Elliptic Theory 2

Recursive Hierarchy in Boltzmann-Poisson Equation

1. Point Vortices

2D Euler Equation (simply connected domain)

$$v_t + (v \cdot \nabla) v = -\nabla p$$
$$\nabla \cdot v = 0$$
$$\nu \cdot v|_{\partial\Omega} = 0$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}$$ gradient

vorticity

$$\omega = \nabla^\perp v$$
$$\nabla^\perp = \begin{pmatrix} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{pmatrix}$$
$$x = (x_1, x_2)$$

vortex equation

$$\omega_t + v \cdot \nabla \omega = 0$$
$$v = \nabla^\perp \psi$$
$$\Delta \psi = -\omega$$ stream function
$$\psi|_{\partial\Omega} = 0$$

$$\nabla^\perp \cdot \nabla^\perp = \Delta$$

point vortices

$$\omega(dx,t) = \sum_{i=1}^{N} \alpha_i \delta_{x_i(t)}(dx)$$

Kirchhoff equation

$$\alpha_i \frac{dx_i}{dt} = \nabla_i^\perp H, \; 1 \le i \le N$$

Hamiltonian

$$H = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

Green's function

$$-\Delta G(x,x') = \delta_{x'}(dx)$$
$$G(x,x')|_{\partial\Omega} = 0$$
$$(x,x') \in \overline{\Omega} \times \Omega$$

Robin function

$$R(x) = \left[G(x,x') + \frac{1}{2\pi} \log|x - x'| \right]_{x'=x}$$

Onsager 49

Hamiltonian $$H = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

$H = \hat{H}_N(x_1, \cdots, x_N)$ $N \gg 1$ total energy

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad 1 \leq i \leq N$$

$$p_i = p_i(t), \; q_i = q_i(t) \; \in \mathbf{R}^2$$

micro-canonical ensemble $\mathbf{R}^{4N}/\{H = E\}$

$$x = (q_1, \ldots, q_N, p_1, \ldots, p_N)$$

co-area formula $$dx = dE \cdot \frac{d\Sigma(E)}{|\nabla H|}$$

$$d\Sigma(E) \; \leftrightarrow \; \{x \in \mathbf{R}^{4N} \mid H(x) = E\}$$

→ canonical ensemble

thermal equilibrium

Gibbs measure $$\mu^{E,N} = \frac{1}{W(E)} \cdot \frac{d\Sigma(E)}{|\nabla H|}$$

weight factor $$W(E) = \int_{H=E} \frac{d\Sigma(E)}{|\nabla H|}$$

inverse temperature $$\beta = \frac{\partial}{\partial E} \log W(E) = \frac{\Theta''(E)}{\Theta'(E)}$$

$$\Theta(E) = \int_{H<E} dx = \int_{-\infty}^{E} W(E')dE'$$

↑ bounded monotone

$E \gg 1 \Rightarrow \beta < 0$ ordered structure in <u>negative temperature</u>

Joyce-Montgomery 73

micro-canonical statistics

$$\mathbf{R}^{4N}/\{H=E\}$$

$$x=(q_1,\ldots,q_N,p_1,\ldots,p_N)$$

$$dx = dE\cdot\frac{d\Sigma(E)}{|\nabla H|}$$

$$d\Sigma(E)\ \leftrightarrow\ \{x\in\mathbf{R}^{4N}\mid H(x)=E\}$$

micro-canonical measure

$$d\mu^{E,N}=\frac{1}{W(E)}\cdot\frac{d\Sigma(E)}{|\nabla H|}$$

weight factor

$$W(E)=\int_{\{H=E\}}\frac{d\Sigma(E)}{|\nabla H|}$$

canonical statistics

$$\mathbf{R}^{4N}/\{T\}$$

inverse temperature

$$\beta=1/(kT)$$

canonical measure

$$d\mu^{\beta,N}=\frac{e^{-\beta H}dx}{Z(\beta,N)}$$

weight factor

$$Z(\beta,N)=\int_{\mathbf{R}^{4N}}e^{-\beta H}dx$$

thermo-dynamical relation

$$\beta=\frac{\partial}{\partial E}\log W(E)$$

micro-canonical probability measure

$$\mu^n=\mu^n(dx_1,\ldots,dx_n)$$

one point pdf

$$\rho_1^n(x_i)dx_i = \int_{\Omega^{n-1}}\mu^n(dx_1\ldots dx_{i-1}dx_{i+1}dx_n)$$

↑

equal a priori probability (independent of i)

k-point reduced pdf

$$\rho_k^n(x_1,\ldots,x_k)dx_1\ldots dx_k = \int_{\Omega^{n-k}}\mu^n(dx_{k+1},\ldots,dx_n)$$

stationary point vortices

$$\omega_N(x)dx=\sum_{i=1}^N\alpha\delta_{x_i}(dx)$$

⟶

$$\begin{aligned}\langle\omega_N(x)\rangle &= \sum_{i=1}^N\int_{\Omega^N}\alpha\delta(x_i-x)\mu^N(dx_1\ldots dx_N)\\ &= N\alpha\rho_1^N(x)\end{aligned}$$

phase mean

high energy limit
(single intensity)

$$\alpha_i = \hat{\alpha},\ N \uparrow +\infty,\ \hat{\alpha}N = 1$$
$$\hat{H}_N = H,\ \hat{\alpha}^2 N \hat{\beta} = \beta$$

→ two point pdf compatibility

$$\hat{H}_N(x_1, \ldots, x_N) = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

Boltzmann $\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}$

duality ↕

Poisson $\psi = \int_\Omega G(\cdot, x')\rho(x')dx'$

energy $\tilde{E} = H$

inverse temperature $\tilde{\beta} = \frac{\partial}{\partial \tilde{E}} \log W(\tilde{E})$

weight factor $W(\tilde{E}) = \int_{H=\tilde{E}} \frac{d\Sigma_{\tilde{E}}}{|\nabla H|}$

mean field limit $\lim_{N\to\infty} \langle \omega_N(x) \rangle = \rho(x) = \lim_{N\to\infty} N\alpha\rho_1^N(x)$

propagation of chaos
(factorization property)

$$\rho_k^N \rightharpoonup \rho^{\otimes k} = \Pi_{i=1}^k \rho(x_i)$$

(Suzuki's uniqueness theorem)

<u>rigorous derivation</u>

Caglioti-Lions-Marchioro-Pulvirenti 92, 95. Kiessling 93

1. Bounded Boltzmann weight factors {z}
2. Uniqueness of the solution to the limit equation

→

1. convergence to the limit
2. canonical-micro canonical equivalence in the limit
3. propagation of chaos

OK if $\beta > -8\pi$

Theorem A [S. 92]

$0 < \lambda < 8\pi \Rightarrow \exists 1$ solution

Boltzmann Poisson Equation

$\Omega \subset \mathbf{R}^2$ bounded domain $\partial\Omega$ smooth

$\lambda > 0$ constant

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega, \ v = 0 \text{ on } \partial\Omega$$

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}, \ \lambda = -\beta$$

$$\psi = \int_\Omega G(\cdot, x')\rho(x')dx'$$

mean field equation in stream function

Impact to the Elliptic Theory

Theorem B [Nagasaki-S. 90a]

$\{(\lambda_k, v_k)\}$ solution sequence s.t.

$\lambda_k \to \lambda_0 \in [0, \infty), \ \|v_k\|_\infty \to \infty$

$\Rightarrow \lambda_0 = 8\pi N, \ N \in \mathbf{N}$

quantized blowup mechanism

$\exists$sub-sequence, $\exists \mathcal{S} \subset \Omega, \ \sharp\mathcal{S} = N$, s.t.

$v_k \to v_0$ loc. unif. in $\overline{\Omega} \setminus \mathcal{S}$

$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0)$$

recursive hierarchy

$$\nabla_{x_i} H_N(x_1^*, \cdots, x_N^*) = 0, \ 1 \leq i \leq N$$

$G = G(x, x')$ the Green's function

$\mathcal{S} = \{x_1^*, \ldots, x_N^*\}$

$$H_N(x_1, \ldots, x_N) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$$

$$R(x) = \left[G(x, x') + \frac{1}{2\pi}\log|x - x'|\right]_{x=x'}$$

1. non-radial bifurcation on annulus (S.S. Lin 89 Nagasaki-S. 90b)

2. effective bound of blowup points for simply-connected domain (S.-Nagasaki 89 Grossi-F.Takahashi 10)

3. classification of singular limits (Nagasaki-S. 90a)

4. spherical mean value theorem (S. 90)

5. localization (Brezis-Merle 91)

6. entire solution (W. Chen-C. Li 91)

7. sup + inf inequality (Shafrir 92)

8. uniqueness (S. 92)

9. field-particle duality (S. 92 Wolansky 92)

10. singular perturbation (Weston 78 Moseley 83 S. 93 Baraket-Pacard 98 Esposito-Grossi-Pistoia 05 del Pino-Kowarzyk-Musso 05)

11. blowup analysis (Li-Shafrir 94)

12. Chern-Simons theory (Tarantello 96)

13. global bifurcation (S.-Nagasaki 89 Mizoguchi-S. 97 Chang-Chen-Lin 03)

14. min-max solution (Ding-Jost-Li Wang 99)

15. local uniform esitmate (Y.Y. Li 99)

16. variable coefficient (Ma-Wei 01)

17. refined asymptotics (Chen-Lin 02)

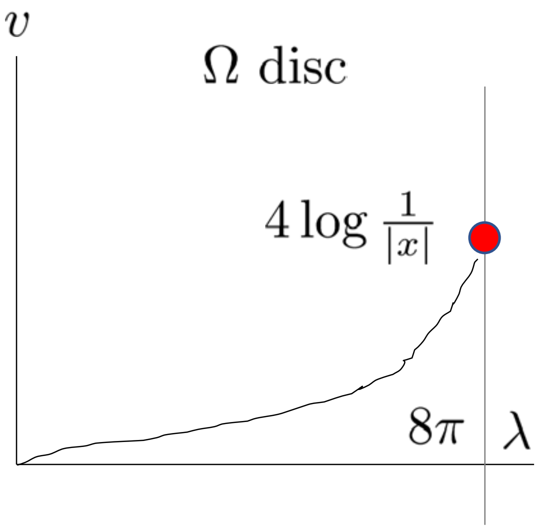
18. topological degree (Li 99 C.C. Chen-C.S. Lin 03 Malchiodi 08)

19. asymptotic non-degeneracy (Gladiali-Grossi 04 Grossi-Ohtsuka-S. 11)

20. isoperimetric profile (Lin-Lucia 06)

21. deformation lemma (Lucia 07)

22. Morse index (Gladiali-Grossi 09)



$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega \subset \mathbf{R}^2$$
$$v|_{\partial\Omega} = 0$$

2. Boltzmann-Poisson Equation

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \quad v|_{\partial\Omega} = 0$$

L. Onsager 49

point vortices
ordered structure in negative temperature

Poisson

$$-\Delta v = u$$
$$v|_{\partial\Omega} = 0$$

Boltzmann

$$u = \frac{\lambda e^v}{\int_\Omega e^v}$$

$$G(x,x') = G(x',x) \quad \text{Green}$$

$$R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x-x'| \right]_{x'=x} \quad \text{Robin}$$

Theorem 1 (Nagasaki-S. 90)

$$\{(\lambda_k, v_k)\}, \ \lambda_k \to \lambda_0 \in (0,\infty), \ \|v_k\|_\infty \to \infty$$

$$\Rightarrow \ \lambda_0 = 8\pi\ell, \ \ell \in \mathbf{N}, \ \exists \mathcal{S} \subset \Omega, \ \sharp\mathcal{S} = \ell$$

$v_k \to v_0$ loc. unif. in $\overline{\Omega} \setminus \mathcal{S}$ (sub-sequence)

$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x,x_0), \ \mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$$

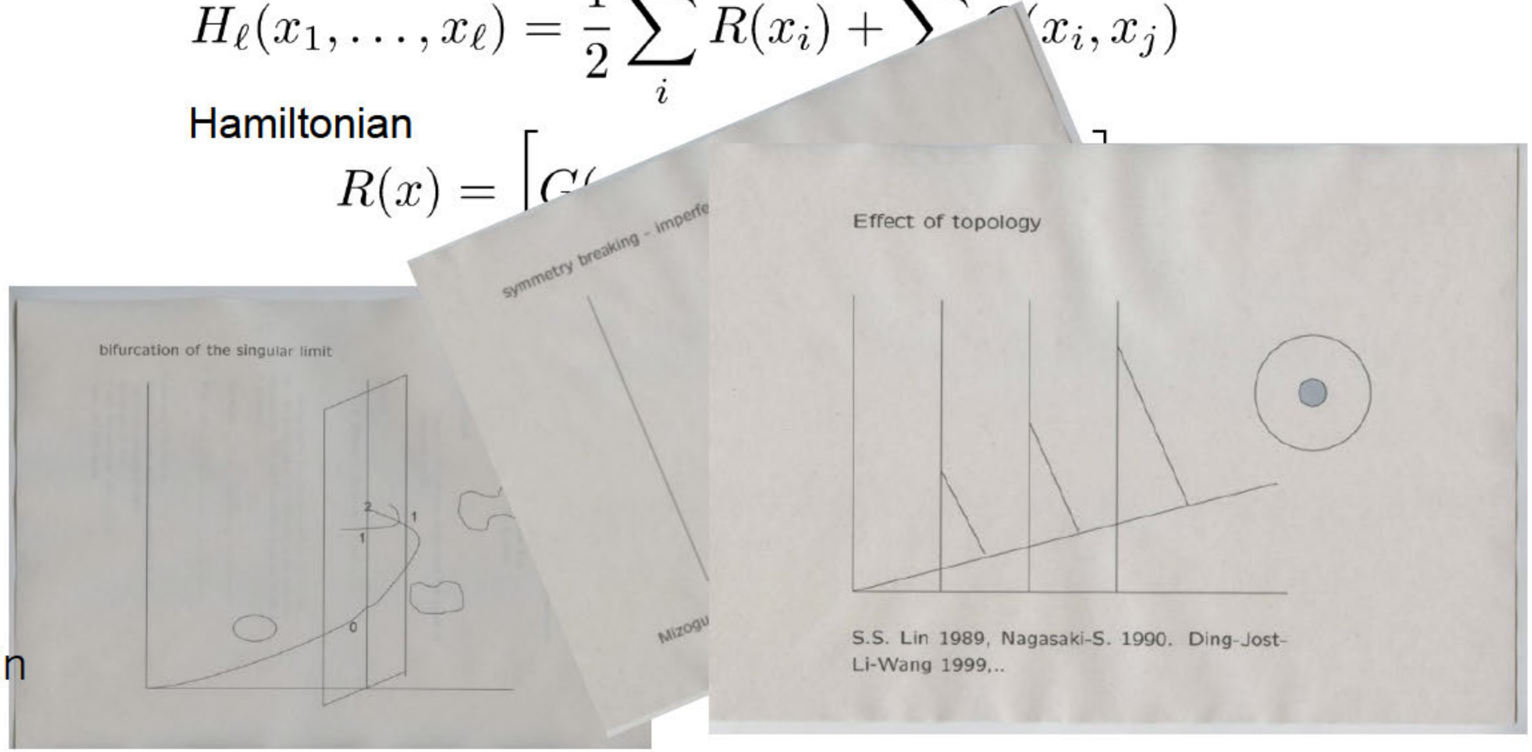
singular limit — blowup set

$$\nabla H_\ell|_{(x_1,\cdots,x_\ell)=(x_1^*,\cdots,x_\ell^*)} = 0$$

$$H_\ell(x_1,\ldots,x_\ell) = \frac{1}{2}\sum_i R(x_i) + \sum \ldots (x_i, x_j)$$

Hamiltonian

$$R(x) = \left[G(\ldots \right.$$



complex structure (Liouville integral)

$$-\Delta v = \sigma e^{v}$$

$$\Leftrightarrow \ \exists F = F(z), \ z \in \Omega \subset \mathbf{R}^2 \cong \mathbf{C}$$ meromorphic

$$\rho(F) = \left(\frac{\sigma}{8}\right)^{1/2} e^{v/2} = \frac{|F'|}{1+|F|^2}$$ spherical derivative

$$-\Delta v = \sigma e^{v}, \ v|_{\partial\Omega} = 0 \ \Leftrightarrow \ \rho(F)|_{\partial\Omega} = \left(\frac{\sigma}{8}\right)^{1/2}$$

$$\hat{F} = \sqrt{8} \circ F : \Omega \to S^2$$ conformal

$$\left.\frac{d\Sigma}{ds}\right|_{\partial\Omega} = \sigma^{1/2}$$

$(S^2, d\Sigma)$ round sphere

$|S^2| = 8\pi$

$$\int_{\partial\Omega} \frac{d\Sigma}{ds} ds = |\partial\Omega| \, \sigma^{1/2}$$

immersed length of $\hat{F}(\partial\Omega)$

Proof of Theorem (90)

1. Liouville integral
2. boundary reflection
3. elliptic regularity

4. complex function theory
4-1. maximum principle
4-2. Montel's theorem
4-3. theorem of coincidence
4-4. residue analysis

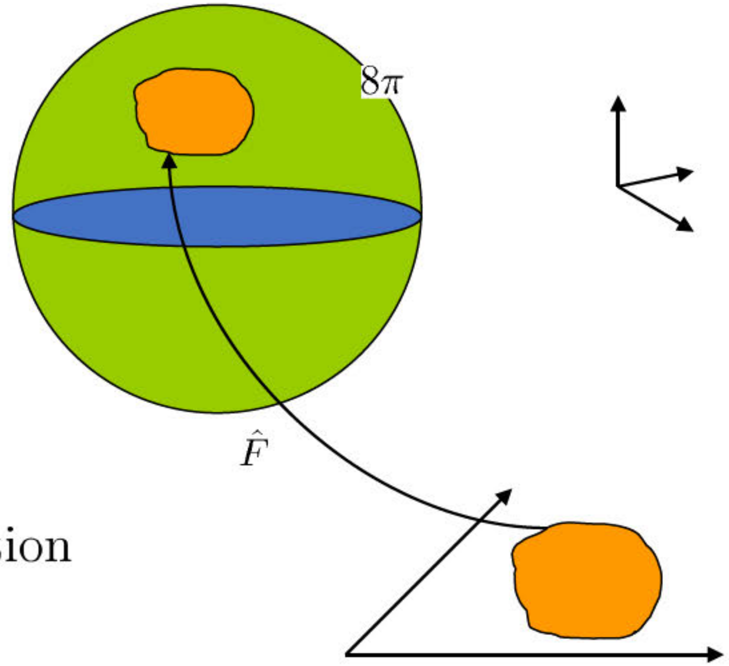
$$\int_{\Omega} \left(\frac{d\Sigma}{ds}\right)^2 dx = 8 \int_{\Omega} \rho(F)^2 dx = \int_{\Omega} \sigma e^{v}$$

immersed area of $\hat{F}(\Omega)$

$$\lambda = \int_{\Omega} \sigma e^{v} \to 8\pi\ell$$

$\Leftrightarrow$ total mass quantization

due to ℓ-covering



Blowup analysis

$\Omega \subset \mathbf{R}^2$: open set, $V \in C(\overline{\Omega})$

$-\Delta v = V(x)e^v, \ 0 \le V(x) \le b \quad \text{in } \Omega$

$\int_\Omega e^v \le C$

Theorem 2 [Li-Shafrir 94]

$\{(V_k, v_k)\}$ solution sequence

$V_k \to V$ loc. unif. in Ω

$\Rightarrow$ $^\exists$ sub-sequence with the alternatives;

1. $\{v_k\}$: loc. unif. bdd in Ω

2. $\exists \mathcal{S} \subset \Omega, \sharp\mathcal{S} < +\infty$

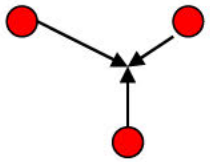
$v_k \to -\infty$ loc. unif. in $\Omega \setminus \mathcal{S}$

$\mathcal{S} = \{x_0 \in \Omega \mid \exists x_k \to x_0, \ v_k(x_k) \to +\infty$

$$V_k(x)e^{v_k}dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) \text{ in } \mathcal{M}(\Omega)$$

$m(x_0) \in 8\pi\mathbf{N}$

3. $v_k \to -\infty$ loc. unif. in Ω



Comments

1. mass quantization for variable coefficients without boundary condition
2. possible collapse collision
3. many applications together with the proof

prescaled analysis ...Brezis-Merle 91

linear theory $\Rightarrow$

1, 2 with $m(x_0) \ge 4\pi$ (rough estimate), 3

2... localized to $B = B(0, R)$

$-\Delta v_k = V_k(x)e^{v_k}$, $V_k(x) \ge 0$ in B

$V_k \to V$ unif. in $\overline{B}$, $\max_{\overline{B}} v_k \to +\infty$

$\max_{\overline{B}\setminus B_r} v_k \to -\infty$, $^\forall r \in (0, R)$

$\lim_k \int_B V_k e^{v_k} = \alpha$, $\int_B e^{v_k} \le C$

$\Rightarrow \alpha \in 8\pi\mathbf{N}$

Boltzmann-Poisson-Gel'fand equation

$$-\Delta v = \lambda e^{v}, \ v|_{\partial\Omega} = 0$$

$$\{(\lambda_k, v_k)\}, \ \lambda_k \to 0 \ \Rightarrow \ \text{(sub-sequence)}$$

$$\lambda_k \int_\Omega e^{v_k} \to 8\pi\ell, \ \ell = 0, 1, 2, \cdots, +\infty$$

$$0 < \ell < +\infty \ \Rightarrow \ \exists \mathcal{S} \subset \Omega, \ \sharp\mathcal{S} = \ell$$

$$v_k \to v_0 \text{ loc. unif. in } \overline{\Omega} \setminus \mathcal{S} \qquad \mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$$

$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0) \qquad x_* = (x_1^*, \cdots, x_\ell^*)$$

$$\nabla H_\ell(x_*) = 0, \ H_\ell(x_1, \ldots, x_\ell) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$$

Theorem 3 (Gladiali-Grossi-Ohtsuka-S. 14) $k \gg 1$

(augmented) Morse indices

$$\ell + \mathrm{ind}_M\{-H_\ell(x_*)\} \leq \mathrm{ind}_M(v_k)$$

$$\mathrm{ind}_M^*(v_k) \leq \ell + \mathrm{ind}_M^*\{-H_\ell(x_*)\}$$

Corollary (Gladiali-Grossi 09) x_* non-degenerate

⇒ $v_k, \ k \gg 1$ non-degenerate

Theorem 2 (Baraket-Pacard 98)

$(x_1^*, \ldots, x_\ell^*) \in \Omega \times \ldots \times \Omega$

non-degenerate critical point of $H_\ell(x_1, \ldots, x_\ell)$

∃ sequence of ℓ point blow up solutions

Remark

1. only one point blowup and ∃1 blowup spot for convex domain
2. effective bound of the number of blowup points for simply connected domain
3. domain homology and Hamiltonian (Cao 10)
4. inhomogeneous coefficients, equations on manifold, etc. (Ohtsuka-Sato-S.)
5. one-point blowup case
6. refined asymptotics with Morse index correspondence
7. asymptotic non-degeneracy in multi-blowup

3. Asymptotic non-degeneracy

$$-\Delta v = \lambda e^v \text{ in } \Omega, \ \ v|_{\partial\Omega} = 0$$

$$\lambda_k \to 0, \ \lambda_k \int_\Omega e^{v_k} \to 8\pi$$

$$v_k(x) \to 8\pi G(x, x_0), \ x \in \overline{\Omega} \setminus \{x_0\}$$ locally uniformly

$$\nabla R(x_0) = 0$$

Theorem (corollary of Theorem 3)

$x_0 \in \Omega$ non-degenerate critical point of $R(x)$

$\Longrightarrow$ $-\Delta_D - \lambda_k e^{v_k}, \ 0 < \sigma_k \ll 1$ non-degenerate

Proof. otherwise

$$\exists \lambda_k \downarrow 0, \ v_k, \ w_k, \ \ -\Delta v_k = \lambda_k e^{v_k} \text{ in } \Omega, \ \ v_k|_{\partial\Omega} = 0$$

$$-\Delta w_k = \lambda_k e^{v_k} w_k \text{ in } \Omega, \ \ w_k|_{\partial\Omega} = 0, \ \ \|w_k\|_\infty = 1$$

$$v_k(x_k) = \|v_k\|_\infty, \ x_k \to x_0$$

drop k

Green $$\int_{\partial\Omega} w\frac{\partial v_i}{\partial\nu} - v_i \frac{\partial w}{\partial \nu} ds = 0, \ \ v_i = \frac{\partial v}{\partial x_i}$$

scaling $$\delta_k^2 \lambda_k e^{v_k(x_k)} = 1$$

sub-sequence ~ locally uniformly in $\mathbf{R}^2$

$$\tilde{v}_k(x) = v_k(\delta_k x + x_k) - v(x_k) \to v_0(x)$$

$$\tilde{w}_k(x) = w_k(\delta_k x + x_k) \to w_0(x)$$

$$-\Delta v_0 = e^{v_0} \text{ in } \mathbf{R}^2, \ \ \int_{\mathbf{R}^2} e^{v_0} < +\infty$$

$$-\Delta w_0 = e^{v_0} w_0 \text{ in } \mathbf{R}^2, \ \ \|w_0\|_\infty \le 1$$

Liouville property – Baraket-Pacard 98

$$w_0(x) = a \cdot \frac{x}{1+|x|^2} + b\frac{8-|x|^2}{8+|x|^2}, \ a \in \mathbf{R}^2, \ b \in \mathbf{R}$$

Lemma 1 (Nagasaki-S.)

$$v_{ki} \to 8\pi \frac{\partial G}{\partial x_i}(\cdot, x_0)$$ locally uniformly $\text{(except for } x_0)$

Lemma 2 (Gladiali-Grossi 09)

$$\delta_k^{-1} w_k \to 2\pi a \cdot \nabla_{x'} G(\cdot, x_0)$$ locally uniformly

Step 1

$$w_k = \gamma_k \{G(\cdot, x_0) + o(1)\} + 2\pi \delta_k a \cdot \nabla_{x'} G(\cdot, x_0) + o(\delta_k)$$

$$\gamma_k = \int_{\Omega \cap B(x_0, R)} \lambda_k e^{v_k} w_k dx'$$

1. removable singularity theory

$w_k \to 0$ locally uniformly

2. Green's formula

$$w_k(x) = \int_\Omega G(x, x') \lambda_k e^{v_k(x')} w_k(x') dx'$$

3. localization around $x' = x_0$

4. Y.Y. Li's estimate $|x - x_0| \geq \delta^k, \ 0 < k < 1/4$

5. Taylor's expansion $G(x, x'), \ x' = x_0, \ |x' - x_0| < \delta^k$

Step 2

$$\overline{w_k}(x = (x - x_0) \cdot \nabla v_k + 2, \ -\Delta \overline{w_k} = \lambda_k e^{v_k} \overline{w_k}$$

$$\int_{\partial B_R(x_0)} \frac{\partial \overline{w_k}}{\partial \nu} w_k - \overline{w_k} \frac{\partial w_k}{\partial \nu} d\sigma = 0$$ ⟶ $\gamma_k = o(\delta_k)$

completion of the proof

non-degeneracy + Green+ $$\int_{\partial\Omega} \frac{\partial G}{\partial x_i}(x, y) \frac{\partial}{\partial \nu_x} \frac{\partial}{\partial y_j} G(x, y) ds_x = -\frac{1}{2} \frac{\partial^2 R}{\partial y_i \partial y_j}(y)$$

⟶ a=0, b=0 ⟶ $|\exists \tilde{x}_k| \to +\infty, \ w_k(\tilde{x}_k) = 1$

exclude by

1. Kelvin transformation
2. Y.Y. Li's estimate
3. maximum principle

Open questions

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \quad v|_{\partial\Omega} = 0$$

$$\{(\lambda_k, v_k)\}, \ \lambda_k \to 8\pi, \ \|v_k\|_\infty \to +\infty$$

$$v_k \to v_0 \text{ loc. unif. in } \overline{\Omega} \setminus \mathcal{S}$$

$$v_0(x) = 8\pi G(x, x_0), \ \nabla R(x_0) = 0$$

$g : B = B(0,1) \to \Omega$ conformal

$$g(z) = x_0 + \sum_{k=1}^{\infty} a_k z^k$$

$\nabla R(x_0) = 0$

$\Leftrightarrow \ a_2 = 0$

$$\exists \nabla^2 R(x_0)^{-1} \ \Leftrightarrow \ |a_3/a_1| \neq 1/3$$

$$\lambda = 8\pi + C\sigma_k + o(\sigma_k), \ \sigma_k = \frac{\lambda_k}{\int_\Omega e^{v_k}} \to 0$$

$$\frac{C}{\pi} = -|a_1|^2 + \sum_{k=3}^{\infty} \frac{k^2}{k-2} |a_k|^2$$

$|a_3/a_1| \neq 1/3, \ C \neq 0$ **Conjecture**

$\longrightarrow$ $v_k, \ k \gg 1$ $\mathcal{L}$ non-degenerate

Variation functional $$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v, v \in H_0^1(\Omega)$$

Quadratic form $$Q(\varphi, \varphi) = \frac{d^2}{ds^2} J_\lambda(v + s\varphi)\Big|_{s=0}$$

$\varphi \in H_0^1(\Omega)$

$p = \frac{\lambda e^v}{\int_\Omega e^v}]$

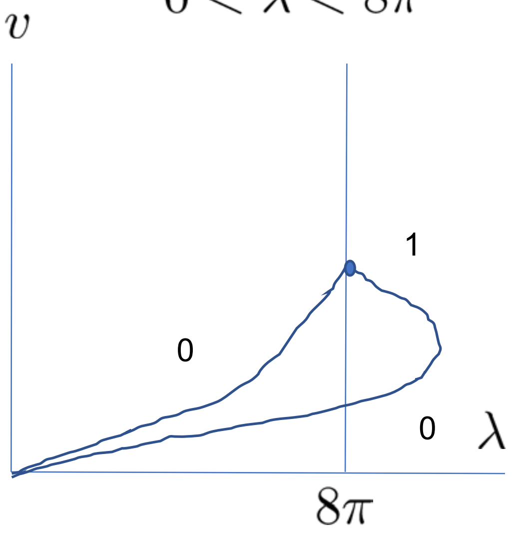
$$= (\nabla\varphi, \nabla\varphi) - \int_\Omega p\varphi^2 + \frac{1}{\lambda}\left(\int_\Omega p\varphi\right)^2$$

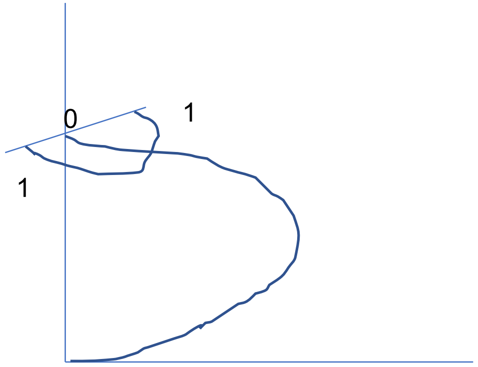
Linearized operator $$\mathcal{L}\psi = -\Delta\psi - p\psi + \frac{1}{\lambda}\left(\int_\Omega p\psi\right) p$$

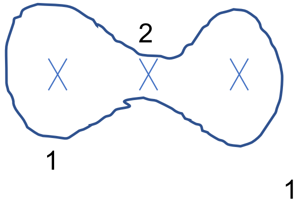
$$D(\mathcal{L}) = H_0^1(\Omega) \cap H^2(\Omega)$$

Theorem 3 (S. 92, Bartoulucci-Lin 15)

$0 < \lambda < 8\pi$ $\longrightarrow$ non-degenerate







Gladial-Grossi 04
Sato-S. 07
Grossi-Ohtsuka-S. 11
Ohtsuka-Sato-S. 13