Elliptic Theory 1

Interface Vanishing of Non-stationary Maxwell Equation

1. Introduction

Non-stationary Maxwell equation

E: electric field, B: magnetic field, J : current density, ρ : electric charge

$$\nabla \times B - \frac{\partial E}{\partial t} = J, \ \nabla \cdot E = \rho$$
$$\nabla \times E + \frac{\partial B}{\partial t} = 0, \ \nabla \cdot B = 0 \quad \text{in } \Omega$$

$(x,t) \in \Omega \subset \mathbf{R}^4$, domain $\nabla \cdot$ divergence

$x = (x_1, x_2, x_3)$, $\nabla = \nabla_x$ $\nabla \times$ rotation

Definition $\Omega \subset \mathbf{R}^n$ region with interface ⟺ $\exists \mathcal{M}, \Gamma \equiv \Omega \cap \mathcal{M} \neq \emptyset$

smooth non-compact hyper-surface without boundary

⟶ $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-, \quad \Gamma_\pm = \partial\Omega_\pm \setminus \partial\Omega (= \Gamma)$

Assumption   discontinuity of permeability, electric conductivity ⟶ interface of J, ρ

(magnetoencephalography)

conclusion   interface vanishing of some components of B, E

Theorem 1 $\Omega \subset \mathbf{R}^4$ region with interface

$$E, B \in H^1(\Omega)^3,\ J \in L^2(\Omega)^3,\ \rho \in L^2(\Omega)$$

Sobolev space

$$\nabla \times B - \frac{\partial E}{\partial t} = J,\ \nabla \cdot E = \rho$$
$$\nabla \times E + \frac{\partial B}{\partial t} = 0,\ \nabla \cdot B = 0 \quad \text{in } \Omega$$

$$\nabla \times J \in L^2(\Omega_\pm)^3$$
$$\frac{\partial J}{\partial t} + \nabla\rho \in L^2(\Omega_\pm)^3$$

in the sense of distributions

→

$$(-\partial_t^2 + \Delta_x)(\nu^0 B + \tilde{\nu} \times E) \in L^2(\Omega)^3$$
$$(-\partial_t^2 + \Delta_x)(\tilde{\nu} \cdot B) \in L^2(\Omega)$$

$$\nu = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \\ \nu^0 \end{pmatrix}, \quad \tilde{\nu} = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix}$$

in the sense of distributions

outer normal unit on Γ_-

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H2 singularities of the above components of electric magnetic fields pass through the interface with light velocity

Remark

$$(-\partial_t^2 + \Delta_x)E \in L^2(\Omega_\pm)^3$$
$$(-\partial_t^2 + \Delta_x)B \in L^2(\Omega_\pm)^3$$

Interface vanishing does not occur to all components

Corollary $\Omega \subset \mathbf{R}^3$ region with interface ν outer unit normal on Γ_-

1. $B \in H^1(\Omega)^3,\ \nabla \times B = J,\ \nabla \cdot B = 0 \text{ in } \Omega$

$\nabla \times J \in L^2(\Omega_\pm)^3 \quad \Rightarrow \quad \Delta(\nu \cdot B) \in L^2(\Omega)$

Kobayashi-S.-Watanabe 03

2. $E \in H^1(\Omega)^3,\ \nabla \times E = 0,\ \nabla \cdot E = \rho \text{ in } \Omega$

$\nabla \rho \in L^2(\Omega_\pm)^3 \quad \Rightarrow \quad \Delta(\nu \times E) \in L^2(\Omega)$

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layer potential, 3-D vector analysis, differential forms

$H^q(\Lambda^p(D))$ p-forms with Hq coefficients on $D = \Omega,\ \Omega_\pm$

$$B \in H^0(\Lambda^1(D)),\ B^\nu = (B,\nu)\nu,\ B^\tau = B - B^\nu,\ \nu = \sum_i \nu^i dx_i$$

Theorem 2 $\Omega \subset \mathbf{R}^n$ region with interface ν outer unit normal on Γ_- $B, E \in H^1(\Lambda^1(\Omega))$

1. $dB = J \in L^2(\Lambda^2(\Omega)),\ \delta B = 0,\ \delta J \in L^2(\Lambda^1(\Omega_\pm)) \ \Rightarrow \ \Delta B^\nu \in L^2(\Lambda^1(\Omega))$

2. $dE = 0,\ \delta E = g \in L^2(\Lambda^0(\Omega)),\ dg \in L^2(\Lambda^1(\Omega_\pm)) \ \Rightarrow \ \Delta E^\tau \in L^2(\Lambda^1(\Omega))$

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strategy

1. Maxwell equation → 2-form equation on the Minkowski space
2. Interface vanishing of 2-form on Euclidean space
3. Interface vanishing on Minkowski metric

obstruction

1. Normal and tangential components of 2-forms are not defined
2. Traces on interface are beyond functions for $d\omega,\ \delta\omega;\ \omega \in H^1(\Lambda^p(D))$

Theorem 3

$\Omega \subset \mathbf{R}^n$ region with interface ν outer unit normal on Γ_- $\omega \in H^1(\Lambda^2(\Omega))$

$$d\omega = \theta,\ \delta\omega = 0,\ \delta\theta \in L^2(\Lambda^2(\Omega_\pm)) \ \Rightarrow\ \Delta(\nu, \hat{\omega}^i) \in L^2(\Lambda^0(\Omega)),\ 1 \leq i \leq n$$

$$\hat{\omega}^i = \sum_\ell \tilde{\omega}^{\ell i} dx_i, \quad \tilde{\omega}^{ij} = \begin{cases} \omega^{ij}, & i < j \\ 0, & i = j \\ -\omega^{ij}, & i > j \end{cases}, \quad \omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j$$

Related references

1. T. Suzuki, Mean Field Theories and Dual Variation, 2nd edition, Mathematical Structures of the Mesoscopic Model, Atlantis Press, Paris, 2015.
2. 鈴木貴, 数理医学入門, 共立出版, 2015.

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2. Preliminaries

$D \subset \mathbf{R}^n$ open set

$H^q(\Lambda^p) = H^q(\Lambda^p(\Omega)) = \{p\text{-forms} \mid \text{coefficients are in } H^q\}$

$\Lambda^p = \Lambda^p(D) = \{p\text{-forms}\},\ \wedge$ wedge product, $d : \Lambda^p \to \Lambda^{p+1}$ outer derivative

$L^2(\Lambda^0) = L^2(D)$

$\alpha = \sum_\ell \alpha^\ell dx_\ell,\ \beta = \sum_\ell \beta^\ell dx_\ell$ 1-forms $\Longrightarrow$ $(\alpha, \beta) = \sum_\ell \alpha^\ell \beta^\ell$

$\lambda = \alpha_1 \wedge \cdots \wedge \alpha_p,\ \mu = \beta_1 \wedge \cdots \wedge \beta_p$ p-forms $\Longrightarrow$ $(\lambda, \mu) = \det\ ((\alpha_i, \beta_j))_{i,j}$

$* : \Lambda^p(D) \to \Lambda^{n-p}(D)$ Hodge operator $\quad \omega \wedge \tau = (*\omega, \tau)\ dx_1 \wedge \cdots \wedge dx_n,\ \omega \in \Lambda^p(D),\ \tau \in \Lambda^{n-p}(\Lambda)$

$\longrightarrow\ *(dx_{j_1} \wedge \cdots \wedge dx_{j_p}) = \text{sgn}\ \sigma \cdot dx_{j_{p+1}} \wedge \cdots \wedge dx_{j_n},\ \sigma : (1, \cdots, n) \mapsto (j_1, \cdots, j_n)$

co-derivative

$\delta = (-1)^p *^{-1} d* : \Lambda^p(D) \to \Lambda^{p-1}(D)$

$$B = \sum_i B^i dx_i$$

$$\Rightarrow\ \delta B = -\sum_i B^i_i$$

$$\omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j$$

$$\Rightarrow\ \delta\omega = -\sum_{i,\ell} \tilde{\omega}^{\ell i}_\ell dx_i$$

$$\tilde{\omega}^{ij} = \begin{cases} \omega^{ij}, & i < j \\ 0, & i = j \\ -\omega^{ij}, & i > j \end{cases}$$

Laplacian

$-\Delta = \delta d + d\delta : \Lambda^p \to \Lambda^p$

$D \subset \mathbf{R}^n$ Lipschitz domain $\exists\gamma : H^1(D) \to H^{1/2}(\partial D)$ trace operator $H^{1/2}(\partial D) \cong H^1(D)/H_0^1(\Omega)$

$\longrightarrow$ $C^\infty(\overline{D}) \subset H^1(D)$ dense write $\varphi|_{\partial D} = \gamma\varphi, \quad \varphi \in H^1(D)$

ν outer unit normal vector

$\nu ds = (*dx_1, \cdots, *dx_n)$ vector area element

Lemma 1 $B \in \Lambda^1(D),\ C \in \Lambda^2(D)$ $\longrightarrow$ $*B = (B,\nu)\ ds,\ \ B \wedge *C = (\nu \wedge B, C)\ ds$

write $\displaystyle\int_D \cdots\ dx_1 \wedge \cdots \wedge dx_n = \int_D, \qquad \int_{\partial D} \cdots\ ds = \int_{\partial D},$

Lemma 2 $\varphi \in H^1(\Lambda^0)$, $B \in H^1(\Lambda^1)$, $J \in H^1(\Lambda^2)$ $\longrightarrow$

$$\int_D (\delta B, \varphi) = \int_D (B, d\varphi) - \int_{\partial D} (B,\nu)\varphi \qquad \text{Gauss}$$

$$\int_D (dB, J) = \int_D (B, \delta J) + \int_{\partial D} (\nu \wedge B, J) \qquad \text{Stokes}$$

Lemma 3 $\quad p \in H^1(\Lambda^0) \quad H^{-1/2}(\partial D) = H^{1/2}(\partial D)'$

1. $\Delta p \in H^1(D)' \;\Rightarrow\; (dp, \nu)|_{\partial D} \in H^{-1/2}(\partial D)$

$$\langle (dp,\nu), \varphi \rangle = \int_D (dp, d\varphi) + \langle \Delta p, \varphi \rangle, \;\forall \varphi \in H^1(D)$$

2. $\nu \wedge dp|_{\partial D} \in H^{-1/2}(\Lambda^2(\partial D))$

$$\langle \nu \wedge dp, J \rangle = -\int_D (dp, \delta J), \;\forall J \in H^1(\Lambda^2(D))$$

$\Omega \subset \mathbf{R}^n$ region with interface $\quad \nu$ outer unit normal on Γ_-

Notation $\quad$ f: 0- form $\quad f_i = \dfrac{\partial f}{\partial x_i}$

identify $\quad$ 1-form ⟷ vector field

$$(\nu, d)f = (\nu, df) = \sum_i \nu^i f_i = \frac{\partial f}{\partial \nu}$$

Lemma 4

$$p \in H^0(\Lambda^0(\Omega)) \;\Rightarrow\; [\nu \wedge dp]_-^+ = 0, \;\; H^{-1/2}(\Lambda^2(\Gamma))$$

$$[\nu \wedge dp]_-^+ = \nu \wedge dp|_{\Gamma_+} - \nu \wedge dp|_{\Gamma_-}$$

proof $\quad J \in C_0^\infty(\Omega) \quad$ 2-form

$$\pm\langle \nu \wedge dp, J \rangle = \int_{\Omega_\pm} (dp, \delta J)$$

$$\longrightarrow \quad [\langle \nu \wedge dp, J \rangle]_-^+ = \int_\Omega (dp, \delta J)$$

Stokes

$$\int_\Omega (dB, J) = \int_\Omega (B, \delta J) + \int_{\partial\Omega} (\nu \wedge B, J)$$

$B = dp, \; p \in H^2(\Omega) \quad \longrightarrow \quad \int_\Omega (dp, \delta J) = 0$

$C^\infty(\overline{D}) \subset H^1(D) \quad$ dense

Lemma 5 $\omega=\sum_{i<j}\omega^{ij}dx_i\wedge dx_j\in H^1(\Lambda^2(\Omega))$

$$\hat\omega^i=\sum_\ell\tilde\omega^{\ell i}dx_i,\quad \tilde\omega^{ij}=\begin{cases}\omega^{ij}, & i<j\\ 0, & i=j\\ -\omega^{ij}, & i>j\end{cases}$$

→

$$\left[\delta\omega+\sum_i[(\nu,d)(\nu,\hat\omega^i)dx_i\right]_-^+=0\quad\text{in } H^{-1/2}(\Gamma)$$

Notation $A\sim B\Leftrightarrow A-B\in H^1(\Omega)$

→ $[A-B]_-^+=0\quad\text{in } H^{1/2}(\Gamma)$

Proof show;

$$\left[\sum_\ell\tilde\omega^{\ell i}_\ell-\frac{\partial}{\partial\nu}(\nu,\hat\omega^i)\right]_-^+=0\ \text{ in } H^{-1/2}(\Gamma),\ 1\le i\le n$$

i: fix $B^\ell=\tilde\omega^{\ell i},\ B=\sum_\ell B^\ell dx_\ell(=\hat\omega^i)$

→ $\sum_\ell\tilde\omega^{\ell i}_\ell-\frac{\partial}{\partial\nu}(\nu,\hat\omega^i)=\sum_\ell\{B^\ell_\ell-\nu^\ell(\nu,B)_\ell\}$

$(\nu,B)=\sum_k\nu^kB^k$

$$\sum_\ell\tilde\omega^{\ell i}_\ell-\frac{\partial}{\partial\nu}(\nu,\hat\omega^i)\sim\sum_\ell B^\ell_\ell-\sum_{\ell,k}\nu^\ell\nu^kB^k_\ell$$

$$=\sum_\ell B^\ell_\ell-\sum_{k,\ell}\nu^k\nu^\ell B^\ell_k=\sum_\ell\{B^\ell_\ell-\nu^\ell(\nu,d)B^\ell\}$$

$p=B^\ell$ → $B^\ell_\ell-\nu^\ell(\nu,d)B^\ell=p_\ell-\nu^\ell(\nu,d)p$

$$=\sum_k\{(\nu^k)^2p_\ell-\nu^\ell\nu^kp_k\}=\sum_k\nu^k(\nu^kp_\ell-\nu^\ell p_k)$$

Lemma 4

$$[B^\ell_\ell-\nu^\ell(\nu,d)B^\ell]_-^+=\sum_k\nu^k\left[\nu^kp_\ell-\nu^\ell p_k\right]_-^+=0$$

on $H^{-1/2}(\Gamma)$

Proof of Theorem 3

$\omega \in H^1(\Lambda^2(\Omega))$

$$d\omega = \theta \in L^2(\Omega), \delta\theta \in H^0(\Lambda^2(\Omega_\pm))$$

$$\Rightarrow \ -\Delta\omega = (d\delta + \delta d)\omega = \delta\theta \in L^2(\Omega_\pm)$$

$$-\Delta(\nu, \hat{\omega}^i) = \exists h^i_\pm \text{ in } L^2(\Omega_\pm)$$
$$\exists h^i \in L^2(\Omega),\ h^i = h^i_\pm \text{ in } \Omega_\pm$$

Lemma 5

$$\delta\omega = 0 \ \Rightarrow \ \left[\frac{\partial}{\partial\nu}(\nu, \hat{\omega}^i)\right]_-^+ = 0 \text{ in } H^{-1/2}(\Gamma)$$

Gauss

$$\int_\Omega h^i\varphi = \int_\Omega (-\Delta\varphi)\cdot(\nu, \hat{\omega}^i),\ \forall\varphi \in C_0^\infty(\Omega)$$

➡ $-\Delta(\nu, \hat{\omega}^i) = h^i \in L^2(\Omega)$

Proof of Theorem 1

$\mathbf{R}^4 \cong \mathbf{R}^{3,1}$ Minkowski space

$$\alpha = \sum_{i=1}^3 \alpha^i dx_i + \alpha^0 dx_0,\ \beta = \sum_{i=1}^3 \beta^i dx_i + \beta^0 dx_0 \quad \text{1-forms}$$

$$(\alpha, \beta) = -\sum_{i=1}^3 \alpha^i\beta^i + \alpha^0\beta^0 \qquad x = (x_1, x_2, x_3),\ t = x_0$$

$$d\delta + \delta d = -\frac{\partial^2}{\partial t^2} + \Delta_x : \Lambda^p(D) \to \Lambda^p(D)$$

Maxwell equation $\quad d\omega = 0,\ d * \omega = -j \text{ in } \Omega$

$$\omega = E^1 dx_0 \wedge dx_1 + E^2 dx_0 \wedge dx_2 + E^3 dx_0 \wedge dx_3$$
$$-B^1 dx_2 \wedge dx_3 - B^2 dx_3 \wedge dx_1 - B^3 dx_1 \wedge dx_2$$
$$j = J^1 dx_0 \wedge dx_2 \wedge dx_3 + J^2 dx_0 \wedge dx_3 \wedge dx_1$$
$$+J^3 dx_0 \wedge dx_1 \wedge dx_2 + \rho dx_1 \wedge dx_2 \wedge dx_3$$

$$E = \begin{pmatrix} E^1 \\ E^2 \\ E^3 \end{pmatrix}, \quad B = \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}, \quad J = \begin{pmatrix} J^1 \\ J^2 \\ J^3 \end{pmatrix}.$$