightarrow$ $A(dy',s') = 8\pi\sum_{j=1}^{\ell}\delta_{y_j'(s')}(dy')$ sub-collapse

absolutely continuous part

translation limit

$$\hat{A}_k(dy',s') = \tilde{A}(dy',s'+s_k),\ \ s_k'\downarrow -\infty$$

$\longrightarrow$ $m(x_0) = 8\pi\ell$ collapse mass quantization

residual vanishing $A^{ac}(dy', s') = 0 \Leftrightarrow \zeta^{ac}(dy, s) = 0$

$$A_{s'} + \beta\nabla \cdot (A\nabla'^{\perp}\Gamma * A) = \nabla' \cdot (\nabla' A - A\nabla'\Gamma * A)$$

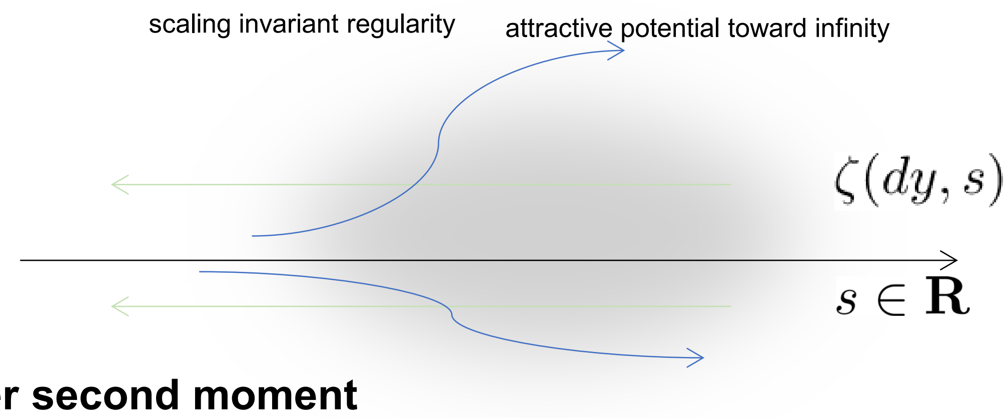
1st envelope

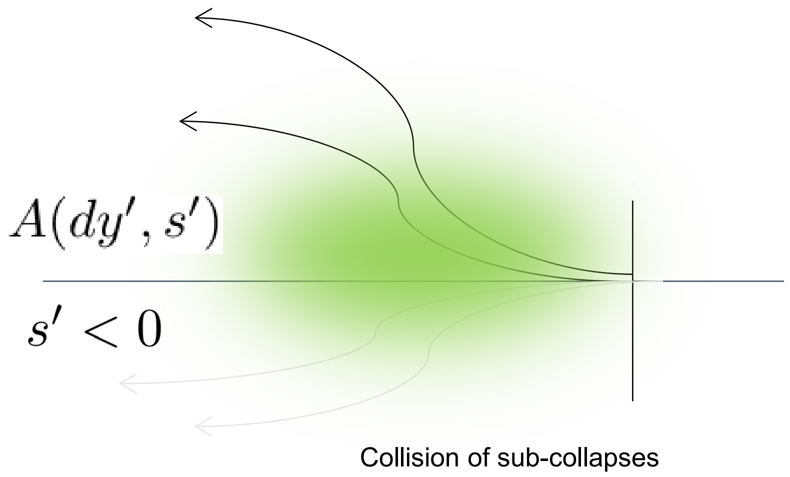
$$m(x_0) = \zeta(\mathbf{R}^2, s)$$

2nd envelope

$$\langle |y|^2, \zeta(dy, s)\rangle \leq C$$

$$\zeta_s + \beta\nabla \cdot (\nabla^{\perp}\Gamma * \zeta) = \nabla \cdot (\nabla\zeta - \zeta\nabla(\Gamma * \zeta + |y|^2/4))$$





outer second moment

$$\frac{d}{ds}\langle\varphi, \zeta\rangle \geq \langle\Delta\varphi - C\varphi_r + \frac{1}{2}r\varphi_r, \zeta\rangle, \ \varphi = \varphi(r)$$

$$\varphi(r) = \xi(r/R), \ \xi(r) = (r^2 - 1)_+$$

$$R \gg 1 \ \Rightarrow \ \Delta\varphi + \frac{1}{2}r\varphi_r \geq C\varphi_r, \ r \geq R$$

$$\frac{d}{ds}\langle(\frac{|y|^2}{R^2} - 1)_+, \zeta(dy, s)\rangle \geq 0 \ \longrightarrow \ \zeta(dy, s) = \zeta^s(dy, s)$$

uniform estimate of self-interaction part ←

improved regularity

$$\zeta(B(y_0, 2r), s) < \varepsilon_0$$

$$\longrightarrow \ \|\zeta(\cdot, s)\|_{L^\infty(B(y_0, r))} \leq Cr^{-2}$$

Improved regularity

Lemma $\exists(\varepsilon_0, R) > 0$

$$u_t + \beta\nabla\cdot(u\nabla^\perp\Gamma * u) = \Delta u - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^2\times(0,T)$$

$$u|_{t=0} = u_0(x) \geq 0$$

$$\exists\varepsilon_0,\ R,\ t_0 > 0,\ \forall x_0 \quad \|u_0\|_{L^1(B(x_0,8R))} < \varepsilon_0/2$$

$$\longrightarrow \quad \forall\tau\in(0,t_0)$$

$$\sup_{\tau\leq t<t_0}\|u(\cdot,t)\|_{L^\infty(B(x_0,R))} < +\infty$$

Proof $\exists t_1\in(0,T),\ \|u_0\|_{L^1(B(x_0,8R))} < \varepsilon_0/2$

$$\longrightarrow \sup_{t\in(0,t_1)}\|u(\cdot,t)\|_{L^1(B(x_0,4R)} < \varepsilon_0$$

$$\longrightarrow \frac{d}{dt}\int_{\mathbf{R}^2} u(\log u - 1)\varphi\ dx + \frac{1}{8}\int_{\mathbf{R}^2} u^{-1}|\nabla u|^2\varphi\ dx$$
$$\leq C_\varphi,\ 0\leq t<t_1,\ \varphi = \varphi_{x_0,R}$$

$\|u(\cdot,t)\|_{L\log L(B(x_0,2R))}$ Involved by the initial value!

Gagliardo-Nirenberg

$$\longrightarrow \frac{dJ}{dt} + 3J^{3/2} \leq C_R,\ J = \int_\Omega u(\log u - 1) + 1\ dx$$

$$\frac{d}{dt}t^{-2} + 3(t^{-2})^{3/2} = t^{-3}$$

$$J(t)\leq t^{-2},\ 0<t\leq\min\{t_1,t_0\},\ t_0^{-3} = C_R \quad \square$$

parabolic regularity

scaling

Theorem 4 $\exists\varepsilon_0,\ \sigma_0,\ C$

$$u_t + \beta\nabla\cdot(u\nabla^\perp\Gamma * u) = \Delta u - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^2\times(-T,T)$$

weak solution generated by classical solutions

$$\|u_0\|_{L^1(B(x_0,2R))} < \varepsilon_0,\ u_0 = u|_{t=0} \ \Rightarrow$$

$$\sup_{t\in[-\sigma_0R^2,\sigma_0R^2]\cap(-T,T)}\|u(\cdot,t)\|_{L^\infty(B(x_0,R))} \leq CR^{-2}$$

scaling invariant regularity (reverse scaling back)

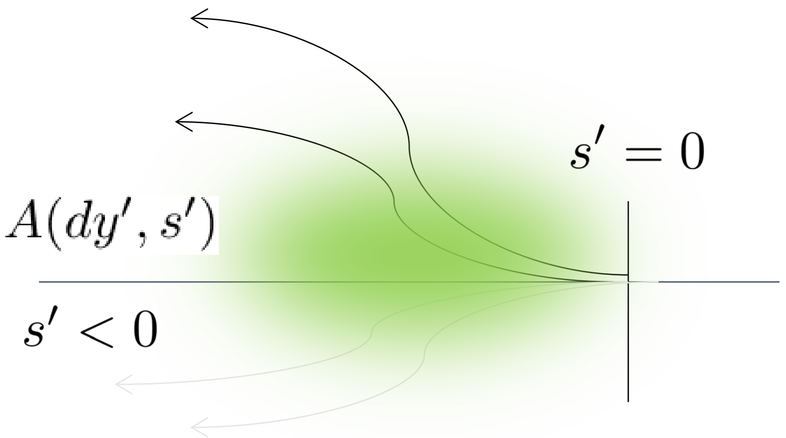
$$\zeta(B(y_0,2r),s) < \varepsilon_0 \ \Rightarrow\ \|\zeta(\cdot,s)\|_{L^\infty(B(y_0,r))} \leq Cr^{-2}$$

Sub-collapse dynamics

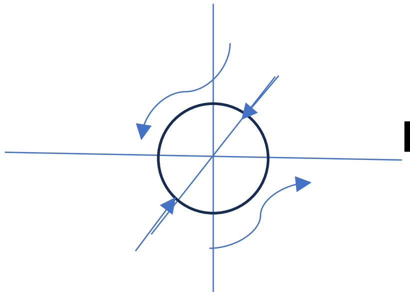
$$\zeta(dy, s) = \sum_{j=1}^{\ell} 8\pi\delta_{y_j(s)}(dy) \quad \Longrightarrow \quad \textbf{scaling back}$$

$$A(dy', s') = \sum_{j=1}^{\ell} 8\pi\delta_{y'_j(s')}(dy')$$

$$A_{s'} = \nabla' \cdot (\nabla' A - A\nabla'\Gamma * A) \quad \text{in } \mathbf{R}^2 \times (-\infty, 0)$$



Collision of sub-collapses



Tracing sub-collapses by the local second moment

simple blowup point

$$\ell = 1 \ \Rightarrow \ \zeta(dy, s) = 8\pi\delta_0(dy)$$

in dynamics

recursive hierarchy $\ell \geq 2$

$$\frac{dy'_j}{ds'} + 8\pi\beta\nabla_j^{\perp} H_\ell^0(y'_1, \cdots, y'_\ell) = 8\pi\nabla_j H_\ell^0(y'_1, \cdots, y'_\ell)$$

$$H_\ell^0(y'_1, \cdots, y'_\ell) = \sum_{1 \leq j < k \leq \ell} \Gamma(y'_j - y'_k)$$

$$\Gamma(y') = \frac{1}{2\pi} \log \frac{1}{|y'|}$$

Remark $\ell = 2 \ \Longrightarrow$

$$y'_\pm(s') = -2(-s')^{1/2} e^{\pm\imath(\frac{\beta}{2}\log(-s') + c)}$$

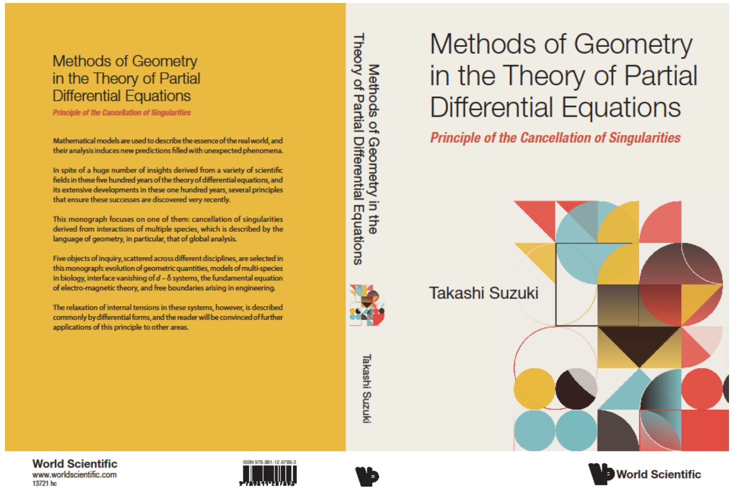
c.f. blowup rate   Herero-Velazquez 96, Mizoguchi 20, Collot-Ghoul-Masmoudi-Nguyen 22

Summary

1. Thermal equilibrium of the point vortex mean field is described by the Boltzmann Poisson equation.
2. Onsager's conjecture of the formation of an ordered structure in negative inverse temperature is realized as the Hamiltonian recurrence with quantized blowup mechanism.
3. Smoluchowski Poisson equation is the fundamental equation for canonical ensembles of Newtonian particles. With the vorticity term it is the kinetic model from quasi-stationary state to stationary state of many point vortices.
4. The stationary state is the Boltzmann Poisson equation of which total set of solutions controls the global-in-time dynamics (potentials of self-organization).
5. As a consequence there is a quantization with Hamiltonian control in blowup in finite and infinite time.
6. If blowup in finite time occurs there is a formation of sub-collapses of normalized masses with a possible collision.
7. The residual part other than sub-collapses vanishes in the whole space of rescaled variables, called the parabolic envelope, while the motion of sub-collapses is controlled by the point vortex Hamiltonian in the rescaled variables.
8. As a consequence any blowup point is of type II, and if the free energy is bounded then it is simple, whereby the local free energy in the parabolic envelope diverges to plus infinity (emergence).
9. Blowup in infinite time, on the other hand, occurs only when the initial mass is quantized, whereby there is a formation of collapses with a normalized mass of which kinetics is subject to the anti-gradient system of Hamiltonian to create a clinic orbit of its critical points.
10. A relative of the Smoluchowski-Poisson equation is the simplified system of chemotaxis, where the Poisson part is modified.
11. The total set of stationary solutions, however, is quite different according to the form of the Poisson part.
12. There is a dis-quantized blowup mechanism if the model is provided with the relaxation time, which is nothing but the model B – model A equation derived from the Lagrangian associated with the Toland duality.
13. Several models in non-equilibrium thermo-dynamics are provided with the structure of semi-unfolding-minimality between the Lagrangian and the field functional, which induces a general criterion of the dynamical stability of the critical point if it is analytic.
14. Higher dimensional analogous of the 2D Smoluchowski Poisson equation is a degenerate parabolic equation associated with the Tsallis entropy, where the finiteness of type II blowup points is known.
15. Its stationary state is realized as an elliptic free boundary problem provided with the quantized blowup mechanism controlled by the Hamiltonian.
16. Euler-Smoluchowski-Poisson equation is treated similarly except for the sub-collapse dynamics.

Open questions

1. Real rate of blowup (for the case of sub-collapse collision)
2. Any blowup point is of type II in the higher dimensional degenerate parabolic equation.
3. Hausdorff dimension of the blowup set of the higher dimensional Smoluchowski Poisson equation I less then or equal to (n-2). A partial answer is known.
4. Hamiltonian control of the blowup set is efficient to many elliptic problems.
5. There is a general view of dynamics in the models associated with the Kuhn Tucker duality.



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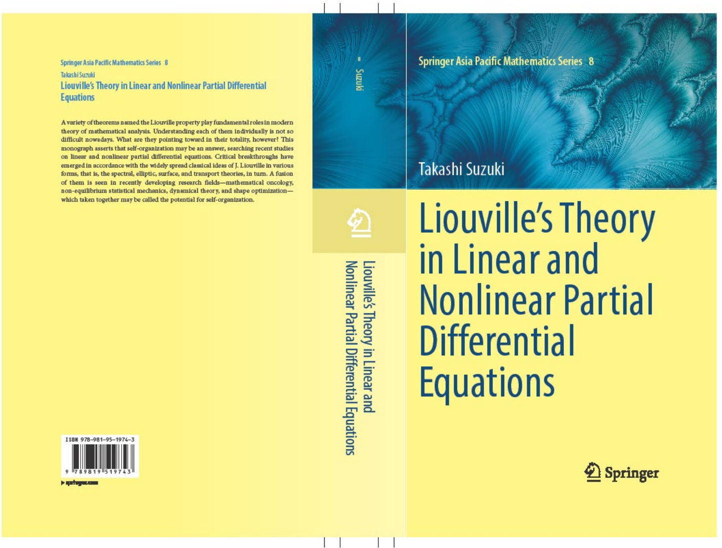

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Senba-S. 01	weak formulation monotonicity formula	formation of collapse weak solution generation
Senba-S. 02a	weak solution	instant blowup for over mass concentrated initial data
Kurokiba-Ogawa 03	scaling invariance	non-existence of over mass entire solution without concentration
Senba-S. 04	backward self-similar transformation scaling limit	
S. 05	parabolic envelope (1) scaling invariance of the scaling limit a local second moment	sub-collapse quantization collapse mass quantization
Senba-Ohtsuka-S. 07	defect moment (1)	radially symmetric dynamics
Senba 07, Naito-S. 08	parabolic envelope (2)	type II blowup rate
S. 08	scaling back	limit equation simplification
Senba-S. 11 (2^{nd} ed.)	translation limit	concentration-cancelation simplification
S. 13a	limit equation classification	boundary blowup exclusion
S. 13b	improved regularity concentration compactness	cloud formation collision of sub-collapses
S. 14	tightness	residual vanishing
S. 18	defect moment (2)	quantization of BUIT
S. 22	outer second moment	residual vanishing in finite time