Smoluchwoski Poisson Equation 2

Potentials of Self-Organization

1. multi-scale model

Keller-Segel 70

$$u_t = \nabla \cdot (d_1(u,v)\nabla u) - \nabla \cdot (d_2(u,v)\nabla v)$$
$$v_t = d_v \Delta v - k_1 vw + k_{-1}p + f(v)u$$
$$w_t = d_w \Delta w - k_1 vw + (k_{-1} + k_2)p + g(v,w)u$$
$$p_t = d_p \Delta p + k_1 vw - (k_{-1} + k_2)p$$

$u = u(x,t)$ cellular slime molds

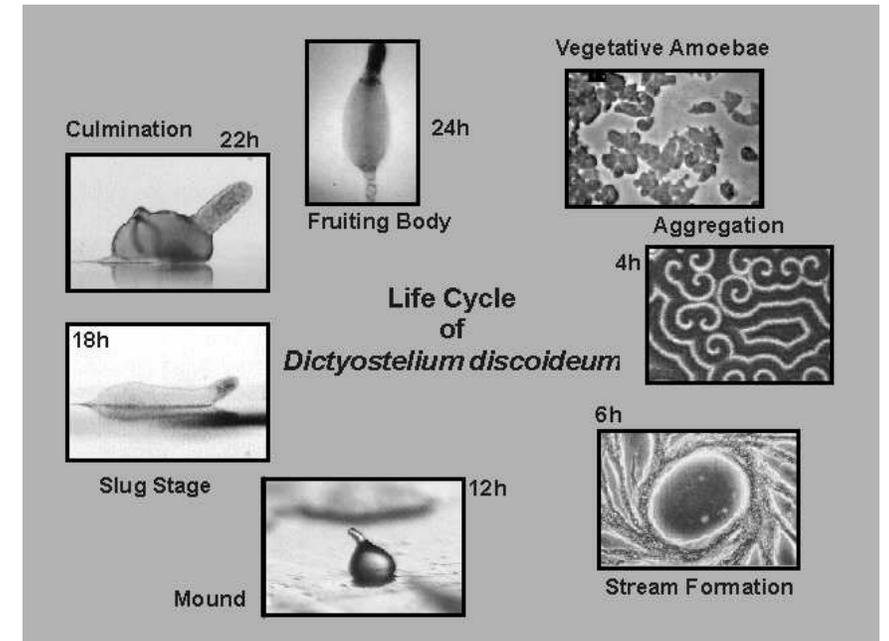$v = v(x,t)$ chemical substances

$w = w(x,t)$ enzymes

$p = p(x,t)$ complices

1. transport, gradient   2. production $u \to (v,w)$

   (a) diffusion u,v,w,p
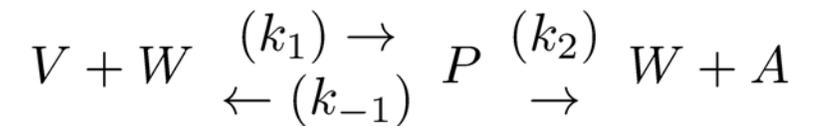
   (b) chemotaxis $v \to u$

Life Cycle of *Dictyostelium discoideum*

Vegetative Amoebae — Aggregation — Stream Formation — Mound — Slug Stage — Fruiting Body — Culmination

moving clustered cells   aggregating cells

3. chemical reaction v, w, p

$$V + W \;\underset{\leftarrow (k_{-1})}{\overset{(k_1)\to}{}}\; P \;\overset{(k_2)}{\to}\; W + A$$

$$v_t = -k_1 vw + k_{-1}p$$
$$w_t = -k_1 vw + (k_{-1} + k_2)p$$
$$p_t = k_1 vw - (k_{-1} + k_2)p$$

Michaelis-Menten reduction

$$v_t = -k_1 vw + k_{-1}p$$
$$w_t = -k_1 vw + (k_{-1} + k_2)p$$
$$p_t = k_1 vw - (k_{-1} + k_2)p$$

$$k_1 vw - (k_{-1} + k_2)p = 0 \quad \text{quasi-static}$$
$$w + p = c \qquad \text{mass conservation}$$

$$\longrightarrow$$

$$u_t = \nabla \cdot (d_1(u, v)\nabla u) - \nabla \cdot (d_2(u, v)\nabla v)$$
$$v_t = d_v \Delta v - k(v)v + f(v)u$$

$$k(v) = \frac{ck_1 k_2}{(k_{-1} + k_2) + k_1 v}$$

Nanjundiah 73

$$d_1(u, v), \ k(v), \ f(v) \quad \text{constants}$$
$$d_2(u, v) = u\chi'(v) \qquad \text{flux=mass} \times \text{velocity}$$
$$\quad \text{sensitivity}$$

$$u_t = d_u \Delta u - \nabla \cdot (u\nabla\chi(v))$$
$$v_t = d_v \Delta v - b_1 v + b_2 u$$

Childress-Percus 81, Jager-Luckhaus 92

$$u_t = \nabla \cdot (\nabla u - u\nabla v), \ \left. \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0$$

$$-\Delta v = u - \frac{1}{|\Omega|} \int_\Omega u, \ \int_\Omega v = 0, \ \left. \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0$$

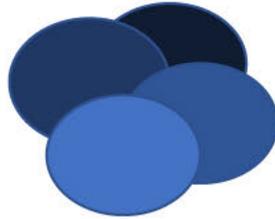Smoluchowski-Poisson equation

Competitive System

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

$$\tau_1 \frac{\partial u_1}{\partial t} = d_1 \Delta u_1 - \chi_1 \nabla \cdot u_1 \nabla v, \quad d_1 \frac{\partial u_1}{\partial \nu} - \chi_1 u_1 \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad u_1|_{t=0} = u_{10}(x) > 0$$
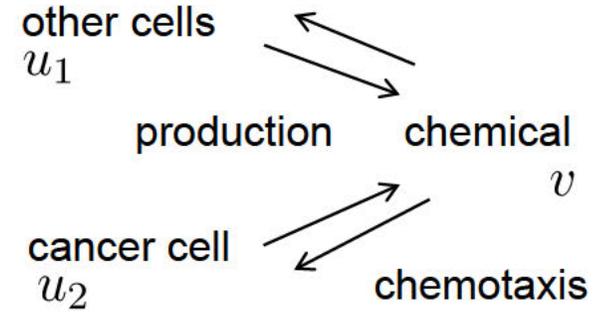
$$\tau_2 \frac{\partial u_2}{\partial t} = d_2 \Delta u_2 - \chi_2 \nabla \cdot u_2 \nabla v, \quad d_2 \frac{\partial u_2}{\partial \nu} - \chi_2 u_2 \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad u_2|_{t=0} = u_{20}(x) > 0$$
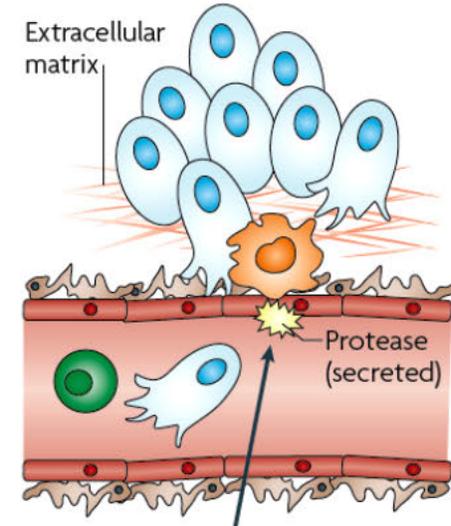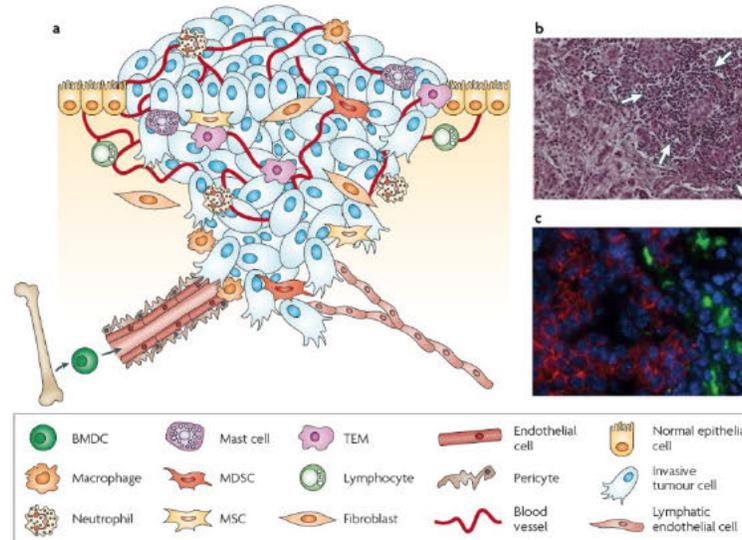
$$-\Delta v = u_1 + u_2, \quad v|_{\partial\Omega} = 0$$

Espejo-Stevens-Velazquez 09
Espejo-Stevens-S. 12

other cells u_1

production   chemical v

cancer cell u_2

chemotaxis

cell sorting

tumor-associated micro environment

BMDC	Mast cell	TEM	Endothelial cell	Normal epithelial cell	
Macrophage	MDSC	Lymphocyte	Pericyte	Invasive tumour cell	
Neutrophil	MSC	Fibroblast	Blood vessel	Lymphatic endothelial cell	

J. Joyce, and J. Pollard. *Nat Rev Cancer* 9: 239-252 (2009)

Extracellular matrix

Protease (secreted)

Protease degradation and tumour cell intravasation
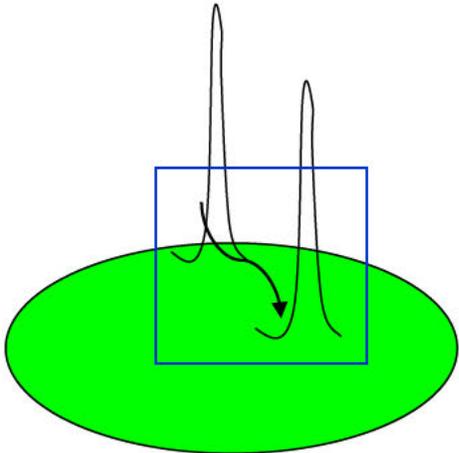
2. Thermo-dynamical structure

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u \nabla v)$$

$$\left.\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0 \qquad \text{transport}$$

closed system

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0 \qquad\qquad \text{potential}$$

Sire-Chavanis 02
motion of the mean field of many self-gravitating Brownian particles
kinetic equation + maximum entropy production

Chavanis 08
relaxation to the equilibrium in the point vortices BBGKY hierarchy
+ factorization

other Poisson parts

a) Debye system (DD model)

$$\Delta v = u, \quad v|_{\partial\Omega} = 0$$

global-in-time existence with compact orbit
Biler-Hebisch-Nadzieja 94

$$\|u\nabla u \cdot \nabla v\|_2 \le C\|u\|_2\|\nabla u\|_2\|\nabla v\|_6$$

b) Childress-Percus-Jager-Luckhaus model (chemotaxis)

$$-\Delta v = u - \frac{1}{|\Omega|}\int u$$

$$\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v = 0$$

blowup threshold
a. Biler 98, Gajewski-Zacharias 98, Nagai-Senba-Yoshida 97
b. Nagai 01, Senba-S. 01b

<u>SP equation</u> $u_t = \nabla \cdot (\nabla u - u \nabla v), \quad -\Delta v = u$

$$\left(\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}, v \right) \Big|_{\partial \Omega} = 0$$

$u = u(x, t) \geq 0$ density

$j = -\nabla u + u \nabla v$ flux (diffusion + chemotaxis)

$u_t + \nabla \cdot j = 0$ conservation law

$v = (-\Delta)^{-1} u$ potential

1. total mass conservation $\dfrac{d}{dt} \|u(t)\|_1 = 0$

attractive (chemotaxis, gravitation)
action at a distance (long range potential)
symmetry (action-reaction)

$G(x, x') = G(x', x)$ Green's function

2. free energy decreasing

$$\frac{d}{dt} \left\{ \int_\Omega u(\log u - 1) - \frac{1}{2} \int\int_{\Omega \times \Omega} G(x, x') u \otimes u \right\} = - \int_\Omega u |\nabla(\log u - v)|^2 \leq 0 \qquad u \otimes u = u(x, t) u(x', t) \, dx dx'$$

3. weak form $\varphi \in C^2(\overline{\Omega}), \; \dfrac{\partial \varphi}{\partial \nu}\Big|_{\partial \Omega} = 0$

$$\begin{aligned} \rho_\varphi(x, x') &= \nabla \varphi(x) \cdot \nabla_x G(x, x') + \nabla \varphi(x') \cdot \nabla_{x'} G(x, x') \\ &\in L^\infty(\Omega \times \Omega) \end{aligned}$$

$$\frac{d}{dt} \int_\Omega \varphi u(\cdot, t) = \int_\Omega \Delta \varphi \cdot u(\cdot, t) + \frac{1}{2} \int\int_{\Omega \times \Omega} \rho_\varphi(x, x') u \otimes u$$

Scaling and Variation

$$u_t = \nabla \cdot (\nabla u - u \nabla \Gamma * u)$$

$$(x, t) \in \mathbf{R}^n \times (0, T)$$

$$-\Delta \Gamma = \delta$$

self-similar transformation

$$u_\mu(x, t) = \mu^2 u(\mu x, \mu^2 t)$$

$$\mu > 0$$

critical dimension

$$u_\mu(x) = \mu^2 u(\mu x), \ \mu > 0$$

$$\|u\|_1 = \|u_\mu\|_1 \equiv \lambda \ \Leftrightarrow \ n = 2$$

Critical mass

$$\mathcal{F}(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2} \langle \Gamma * u, u \rangle$$

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

$$\longrightarrow$$

$$\mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi} \right) \log \mu + \mathcal{F}(u) \qquad \lambda = 8\pi$$

Trudinger-Moser inequality

$$\inf \left\{ \mathcal{F}(u) \mid u \geq 0, \ \|u\|_1 = 8\pi \right\} > -\infty$$

$$u_t = \nabla \cdot (\nabla u - u \nabla v), \ -\Delta v = u$$

$$\left(\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}, v \right) \bigg|_{\partial \Omega} = 0$$

blowup threshold

$$\lambda = \|u_0\|_1 < 8\pi \quad \Rightarrow \quad T = +\infty, \ \|u(t)\|_\infty \leq C$$

$$\|\exists u_0\|_1 > 8\pi, \ T < +\infty$$

3. blowup of the solution

$$\frac{du}{dt} = u^2, \quad u(0) = T^{-1} > 0$$

$$\longrightarrow$$

$$u(t) = (T - t)^{-1}, \quad \lim_{t \uparrow T} u(t) = +\infty$$

quantity distributed in space - time

$\Omega \subset \mathbf{R}^n$ bounded open set, $T > 0$

$u = u(x, t) : \Omega \times [0, T] \to (-\infty, +\infty]$ continuous

$$D(t) = \overline{\{x \in \Omega \mid u(x, t) = +\infty\}}$$

$$D = \bigcup_{0 \leq t \leq T} D(t) \times \{t\} \subset \Omega \times [0, T]$$

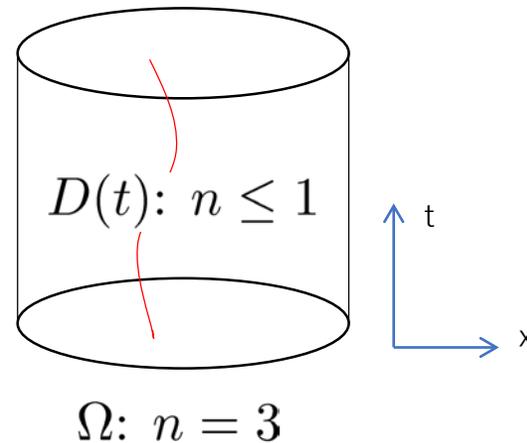$u_t - \Delta u \geq 0$ on $\Omega \times (0, T) \setminus D$

$u = u(x, t)$, Lip. conti. near in $\partial \Omega$ unif. in $t \in [0, T]$

Theorem A [F. Takahashi-S. 08]

$$\int_0^T \mathrm{Cap}_2(D(t)) dt \leq \frac{L^n(\Omega)}{2}$$

~ (n-2) dimensional Hausdorff measure

Temperature infinite region enclosed in a bounded domain in a positive time interval takes a dimension lower than 2

$D(t)$: $n \leq 1$

Ω: $n = 3$

quantized blowup mechanism – spectral level (Boltzmann-Poisson equation)

Theorem B [Nagasaki-S. 90]

$\{(\lambda_k, v_k)\}$ solution sequence

$\lambda_k \to \lambda_0 \in (0, \infty), \; \|v_k\|_\infty \to \infty$

$\Rightarrow$

$\lambda_0 = 8\pi\ell, \; \exists \ell \in \mathbf{N}$

$\exists$sub-sequence, $\exists \mathcal{S} \subset \Omega, \; \sharp \mathcal{S} = \ell$

$v_k \to v_0$ locally uniform in $\overline{\Omega} \setminus \mathcal{S}$

$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0)$ singular limit

$\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$ blowup set

$\nabla_{x_i} H_\ell\big|_{(x_1, \cdots, x_\ell) = (x_1^*, \cdots, x_\ell^*)} = 0, \; 1 \le i \le \ell$

$H_\ell(x_1, \ldots, x_\ell) = \dfrac{1}{2} \sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$
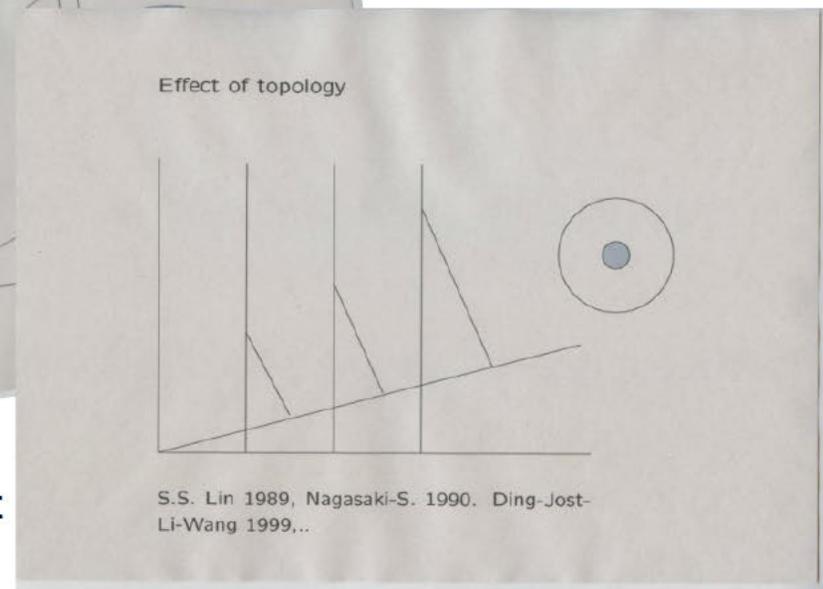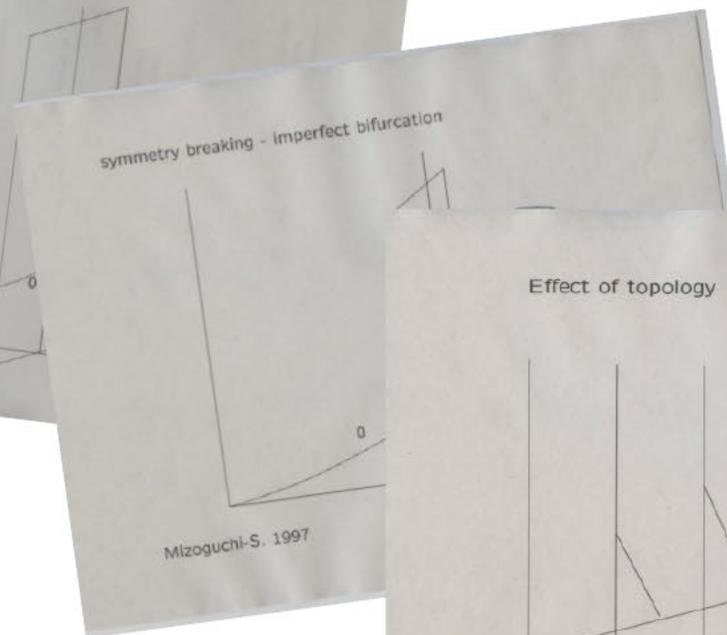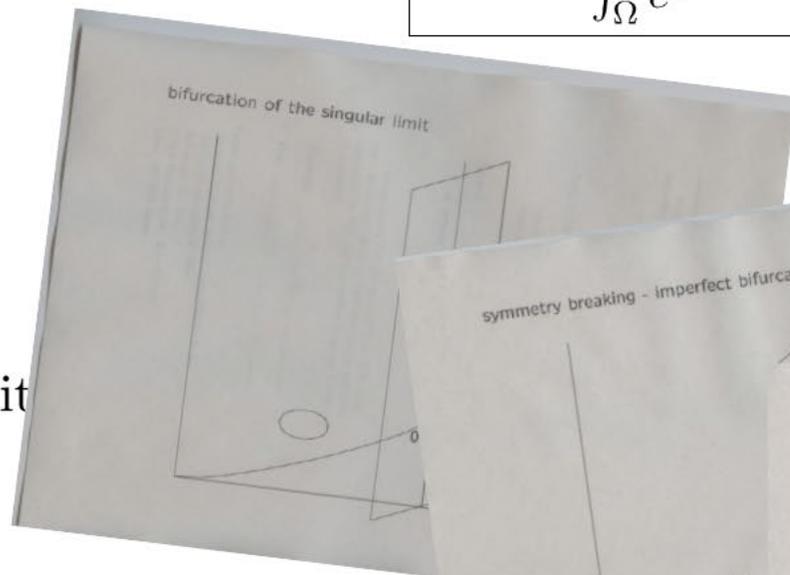
point vortex Hamiltonian

$\lambda = \|u\|_1$

duality

$u \longleftrightarrow v$

$\Omega \subset \mathbf{R}^2$ bounded domain $\partial\Omega$ smooth

$\lambda > 0$ constant

$-\Delta v = \dfrac{\lambda e^v}{\int_\Omega e^v}$ in Ω, $v = 0$ on $\partial\Omega$

bifurcation of the singular limit

symmetry breaking - imperfect bifurcation

Mizoguchi-S. 1997

Effect of topology

S.S. Lin 1989, Nagasaki-S. 1990. Ding-Jost-Li-Wang 1999,..

singular limit

a recursive hierarchy

thermally closed system $\longrightarrow$ total mass conservation
free energy decreasing

Poisson $\qquad -\Delta v = u, \ v|_{\partial\Omega} = 0$

$$u \geq 0, \ \frac{d}{dt}\|u(\cdot, t)\|_1 = 0$$

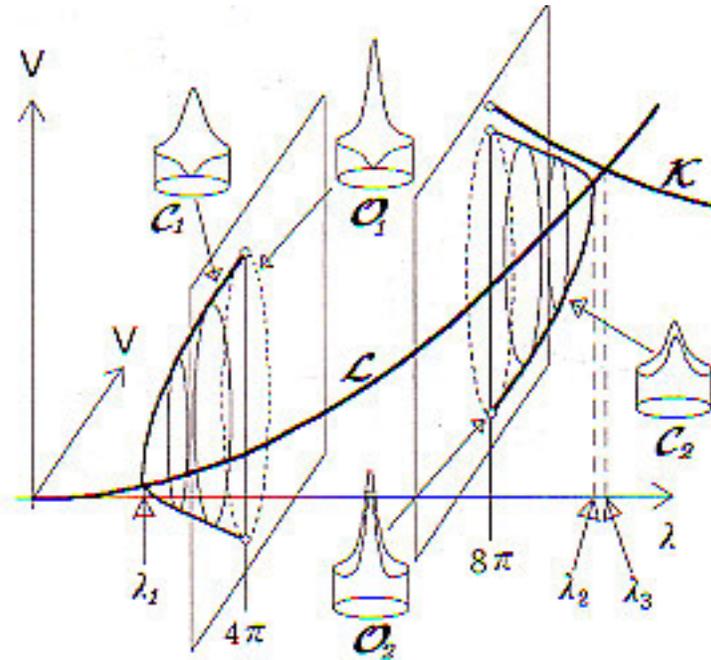$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 dx$$

$$\longrightarrow \quad -\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \ v|_{\partial\Omega} = 0$$

Boltzmann-Poisson equation

stationary state $\longrightarrow$

$$\log u - v = \text{constant}, \ \|u\|_1 = \lambda$$

$$\longrightarrow \quad u = \frac{\lambda e^v}{\int_\Omega e^v}$$

$$-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right), \ \int_\Omega v = 0$$

$$\frac{\partial v}{\partial \nu}\bigg|\partial\Omega = 0$$

Senba-S. 00

$$\lambda = 8\pi, \ 4\pi$$

interior   boundary

$\longrightarrow$ blowup threshold

potentials of self-organization

Theorem B1 (blowup in infinite time)

$$T = +\infty, \ \limsup_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = +\infty$$

$\longrightarrow \quad \lambda \equiv \|u_0\|_1 = 8\pi\ell, \ \exists\ell \in \mathbf{N}$

$\exists x_* \in \Omega^\ell, \ \nabla H_\ell(x_*) = 0 \qquad$ recursive hierarchy

Corollary 1 $\qquad T < +\infty \qquad$ if

(1) $\lambda \notin 8\pi\mathbf{N}, \ \not\exists$ stationary solution or $\mathcal{F}(u_0) \ll -1$

(2) $\lambda \in 8\pi\mathbf{N}, \ \not\exists$ critical point of H_ℓ

Corollary 2 $\qquad \Omega$ convex $\qquad \lambda \neq 8\pi$

$\Rightarrow \ T < +\infty \quad$ or $\quad T = +\infty \quad$ compact orbit

$\downarrow$

$\exists$ stationary solution

c.f. Grossi-F. Takahashi 2010

Theorem B2 (blowup in finite time) $\quad T < +\infty$

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$m(x_0) \in 8\pi\mathbf{N} \qquad\qquad\qquad$ blowup set

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0, \ t_k \uparrow T, \ u(x_k,t_k) \to +\infty\}$$
$$\subset \Omega$$

$$0 < f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$$

S. Liouville's Theory in Linear and Nonlinear PDEs, Springer, 2021. to be published

quantized blowup mechanism in dynamical level

$G = G(x, x')$ Green's function

$R = R(x) \qquad$ Robin function

point vortex Hamiltonian

$$H_\ell(x_1, \cdots, x_\ell) = \frac{1}{2}\sum_{j=1}^{\ell} R(x_j) + \sum_{1 \le i < j \le \ell} G(x_i, x_j)$$

Senba-S. 01	weak formulation	formation of collapse
	monotonicity formula	weak solution generation
Senba-S. 02a	weak solution	instant blowup for over mass concentrated initial data
Kurokiba-Ogawa 03	scaling invariance	non-existence of over mass entire solution without concentration
Senba-S. 04	backward self-similar transformation scaling limit	
S. 05	parabolic envelope (1) scaling invariance of the scaling limit a local second moment	sub-collapse quantization collapse mass quantization
Senba-Ohtsuka-S. 07	defect measure	radially symmetric dynamics
Senba 07, Naito-S. 08	parabolic envelope (2)	type II blowup rate
S. 08	scaling back	limit equation simplification
Senba-S. 11	translation limit	concentration-cancelation simplification
S. 13a	limit equation classification	boundary blowup exclusion
S. 13b	improved regularity concentration compactness	cloud formation
S. 14	tightness	residual vanishing
S. 18	Lioville's formula	quantization of BUIT
S. 21	outer second moment	residual vanishing in finite time