Smoluchowski-Poisson Equation 3

Hamiltonian Control in Three Phases of Time Evolution

1. The model – statistical mechanics

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega} = 0, \;\; u|_{t=0} = u_0(x) > 0$$

2. Poisson Part

$$-\Delta v = u, \;\; v|_{\partial\Omega} = 0$$

Green's function

$$G(x, x') = G(x', x)$$

$u = u(x,t) \geq 0$	density
$j = -\nabla u + u\nabla v$	flux (diffusion v.s. chemotaxis)
$u_t + \nabla \cdot j = 0$	conservation law
$v = (-\Delta)^{-1} u$	potential

attractive (chemotaxis, gravitation)
action at a distance (long range potential)
symmetry (action-reaction)

Chavanis 08 relaxation to the equilibrium in the point vortices, kinetic equation + maximum entropy production

Sire-Chavanis 02 motion of the mean field of many self-gravitating Brownian particles, BBGKY hierarchy + factorization

canonical ensemble

1. total mass conservation $\frac{d}{dt}\|u(t)\|_1 = 0$

2. free energy decreasing

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u \otimes u$$

$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 \leq 0$$

self-similar transformation

$$u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t), \; \mu > 0$$

$$u_\mu(x) = \mu^2 u(\mu x), \;\; \mu > 0$$

$$\|u\|_1 = \|u_\mu\|_1 \equiv \lambda \;\Leftrightarrow\; n = 2$$ *critical dimension*

$$\mathcal{F}(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2}\langle \Gamma * u, u\rangle, \;\; \Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$\mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi}\right)\log\mu + \mathcal{F}(u)$$ *critical mass* $\lambda = 8\pi$

Theorem A (blowup in infinite time)

$$T=+\infty,\ \limsup_{t\uparrow+\infty}\|u(\cdot,t)\|_\infty=+\infty$$

$\longrightarrow$ $\lambda\equiv\|u_0\|_1=8\pi\ell,\ \exists\ell\in\mathbf{N}$ initial mass quantization

$\exists x_*\in\Omega^\ell\setminus D,\ \nabla H_\ell(x_*)=0$ recursive hierarchy

point vortex Hamiltonian, Robin function, Green function

$$H_\ell(x_1,\cdots,x_\ell)=\frac{1}{2}\sum_j R(x_j)+\sum_{i<j}G(x_i,x_j)$$

Theorem B (blowup in finite time) $T<+\infty$

$$u(x,t)dx\rightharpoonup\sum_{x_0\in\mathcal{S}}m(x_0)\delta_{x_0}(dx)+f(x)dx \text{ in } \mathcal{M}=(\overline{\Omega})C(\overline{\Omega})'$$

$m(x_0)\in 8\pi\mathbf{N}$ collapse mass quantization possibly with sub-collapse collision

$$\mathcal{S}=\{x_0\in\overline{\Omega}\mid\exists x_k\to x_0,\ t_k\uparrow T,\ u(x_k,t_k)\to+\infty\}\subset\Omega$$

blowup set; exclusion of boundary blowup

$0<f=f(x)\in L^1(\Omega)\cap C(\overline{\Omega}\setminus\mathcal{S})$ measure theoretic regular part

Corollary 1 $T<+\infty$ if

(1) $\lambda\notin 8\pi\mathbf{N},\ \nexists$ stationary solution or $\mathcal{F}(u_0)\ll -1$

(2) $\lambda\in 8\pi\ell,\ \ell\in\mathbf{N},\ \nexists$ critical point of H_ℓ

Corollary 2 Ω convex $\lambda\neq 8\pi$

$\Rightarrow\ T<+\infty$ or $T=+\infty$ compact orbit

$\exists$ stationary solution

c.f. Grossi-F. Takahashi

Poisson $v=\int_\Omega G(\cdot,x')u(x')dx'\ \Leftrightarrow\ -\Delta v=u,\ v|_{\partial\Omega}=0$

diagonal $D=\{(x_i)\in\Omega^\ell\mid\exists i\neq j,\ x_i=x_j\}$

Robin function $R(x)=\left[G(x,x')+\frac{1}{2\pi}\log|x-x'|\right]_{x'=x}$

S. Chemotaxis, Reaction, Network, World Scientific, Singapore, 2018

2. Blowup in Finite Time (Proof of Theorem B)

- Rescaled Hamiltonian induces residual vanishing

symmetry of the Green function → weak form (symmetrization)

$$\varphi \in C^2(\overline{\Omega}), \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\frac{d}{dt}\int_\Omega \varphi u(\cdot,t) = \int_\Omega \Delta\varphi \cdot u(\cdot,t) + \frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\varphi(x,x')u\otimes u$$

$$\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x')$$

$$\|\rho_\varphi\|_\infty \le C\|\nabla\varphi\|_{C^1}$$

boundary behavior of the Green function
singularity cancellation by the symmetry

monotonicity formula $\lambda = \|u(\cdot,t)\|_1$

$$\left|\frac{d}{dt}\int_\Omega u\varphi\right| \le C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1}$$

→ **weak continuation**

$$0 \le \exists\mu(dx,t) \in C_*([0,T],\mathcal{M}(\overline{\Omega}))$$

$$u(x,t)dx = \mu(dx,t), \; 0 \le t < T$$

epsilon regularity via Gagliard-Nirenberg inequality

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists\varepsilon_0 \Rightarrow x_0 \notin \mathcal{S}$$

→ **formation of collapse**

$$\mu(\cdot,T) = \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0} + f(x)$$

$$m(x_0) \ge \varepsilon_0, \; 0 \le f \in L^1(\Omega), \; \sharp\mathcal{S} < +\infty$$

some technicalities

1. nice cut-off function

$$0 \leq \varphi \leq 1, \ \varphi = \begin{cases} 1, & x \in B(x_0, \frac{R}{2}) \\ 0, & x \in \mathbf{R}^2 \setminus B(x_0, R) \end{cases}$$

$$x_0 \in \overline{\Omega}, \ 0 < R \ll 1, \varphi = \varphi_{x_0,R} \in \mathcal{Y}$$

$$|\nabla \varphi| \leq CR^{-1}\varphi^{\frac{5}{6}}$$

$$|\nabla^2 \varphi| \leq CR^{-2}\varphi^{\frac{2}{3}} \qquad \mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0\}$$

2. Green's function

$$x' \in \Omega, \ -\Delta G(\cdot, x') = \delta_{x'}, \ \ G(\cdot, x')|_{\partial\Omega} = 0$$

fundamental solution $\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$

$$G = G(x, x') \in C^{2+\theta}(\overline{\Omega} \times \overline{\Omega} \setminus D)$$

$$D = \overline{\{(x, x) \mid x \in \Omega\}}, \ 0 < \theta < 1$$

2.1. interior regularity

$$G(x, x') = \Gamma(x - x') + K(x, x')$$

$$K \in C^{2+\theta,1}(\overline{\Omega} \times \Omega) \cap C^{1,2+\theta}(\Omega \times \overline{\Omega})$$

$$\varphi \in C^2(\overline{\Omega}), \ \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0 \Rightarrow \rho_\varphi \in L^\infty(\Omega \times \Omega)$$

discontinuity at the diagonal

2.2. boundary regularity

$$x_0 \in \partial\Omega, \ X : \overline{\Omega \cap B(x_0, 2R)} \to \overline{\mathbf{R}^2_+}$$ conformal diffeo.

$$X(x_0) = 0, \ \mathbf{R}^2_+ = \{(X_1, X_2) \in \mathbf{R}^2 \mid X_2 > 0\}$$

$$G(x, x') = E(x, x') + K(x, x')$$

$$K \in C^{2+\theta,1} \cap C^{1,2+\theta}(\overline{\Omega \cap B(x_0, R)} \times \overline{\Omega \cap B(x_0, R)})$$

$$E(x, x') = \Gamma(X - X') - \Gamma(X - X'_*)$$

$$X_* = (X_1, -X_2), \ X = (X_1, X_2)$$

weak scaling limit ➡ exclusion of boundary blowup

$$\rho_\varphi(x, x') = \nabla\varphi(x) \cdot \nabla_x G(x, x') + \nabla\varphi(x') \cdot \nabla_{x'} G(x, x')$$

related notions

$0 \le \mu = \mu(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$ weak solution

$\longleftrightarrow \quad 0 \le \exists \mathcal{N} = \mathcal{N}(\cdot,t) \in L^\infty_*([0,T], \mathcal{X}')$

1. $t \in [0,T] \mapsto \langle \varphi, \mu(dx,t)\rangle, \ \varphi \in \mathcal{Y}$ a.c.
2. $\frac{d}{dt}\langle \varphi, \mu\rangle = \langle \Delta\varphi, \mu\rangle + \frac{1}{2}\langle \rho_\varphi, \mathcal{N}(\cdot,t)\rangle$ a.e. $t \in [0,T]$
3. $\mathcal{N}|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu \otimes \mu$

$$\mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0\} \qquad \mathcal{X} = [\mathcal{X}_0]^{L^\infty(\Omega\times\Omega)}$$

$$\mathcal{X}_0 = \{\rho_\varphi + \psi \mid \varphi \in \mathcal{Y}, \ \psi \in C(\overline{\Omega}\times\overline{\Omega})\}$$

$\Longrightarrow$

$$\mu(\overline{\Omega},t) = \mu(\overline{\Omega},0) \equiv \lambda, \ 0 \le t \le T$$

$$\left|\frac{d}{dt}\langle \varphi, \mu(dx,t)\rangle\right| \le C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1}$$

$u = u(x,t)$ classical solution

$\Longrightarrow$

$$\mathcal{N}(\cdot,t) = u(x,t) \otimes u(x',t) \ dxdx'$$

$$\|\mathcal{N}(\cdot,t)\|_{\mathcal{X}'} = \lambda^2, \ \lambda = \|u_0\|_1$$

Theorem $\mu_k(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$

$\mathcal{N}_k \in L^\infty_*([0,T], \mathcal{X}')$ weak solutions

$$0 \le \mu_k(\overline{\Omega},t) \le C$$
$$\|\mathcal{N}_k(\cdot,t)\|_{\mathcal{X}'} \le C$$

$\Longrightarrow$ sub-sequence

$$\mu_k(dx,t) \rightharpoonup \mu(dx,t) \quad \text{in } C_*([0,T], \mathcal{M}(\overline{\Omega}))$$
$$\mathcal{N}_k(\cdot,t) \rightharpoonup \mathcal{N}(\cdot,t) \quad \text{in } L^\infty_*([0,T], \mathcal{X}')$$ weal solution

Theorem $\exists \varepsilon_0, \ \sigma_0, \ C$

via improved epsilon regularity

$$u_t = \Delta u - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^2 \times (-T,T)$$
$$u_0 = u|_{t=0}$$

$$\|u_0\|_{L^1(B(x_0,2R))} < \varepsilon_0, \ u_0 = u|_{t=0} \ \Rightarrow$$

$$\sup_{t\in[-\sigma_0R^2, \sigma_0R^2]\cap(-T,T)} \|u(\cdot,t)\|_{L^\infty(B(x_0,R))} \le CR^{-2}$$

Proof of Theorem B (continued) $x_0 \in \mathcal{S}$

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

backward self-similar transformation

$$y = (x-x_0)/(T-t)^{1/2}, \ s = -\log(T-t)$$
$$z(y,s) = (T-t)u(x,t)$$

weak limit $s_k \uparrow +\infty$ subsequence

$$z(y, s+s_k)dy \rightharpoonup \exists \zeta(dy,s) \text{ in } C_*(-\infty,+\infty; \mathcal{M}(\mathbf{R}^2))$$

limit equation exclusion of boundary blowup $x_0 \in \Omega$

$$\zeta_s = \nabla \cdot (\nabla \zeta - \zeta \nabla(\Gamma * \zeta + |y|^2/4)) \ \text{ in } \mathbf{R}^2 \times (-\infty, +\infty)$$

scaling back

$$\zeta(dy,s) = e^{-s}A(dy', s'), \ y' = e^{-s/2}y, \ s' = -e^{-s}$$

$$A_s = \nabla \cdot (\nabla A - A\nabla\Gamma * A) \text{ in } \mathbf{R}^2 \times (-\infty, 0)$$
$$A = A(dy,s) \geq 0, \ A(\mathbf{R}^2, s) = m(x_0)$$

parabolic envelope

$$m(x_0) = \zeta(\mathbf{R}^2, s) \qquad \langle |y|^2, \zeta(dy,s)\rangle \leq C$$

weak Liouville property

$$a_s = \nabla \cdot (\nabla a - a\nabla\Gamma * a) \text{ in } \mathbf{R}^2 \times (-\infty, +\infty)$$
$$\Rightarrow \ a(\mathbf{R}^2, s) = 0 \text{ or } 8\pi$$

translation limit $\zeta^s(dy,s) = \sum_{j=1}^{\ell} 8\pi\delta_{y_j(s)}(dy)$

scaling invariant regularity (scaling back)

$$\zeta(B(y_0, 2r), s) < \varepsilon_0 \ \Rightarrow \ \|\zeta(\cdot, s)\|_{L^\infty(B(y_0,r))} \leq Cr^{-2}$$

$\longrightarrow$ $|y_j(s)| \leq C$

residual vanishing

1st envelope

$$m(x_0) = \zeta(\mathbf{R}^2, s)$$

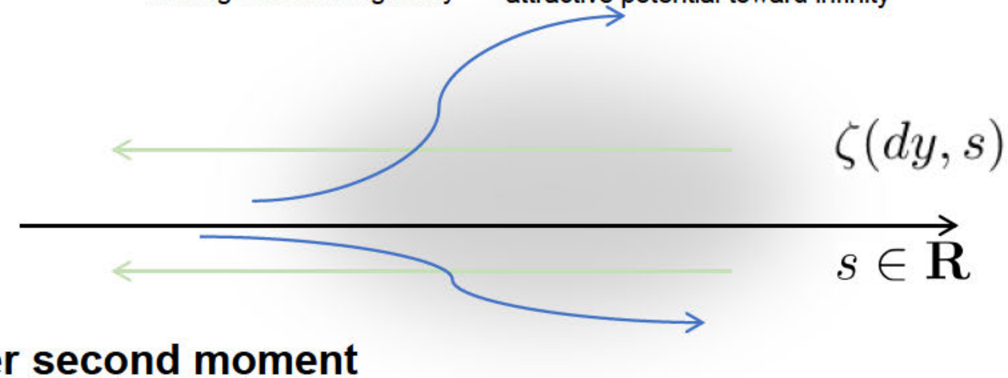
2nd envelope

$$\langle |y|^2, \zeta(dy, s) \rangle \leq C$$

$$\zeta_s = \nabla \cdot (\nabla \zeta - \zeta \nabla (\Gamma * \zeta + |y|^2/4))$$

scaling invariant regularity

attractive potential toward infinity



outer second moment

$$\frac{d}{ds}\langle \varphi, \zeta \rangle \geq \langle \Delta\varphi - C\varphi_r + \frac{1}{2} r\varphi_r, \zeta \rangle, \ \varphi = \varphi(r)$$

$$\varphi(r) = \xi(r/R), \ \xi(r) = r^2 - 1$$

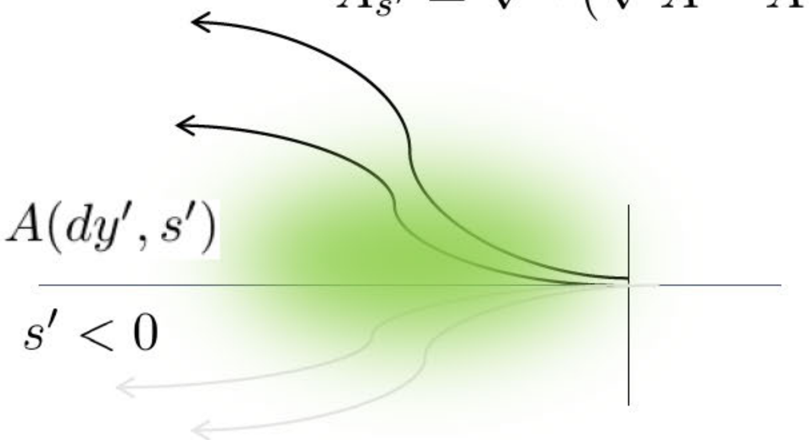
$$R \gg 1 \ \Rightarrow \ \Delta\varphi + \frac{1}{2} r\varphi_r \geq C\varphi_r, \ r \geq R$$

$$\frac{d}{ds}\langle (\frac{|y|^2}{R^2} - 1)_+, \zeta(dy, s) \rangle \geq 0 \quad \longrightarrow \quad \zeta(dy, s) = \zeta^s(dy, s)$$

collapse mass quantization

$$\zeta(\mathbf{R}^2, s) = m(x_0) \in 8\pi\mathbf{N}$$

$$A_{s'} = \nabla' \cdot (\nabla' A - A \nabla' \Gamma * A)$$



Collision of sub-collapses

simple blowup point

$$\ell = 1 \ \Rightarrow \ \zeta(dy, s) = 8\pi\delta_0(dy)$$

recursive hierarchy $\ell \geq 2$

$$\frac{dy'_j}{ds'} = 8\pi \nabla_j H^0_\ell(y'_1, \cdots, y'_\ell)$$

$$A(dy', s') = \sum_{j=1}^{\ell} 8\pi \delta_{y'_j(s')}(dy')$$

$$H^0_\ell(y'_1, \cdots, y'_\ell) = \sum_{1 \leq j < k \leq \ell} \Gamma(y'_j - y'_k), \ \Gamma(y') = \frac{1}{2\pi} \log \frac{1}{|y'|}$$

3. Blowup in infinite time (Proof of Theorem A)

assume $T=+\infty,\ t_k\uparrow+\infty,\ \lim_{k\to\infty}\|u(\cdot,t_k)\|_\infty=+\infty$

subsequence $u(\cdot,t+t_k)dx \rightharpoonup \mu(dx,t)\in C_*(-\infty,+\infty;\mathcal{M}(\overline{\Omega}))$ weak solution

$$\mu(dx,t)=\sum_{x_0\in\mathcal{S}_t} m(x_0)\delta_{x_0}(dx)+f(x,t)dx$$

improved regularity
formation of collapse in infinite time

blowup set

exclusion of boundary blowup

$$m(x_0)\geq\varepsilon_0,\ 0\leq f=f(\cdot,t)\in L^1(\Omega)$$

$$\mathcal{S}_t=\{x_0\in\overline{\Omega}\mid \exists x_k\to x_0,\ \lim_k u(x_k,t+t_k)=+\infty\}\subset\Omega$$

dilation $x_0=0\in\mathcal{S}_0,\ \beta>0$

$$\mu_\beta(dx',t')=\beta^2\mu(dx,t),\ x'=\beta x,\ t'=\beta^2 t$$

$\beta_k\downarrow 0$ subsequence

$$\mu_{\beta_k}(dx,t)\rightharpoonup\tilde{\mu}(dx,t)\in C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$ scaling limit

$$m(x_0)=\tilde{\mu}(\mathbf{R}^2,0)=8\pi\geq\varepsilon_0$$ full orbit of weak solutions on the whole space

Liouville property

collapse mass quantization

local second moment traces the collapse dynamics

$$\sharp\,\mathcal{S}_t\equiv\ell,\ \mu^s(dx,t)=\sum_{i=1}^{\ell}8\pi\delta_{x_i(t)}(dx)$$

$$\frac{dx_i}{dt}=8\pi\nabla_{x_i}H_\ell(x_1,\cdots,x_\ell),\ 1\leq i\leq\ell$$

recursive hierarchy

anti-gradient system

a blowup criterion excludes the collapse collision in infinite time

$$x(t)=(x_i(t))\in\Omega^\ell\setminus D$$ pre-compact

$$D=\{(x_i)\mid\exists i\neq j,\ x_i=x_j\}$$

→

$$\exists x^*\in\Omega^\ell\setminus D,\ \nabla_{x_i}H_\ell(x^*)=0,\ 1\leq i\leq\ell$$

residual vanishing $x_i = x_i(t),\ u_k(x,t) = u(x,t+t_k),\ v_k(x,t) = v(x,t+t_k),\ 0 < r \ll 1$

$$\frac{d}{dt}\int_{B(x_i,r)} |x-x_i|^2 u_k = \int_{B(x_i,r)} \frac{\partial}{\partial t}(|x-x_i|^2 u_k) + \dot{x}_i \cdot \nabla(|x-x_i|^2 u_k)\ dx$$

Liouville's formula

$$= \int_{B(x_i,r)} |x-x_i|^2 u_{kt} + \dot{x}_i \cdot |x-x_i|^2 \nabla u_k\ dx$$

$$\int_{B(x_i,r)} |x-x_i|^2 u_{kt}$$

$$= \int_{B(x_i,r)} |x-x_i|^2 \nabla \cdot (\nabla u_k - u_k \nabla v_k)\ dx$$

$$\le r^2 \int_{\partial B(x_i,r)} \frac{\partial u_k}{\partial \nu} - u_k \frac{\partial v_k}{\partial \nu}\ dS$$

$$+ \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k \nabla v_k\ dx$$

$$= \int_{B(x_i,r)} \cancel{r^2 u_{kt}} + 4u_k + 2(x-x_i)\cdot u_k \nabla v_k\ dx$$

$$\int_{B(x_i,r)} \dot{x}_i \cdot |x-x_i|^2 \nabla u_k$$

$$= \int_{\partial B(x_i,r)} (\dot{x}_i \cdot \nu)|x-x_i|^2 u_k\ dS$$

$$- \int_{B(x_i,r)} 2(x-x_i)\cdot \dot{x}_i u_k$$

$$= \int_{B(x_i,r)} \cancel{r^2 \dot{x}_i \cdot \nabla u_k}$$

$$-2(x-x_i)\cdot \dot{x}_i u_k\ dx$$

$$\frac{d}{dt}\int_{B(x_i,r)} u_k = \int_{B(x_i,r)} \cancel{u_{kt}}$$

$$+\dot{x}_i \cdot \cancel{\nabla u_k}\ dx$$

$$\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2) u_k$$

$$\le \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k \nabla v_k - 2(x-x_i)\cdot \dot{x}_i u_k\ dx$$

$$x_i = x_i(t)$$
$$u_k(x,t) = u(x, t+t_k)$$
$$v_k(x,t) = v(x, t+t_k)$$
$$0 < r \ll 1$$

$\longrightarrow$

$$\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2)u_k$$
$$\le \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k \nabla v_k$$
$$-2(x-x_i)\cdot \dot{x}_i u_k \ dx$$

$$v_k(x,t) = \sum_{i=0}^{3} v_k^i(x,t)$$
$$v_k^0(x,t) = \int_{B(x_i,r)} \Gamma(x-x')u_k(x',t)dx'$$
$$v_k^1(x,t) = \int_{B(x_i,r)} K(x,x')u_k(x',t) \ dx'$$
$$v_k^2(x,t) = \int_{\Omega\setminus \mathcal{S}_t^{2r}} G(x,x')u_k(x',t)dx'$$
$$v_k^3(x,t) = \int_{\mathcal{S}_t^{2r}\setminus B(x_i,r)} G(x,x')u_k(x',t)dx'$$

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$
$$G(x,x') = \Gamma(x-x') + K(x,x')$$

$$2\int_{B(x_i,r)} (x-x_i)\cdot u_k \nabla v_k^0 \ dx$$
$$= -\frac{1}{2\pi}\left(\int_{B(x_i,r)} u_k \ dx\right)^2$$

$$\|u_k(\cdot,t)\|_1 = \lambda, \ K(x,x') \in C^1(\Omega\times\Omega)$$
$$\sup_x \int_\Omega |\nabla_x G(x,x')| \ dx' \le C$$

$\longrightarrow$

$$\|\nabla v^i(\cdot,t)\|_{L^\infty(B(x_i,r))} \le C, \ 1 \le i \le 3$$

$\mathcal{O} = \{x(t)\}$ compact $\longrightarrow$ $|\dot{x}_i| \le C$

away from diagonal and boundary

$$\frac{d}{dt}\int_{B(x_i,r)}(|x-x_i|^2-r^2)u_k$$

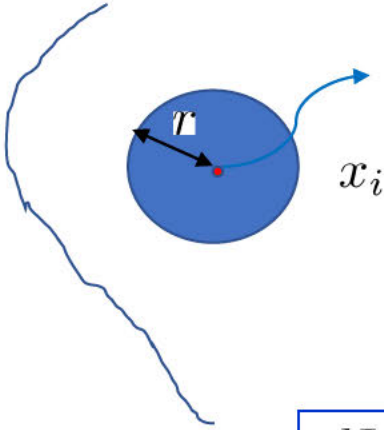
$$\leq 4\int_{B(x_i,r)}u_k-\frac{1}{2\pi}\left(\int_{B(x_i,r)}u_k\right)^2$$

$$+C\int_{B(x_i,r)}|x-x_i|u_k$$

$k\to\infty$

as distributions in time

defect measure



$x_i=x_i(t)$

$$\frac{d}{dt}\int_{B(x_i,r)}(|x-x_i|^2-r^2)f$$

$$\leq 4\left(8\pi+\int_{B(x_i,r)}f\right)$$

$$-\frac{1}{2\pi}\left(8\pi+\int_{B(x_i,r)}f\right)^2$$

$$+C\int_{B(x_i,r)}|x-x_i|f$$

$$r^2\int_{B(x_i,r)}u_k\to 8\pi r^2+\int_{B(x_i,r)}f$$

$0<r\ll 1$

$$\frac{dI}{dt}\leq\int_{B(x_i,r)}-4f+C|x-x_i|f\ dx\leq\frac{2I}{r^2}$$

$$I(t)\equiv\int_{B(x_i,r)}(|x-x_i|^2-r^2)f\leq 0$$

$I(t)\equiv 0$

$f=0$ in $B(x_i,r)$

$f\equiv 0$

bounded free energy →

1. stationary collapse formation in infinite time
2. simple collapse formation in finite time

$\lambda=8\pi$

higher-dimensional analogue – a challenge

1. plasma confinement
2. mean filed limit of self-interacting particles associated with Tsallis entropy
3. incompressible Euler flow with self-gravitation