Smoluchowski Poisson Equation 1

Symmetry of Action-Reaction v.s. Duality of Field-Particles

1. Action Reaction Law

Euler's equation of motion

$$v_t + (v \cdot \nabla) v = -\nabla p$$

$$\nabla \cdot v = 0, \ \ \nu \cdot v|_{\partial\Omega} = 0$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} \end{pmatrix}$$

$$v = \begin{pmatrix} v^1 \\ v^2 \\ 0 \end{pmatrix}, \ v^1 = v^1(x_1, x_2, t), \ v^2 = v^2(x_1, x_2, t)$$

2D $\longrightarrow$ $\nabla \times v = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}, \ \omega = \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}$

vortex equation

$$\omega_t + \nabla \cdot (v\omega) = 0, \nabla \cdot v = 0$$

stream function (on simply connected domain)

$$\nabla \cdot v = \frac{\partial v^1}{\partial x_1} + \frac{\partial v^2}{\partial x_2} = 0 \ \longrightarrow \ v^1 = \frac{\partial \psi}{\partial x_2}, \ v^2 = -\frac{\partial \psi}{\partial x_1}$$

$$\omega_t + \nabla \cdot (\omega \nabla^\perp \psi) = 0, \ -\Delta \psi = \omega$$

boundary condition $\nu \cdot v|_{\partial\Omega} = 0 \ \longrightarrow \ \psi|_{\partial\Omega} = \text{constant}$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}, \ \nabla^\perp = \begin{pmatrix} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{pmatrix}$$

$$\omega_t + \nabla \cdot (\omega \nabla^\perp \psi) = 0, \ \psi(\cdot, t) = \int_\Omega G(\cdot, x')\omega(x', t) dx'$$

Green function $-\Delta_x G(x, x') = \delta_{x'}(dx), \ G(x, x')|_{x \in \partial\Omega} = 0$

symmetry $G(x, x') = G(x', x)$

$$\omega_t + \nabla \cdot (\omega \nabla^\perp \psi) = 0, \;\; \psi = \int_\Omega G(\cdot, x')\omega(x', t) \; dx'$$

weak formulation $\varphi \in C^1(\overline{\Omega}), \;\; \varphi|_{\partial\Omega} = 0$

$$\frac{d}{dt}\int_\Omega \varphi\omega = \frac{1}{2}\int_{\Omega\times\Omega} \rho_\varphi(x, x')\omega \otimes \omega \; dxdx'$$

$$\omega \otimes \omega = \omega(x,t)\omega(x',t) \qquad G(x,x') = G(x',x)$$

$$\int_\Omega \nabla \cdot (\omega\nabla^\perp\psi)\varphi \; dx = -\int_\Omega \omega\nabla^\perp\psi \cdot \nabla\varphi \; dx$$

$$= -\int_{\Omega\times\Omega} \nabla\varphi(x) \cdot \nabla_x^\perp G(x,x')\omega(x,t)\omega(x',t)dxdx'$$

$$= -\frac{1}{2}\int_{\Omega\times\Omega} \rho_\phi(x,x')\omega \otimes \omega \; dxdx'$$

$$G(x,x') \approx \Gamma(x - x'), \;\; \Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$\rho_\varphi(x,x') = \nabla\varphi(x) \cdot \nabla_x^\perp G(x,x') + \nabla\varphi(x') \cdot \nabla_{x'}^\perp G(x,x')$$
$$\in L^\infty(\Omega \times \Omega)$$

system of point vortices $\omega(dx,t) = \sum_{i=1}^N \alpha_i \delta_{x_i(t)}(dx)$

$\varphi = |x-a|^2\varphi_{x_0,R}$ local second moment

p.v. $\Longrightarrow$ Kirchhoff equation $\frac{dx_i}{dt} = \nabla_{x_i}^\perp H_N$

point vortices Hamiltonian

$$H_N(x_1,\ldots,x_N) = \sum_i \frac{\alpha_i^2}{2}R(x_j) + \sum_{i<j}\alpha_i\alpha_j G(x_i,x_j)$$

Robin function $R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x-x'|\right]_{x'=x}$

Point Vortex Mean Field ~ Kinetic Theory

Chavanis 08 Langevin equation $\mu > 0$ mobility

$$\frac{dx_i}{dt} = \alpha \nabla_i^\perp \hat{H}_N - \mu\alpha^2 \nabla_i \hat{H}_N + \sqrt{2\nu} R_i(t), \ 1 \leq i \leq N$$

$\nu > 0$ viscosity of particles

$R_i(t)$ white noise

$$\langle R_i(t) \rangle = 0, \ \langle R_i^\alpha(t) R_j^\beta(t') \rangle = \delta_{ij}\delta_{\alpha\beta}\delta(t-t')$$

$P_N(x_1, \cdots, x_N, t)$ N-pdf

$$\frac{\partial P_N}{\partial t} + \alpha \nabla^\perp \cdot \hat{H}_N \nabla P_N = \nabla \cdot (\nu \nabla P_N + \mu\alpha^2 P_N \nabla \hat{H}_N)$$

BBGKY hierarchy $\{P_i\}_{i=1,2,\cdots,N}$

factorization (propagation of chaos)

$$P_N(x_1, x_2, \cdots, x_N, t) = \prod_{i=1}^N P_1(x_i, t)$$

high-energy limit $\mu\hat{\beta}N\alpha^2 = \nu\beta, \ \alpha N = 1, \ \omega = P_1$

Euler-Smoluchowski-Poisson equation

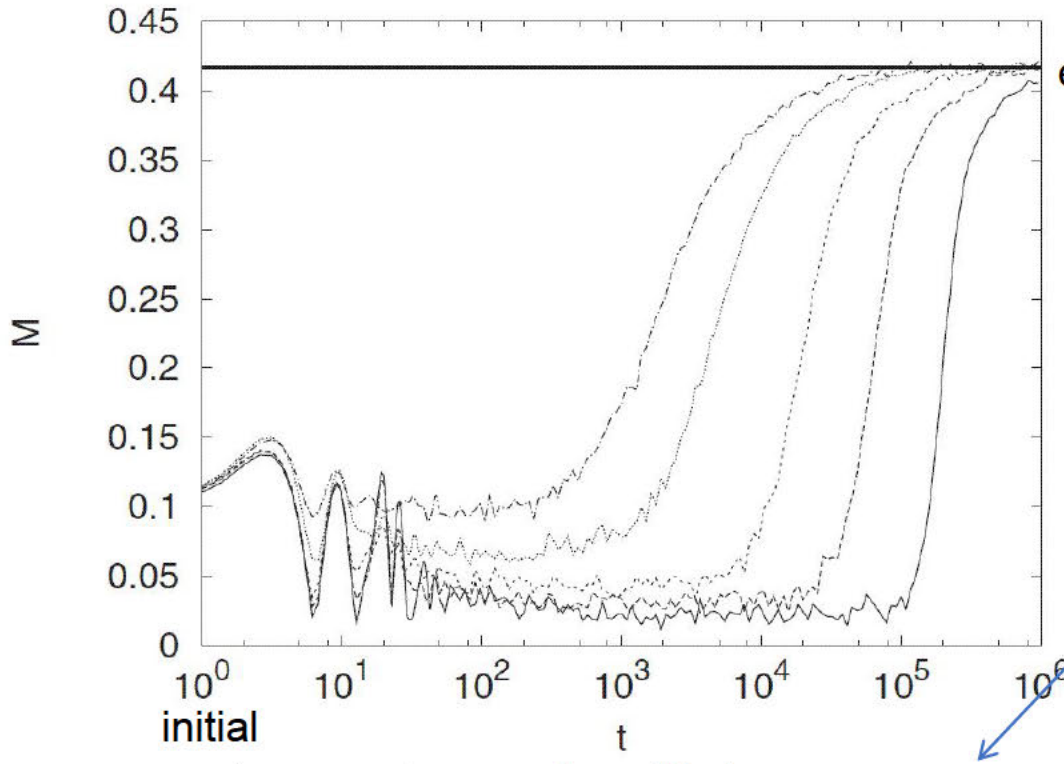
$$\frac{\partial \omega}{\partial t} + \nabla^\perp \psi \cdot \nabla \omega = \nu \nabla \cdot (\nabla \omega + \beta\alpha\omega\nabla\psi)$$

$$-\Delta\psi = \omega, \ \ \psi|_{\partial\Omega} = 0$$

2. Kinetics of Newtonian Point Vortices

Hamilton System of Many Particles with Long Range Interaction

state of the system



equilibrium

Gibbs
Onsager

kinetic

time

initial

relaxation

quasi-equilibrium

relaxation

static

Staniscia-Chavanis-Ninno-Fanelli 09

Lynden-Bell
Miller-Robert-Sommeria

patch model

$$\omega(x,t) = \sum_{i=1}^{N_p} \sigma_i 1_{\Omega_i(t)}(x)$$

static → Robert-Sommeria 91

$$\overline{\omega} = \int_I \sigma p(x,\sigma)\; d\sigma$$

$$\int_I p(x,\sigma)\; d\sigma = 1, \;\forall x$$

$$c(\sigma) - \log(\frac{1}{M(\sigma)} \int_\Omega p(x,0) e^{-\beta\sigma\overline{\psi}})$$

$$\zeta(x) = \log(\int_I e - -c(\sigma) - \beta\sigma\overline{\psi} d\sigma) - 1$$

$$p(x,\sigma) = e^{-c(\sigma)-(\zeta(x)+1)-\beta\sigma\overline{\psi}}$$

$$-\Delta\overline{\psi} = \int_I \sigma \frac{e^{-c(\sigma)-\beta\sigma\overline{\psi}}}{\int_I e^{-c(\sigma')-\beta\sigma'\overline{\psi}} d\sigma'} d\sigma, \quad \overline{\psi}|_{\partial\Omega} = 0$$

maximal entropy production principle

$$D = D(x,t) > 0$$

diffusion coefficient

β inverse temperature

$c(\sigma)$ chemical potential

$M(\sigma)$ patch area

kinetic → Robert-Sommeria 92

$$\frac{\partial p}{\partial t} + \nabla \cdot p\overline{u} = \nabla \cdot D\left(\nabla p + \beta\left(\sigma - \overline{\omega}\right) p \nabla\overline{\psi}\right)$$

$$\overline{\omega} = \int_I \sigma p \; d\sigma = -\nabla^\perp \overline{u} = -\Delta\overline{\psi}, \quad \overline{\psi}|_{\partial\Omega} = 0$$

$$\beta = -\frac{\int_\Omega D \nabla\overline{\omega} \cdot \nabla\overline{\psi}\; dx}{\int_\Omega D\left(\int_I \sigma^2 p\; d\sigma - \overline{\omega}^2\right) |\nabla\overline{\psi}|^2 \; dx}$$

$$\Delta = \nabla^\perp \cdot \nabla^\perp$$

$$\overline{u} = \nabla^\perp \psi$$

point vortex model

$$\omega(dx,t)=\sum_{i=1}^{N}\alpha_i\delta_{x_i(t)}(dx)$$

$$\alpha_i=\tilde{\alpha}_i\alpha,\ \tilde{\alpha}_i\in I=[-1,1]$$

$$\alpha N=1,\ N\to\infty,\ H_N=\text{constant},\ \alpha^2 N\beta_N=\beta$$

$$\rho^{\tilde{\alpha}}(x)=\lim_{N\to\infty}\int_{\Omega^{N-1}}\mu_N^{\beta_N}(dx,dx_2,\cdots dx_N)$$

$$\mu_N^{\beta_N}(dx_1,\cdots dx_N)=\frac{1}{Z(N,\beta_N)}e^{-\beta_N H_N}dx_1\cdots dx_N$$

stationary

$$\overline{\omega}(x)=\int_I\tilde{\rho}^{\tilde{\alpha}}(x)P(d\tilde{\alpha}),\ x\in\Omega$$

$$\int_\Omega\rho^{\tilde{\alpha}}(x)\ dx=1,\quad\forall\tilde{\alpha}\in I$$

→ Sawada-S. 08

$$\overline{\omega}=-\Delta\overline{\psi}$$

$$-\Delta\overline{\psi}=\int_I\tilde{\alpha}\frac{e^{-\beta\tilde{\alpha}\overline{\psi}}}{\int_\Omega e^{-\beta\tilde{\alpha}\overline{\psi}}dx}P(d\tilde{\alpha}),\quad\overline{\psi}|_{\partial\Omega}=0$$

one component case $P(d\tilde{\alpha}) = \delta_1(d\tilde{\alpha}), \ D(x,t) = 1$

$$\omega_t + \nabla\cdot\omega\nabla^\perp\psi = \nabla\cdot(\nabla\omega + \beta\omega\nabla\psi)$$

$$\frac{\partial\omega}{\partial\nu} + \beta\omega\frac{\partial\psi}{\partial\nu}\Big|_{\partial\Omega} = 0, \ \omega|_{t=0} = \omega_0(x) \geq 0$$

$$-\Delta\psi = \omega, \quad \psi|_{\partial\Omega} = 0$$

micro-canonical

$$\beta = -\frac{\int_\Omega \nabla\omega\cdot\nabla\psi}{\int_\Omega \omega|\nabla\psi|^2}$$

$\|\omega(\cdot,t)\|_1 = \lambda$ mass

$\frac{d}{dt}(\omega,\psi) = 0$ energy

$$\frac{d}{dt}\int_\Omega \Phi(\omega) = -\int_\Omega \omega|\nabla(\log\omega + \beta\psi)|^2 \leq 0$$

$\Phi(s) = s(\log s - 1) + 1$ entropy

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canonical

β constant

2D Brownian vortices
Chavanis 08

Euler-Smolchowski-Poisson

$\beta = -8\pi/\lambda$ blowup threshold
S. 14

$\|\omega(\cdot,t)\|_1 = \lambda$ mass

$$\frac{d\mathcal{F}}{dt} = -\int_\Omega \omega|\nabla(\log\omega + \beta\psi)|^2 \leq 0$$

$\mathcal{F}(\omega) = \int_\Omega \Phi(\omega) - \frac{1}{2}((-\Delta)^{-1}\omega,\omega)$ free energy

K. Sawada (2015)

point vortex model ← patch model

kinetic

$$\frac{\partial \rho^{\tilde\alpha}}{\partial t} + \nabla\cdot\rho^{\tilde\alpha}\overline{u} = \nabla\cdot D\left(\nabla\rho^{\tilde\alpha} + \beta\tilde\alpha\rho^{\tilde\alpha}\nabla\overline{\psi}\right)$$

$$\overline{\omega} = \int_I \tilde\alpha\rho^{\tilde\alpha}P(d\tilde\alpha) = -\nabla^{\perp}\overline{u} = -\Delta\overline{\psi}, \quad \overline{\psi}\big|_{\partial\Omega} = 0$$

$$\beta = -\frac{\int_\Omega D\nabla\overline{\omega}\cdot\nabla\overline{\psi}\ dx}{\int_\Omega D\int_I \tilde\alpha^2\rho^{\tilde\alpha}(x)P(d\tilde\alpha)|\nabla\overline{\psi}|^2\ dx}$$

← Robert-Sommeria 92

static

Joyce-Montgomery 73 ・・・Sawada-S. 08 ← Robert-Sommeria 91

3. Methods of Mathematical Modeling

<u>Coarsening key factors</u>

1. supply-consumption

$$u_t = \alpha,\ v_t = -\beta$$

2. production-annihilation

$$u_t = \alpha u,\ v_t = -\beta v$$

3. transport

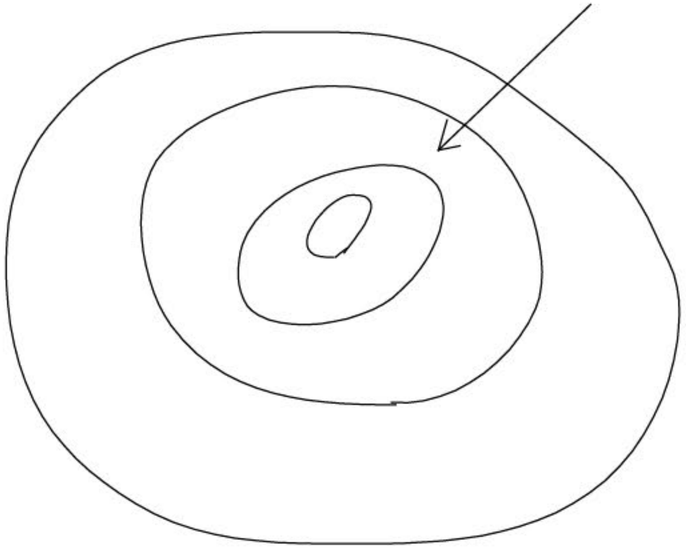
flux

$$u_t = -\nabla \cdot j,\ j = \text{mass} \times \text{velocity}$$

4. gradient

$j = -d_u \nabla u$ diffusion

$j = d_v u \nabla v$ chemotaxis



5. chemical reaction

$$A + B \to C\ (k)$$

→ (mass action)

$$\frac{d[A]}{dt} = -k[A][B]$$
$$\frac{d[B]}{dt} = -k[A][B]$$
$$\frac{d[C]}{dt} = k[A][B]$$

$$u_t = D\Delta u$$
$$u_t = \nabla \cdot (D\nabla u)$$
$$u_t = \Delta(Du)$$

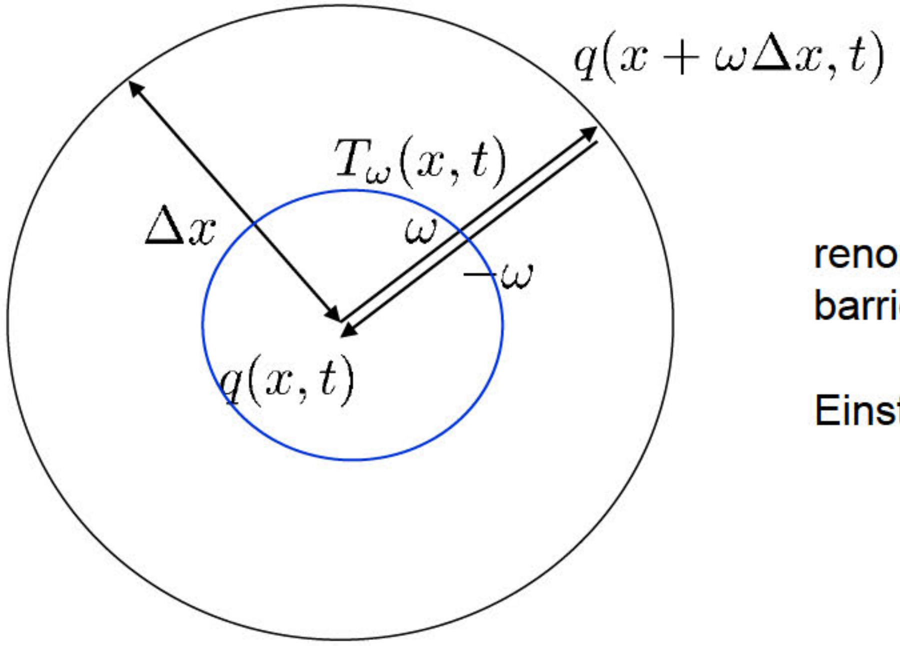
?

Averaging particle movements

$$S^{N-1} = \{\omega \in \mathbf{R}^N \mid |\omega| = 1\}$$

$q = q(x,t)$ particle density $\qquad T = T_\omega(x,t)$ transient probability

master equation
$$\frac{q(x,t+\Delta t) - q(x,t)}{\Delta t} = \int_{S^{N-1}} T_{-\omega}(x+\omega\Delta x, t) q(x+\omega\Delta x, t) d\omega - \int_{S^{N-1}} T_\omega(x,t) d\omega \cdot q(x,t)$$



$$\int_{S^{N-1}} T_\omega(x,t) d\omega = \tau^{-1}$$ τ mean waiting time

renormalization barrier
$$\tau T_\omega(x,t) = \frac{T(x+\omega\frac{\Delta x}{2}, t)}{\int_{S^{N-1}} T(x+\omega'\frac{\Delta x}{2}, t) d\omega'}, \quad T = T(x,t)$$

Einstein formula
$$\tau^{-1}(\Delta x)^2 = 2nD$$

space dimension $\qquad$ diffusion coefficient

spatially-temporally continuous distribution function
Ichikawa-Rouzi-S. 12

→ Smoluchowski equation
$$\frac{\partial q}{\partial t} = D\nabla \cdot (\nabla q - q\nabla \log T)$$

Realizing physical laws

S. Mean Field Theories and Dual Variation, 2nd edition, Atlantis Press, 2015

system	consistency	dynamics	ensemble
isolated	energy	entropy	micro-canonical
closed	temperature	Helmholtz free energy	canonical
open	pressure	Gibbs free energy	grand-canonical

particle density

Smoluchowski

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

duality

$\longleftrightarrow$

field potential

Poisson

$$v = (-\Delta)^{-1}u, \quad \int_\Omega G(\cdot, x')u(x')dx'$$

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \quad \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad \int_\Omega v = 0$$

symmetry

Helmholtz fee energy

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle$$
$$\delta\mathcal{F}(u) = \log u - (-\Delta)^{-1}u$$

Model (B) equation

$$u_t = \nabla u \cdot \nabla \delta\mathcal{F}(u), \quad \frac{\partial}{\partial \nu}\delta\mathcal{F}(u)\Big|_{\partial\Omega} = 0$$

total mass conservation
free energy decreaseing

Duality of Field and Particles

X Banach space/$\mathbf{R}$

$F : X \to (-\infty, +\infty]$ prop. c'x l.s.c.

$\Rightarrow$

Legendre transformation

$F^* : X^* \to (-\infty, +\infty]$ prop. c'x l.s.c.

$$F^*(p) = \sup_{x \in X} \{\langle x, p\rangle - F(x)\}$$

Fenchel-Moreau duality

$F^{**} = F$

$$F^{**}(x) = \sup_{p \in X^*} \{\langle x, p\rangle - F^*(p)\}$$

Toland duality 78, 79

$F, G : X \to (-\infty, +\infty]$ prop. c'x l.s.c.

$J(x) = G(x) - F(x)$

$J^*(p) = F^*(p) - G^*(p)$

$L(x,p) = F^*(p) + G(x) - \langle x, p\rangle$

... Lagrange function

$$\inf_{X \times X^*} L = \inf_X J = \inf_{X^*} J^*$$

free energy $\longleftrightarrow$ field functional

(Legendre transformation / duality)

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle$$

$u \geq 0, \ \|u\|_1 = \lambda$

particle distribution

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v + \lambda(\log\lambda - 1)$$

potential density

$v \in H^1(\Omega), \ \int_\Omega v = 0, \ \|u\|_1 = \lambda$

楕円型統一理論

近平衡力学系標準理論

孤立系の均質化

ネットワークからの創発

歪対称性が導く特異性の解消

領域の形状

場と粒子の双対性

Hamiltonian　　無限時間爆発　　有限時間爆発

凝縮
臨界
爆発

量子化する爆発機構

単純性

エントロピーの排出
（創発性）

重複
衝突

Boltzmann-Poisson方程式 → Smoluchowski Poisson 方程式

閉鎖系
正準集団

非線形スペクトル理論
（自己組織化のポテンシャル）

質量保存
自由エネルギー

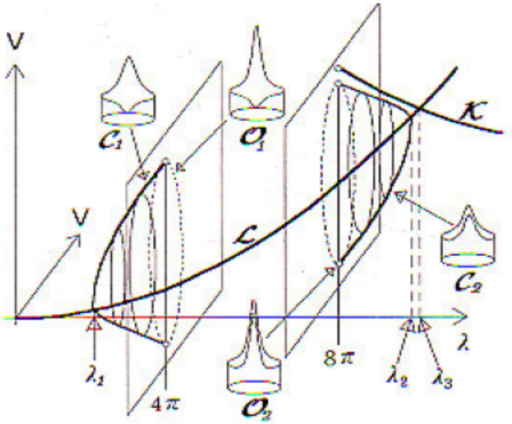
遠隔作用
作用反作用

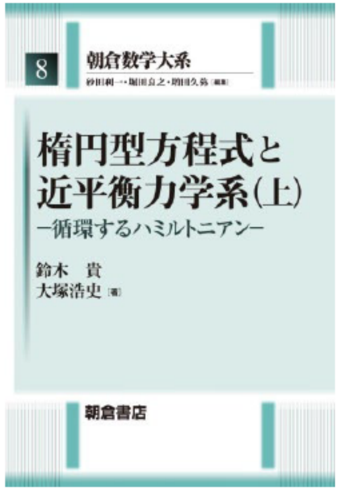
多数粒子
平均場極限
カオスの伝播

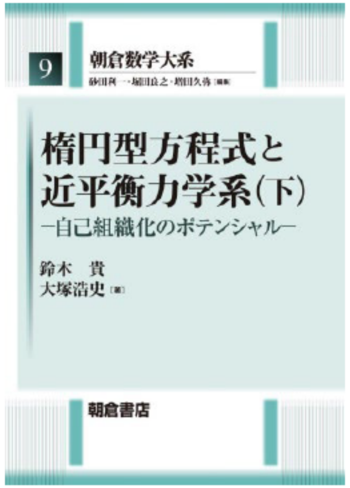
循環的階層

点渦系

孤立系
小正準集団







S. Liouville's Theory in Linear and Nonlinear PDEs (Springer)