Smoluchowski Poisson Equation 1

Symmetry of Action-Reaction v.s. Duality of Field-Particles

1. Action Reaction Law

Euler's equation of motion

$$v_t + (v \cdot \nabla) v = -\nabla p$$

$$\nabla \cdot v = 0, \ \ \nu \cdot v|_{\partial\Omega} = 0$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} \end{pmatrix}$$

$$v = \begin{pmatrix} v^1 \\ v^2 \\ 0 \end{pmatrix}, \ v^1 = v^1(x_1, x_2, t), \ v^2 = v^2(x_1, x_2, t)$$

2D $\longrightarrow$ $$\nabla \times v = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}, \ \omega = \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}$$

vortex equation

$$\omega_t + \nabla \cdot (v\omega) = 0, \nabla \cdot v = 0$$

stream function (on simply connected domain)

$$\nabla \cdot v = \frac{\partial v^1}{\partial x_1} + \frac{\partial v^2}{\partial x_2} = 0 \ \longrightarrow \ v^1 = \frac{\partial \psi}{\partial x_2}, \ v^2 = -\frac{\partial \psi}{\partial x_1}$$

$$\omega_t + \nabla \cdot (\omega \nabla^\perp \psi) = 0, \ -\Delta \psi = \omega$$

boundary condition $\nu \cdot v|_{\partial\Omega} = 0 \ \longrightarrow \ \psi|_{\partial\Omega} = \text{constant}$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}, \ \nabla^\perp = \begin{pmatrix} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{pmatrix}$$

$$\omega_t + \nabla \cdot (\omega \nabla^\perp \psi) = 0, \ \psi(\cdot, t) = \int_\Omega G(\cdot, x')\omega(x', t)dx'$$

Green function $-\Delta_x G(x, x') = \delta_{x'}(dx), \ \ G(x, x')|_{x \in \partial\Omega} = 0$

symmetry $G(x, x') = G(x', x)$

$$\omega_t + \nabla \cdot (\omega \nabla^\perp \psi) = 0$$

$$\psi = \int_\Omega G(\cdot, x')\omega(x', t)\ dx'$$

weak formulation $\quad \varphi \in C^1(\overline{\Omega}),\ \ \varphi|_{\partial\Omega} = 0$

$$\frac{d}{dt}\int_\Omega \varphi\omega = \frac{1}{2}\int_{\Omega\times\Omega} \rho_\varphi(x, x')\omega \otimes \omega\ dxdx'$$

$$\omega \otimes \omega = \omega(x, t)\omega(x', t) \qquad G(x, x') = G(x', x)$$

$$\begin{aligned} \rho_\varphi(x, x') &= \nabla\varphi(x) \cdot \nabla_x^\perp G(x, x') + \nabla\varphi(x') \cdot \nabla_{x'}^\perp G(x, x') \\ &\in L^\infty(\Omega \times \Omega) \end{aligned}$$

$$G(x, x') \approx \Gamma(x - x'),\ \Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

system of point vortices $\quad \omega(dx, t) = \sum_{i=1}^N \alpha_i \delta_{x_i(t)}(dx)$

$\varphi = |x - a|^2 \varphi_{x_0, R}$ local second moment

p.v. ⟶ Kirchhoff equation $\quad \dfrac{dx_i}{dt} = \nabla_{x_i}^\perp H_N$

point vortices Hamiltonian

$$H_N(x_1, \ldots, x_N) = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i\alpha_j G(x_i, x_j)$$

Robin function $\quad R(x) = \left[G(x, x') + \dfrac{1}{2\pi}\log|x - x'|\right]_{x'=x}$

Point Vortex Mean Field ~ Kinetic Theory

Chavanis 08 Langevin equation $\mu > 0$ mobility

$$\frac{dx_i}{dt} = \alpha \nabla_i^\perp \hat{H}_N - \mu\alpha^2 \nabla_i \hat{H}_N + \sqrt{2\nu} R_i(t),\ 1 \leq i \leq N$$

$\nu > 0$ viscosity of particles

$R_i(t)$ white noise

$$\langle R_i(t) \rangle = 0,\ \langle R_i^\alpha(t) R_j^\beta(t') \rangle = \delta_{ij}\delta_{\alpha\beta}\delta(t-t')$$

$P_N(x_1, \cdots, x_N, t)$ N-pdf

$$\frac{\partial P_N}{\partial t} + \alpha \nabla^\perp \cdot \hat{H}_N \nabla P_N = \nabla \cdot (\nu \nabla P_N + \mu\alpha^2 P_N \nabla \hat{H}_N)$$

BBGKY hierarchy $\{P_i\}_{i=1,2,\cdots,N}$

factorization (propagation of chaos)

$$P_N(x_1, x_2, \cdots, x_N, t) = \prod_{i=1}^N P_1(x_i, t)$$

high-energy limit $\mu\hat{\beta}N\alpha^2 = \nu\beta,\ \alpha N = 1,\ \omega = P_1$

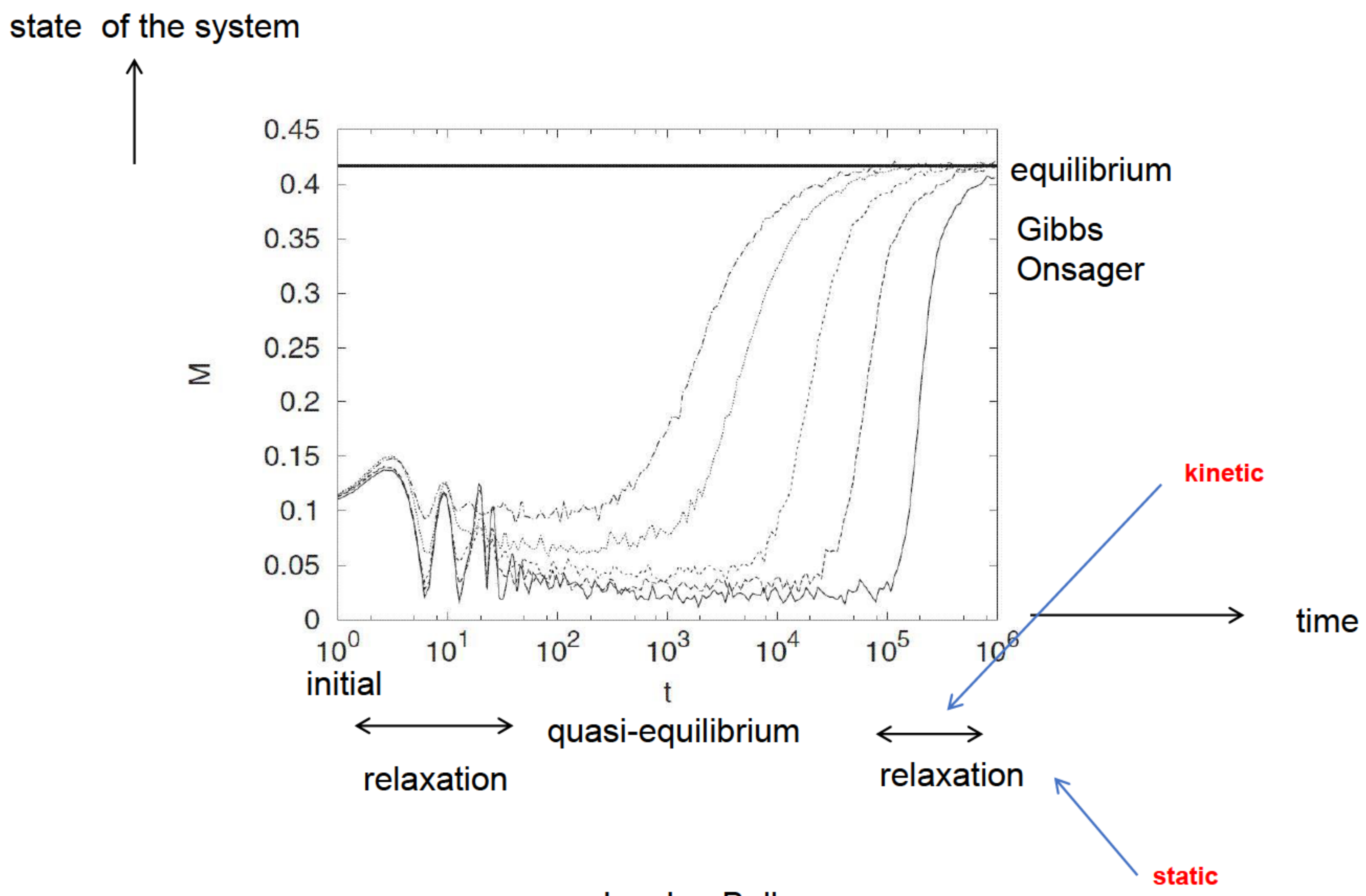
Euler-Smoluchowski-Poisson equation

$$\frac{\partial \omega}{\partial t} + \nabla^\perp \psi \cdot \nabla \omega = \nu \nabla \cdot (\nabla \omega + \beta\alpha\omega \nabla \psi)$$

$$-\Delta \psi = \omega,\ \ \psi|_{\partial\Omega} = 0$$

2. Kinetics of Newtonian Point Vortices

Hamilton System of Many Particles with Long Range Interaction



Staniscia-Chavanis-Ninno-Fanelli 09

Lynden-Bell
Miller-Robert-Sommeria

patch model

$$\omega(x,t) = \sum_{i=1}^{N_p} \sigma_i 1_{\Omega_i(t)}(x)$$

$$c(\sigma) - \log(\frac{1}{M(\sigma)} \int_\Omega p(x,0) e^{-\beta\sigma\overline{\psi}})$$

$$\zeta(x) = \log(\int_I e - -c(\sigma) - \beta\sigma\overline{\psi} d\sigma) - 1$$

static

Robert-Sommeria 91

$$\overline{\omega} = \int_I \sigma p(x,\sigma)\ d\sigma$$

$$\int_I p(x,\sigma)\ d\sigma = 1,\ \forall x$$

$$p(x,\sigma) = e^{-c(\sigma)-(\zeta(x)+1)-\beta\sigma\overline{\psi}}$$

$$-\Delta\overline{\psi} = \int_I \sigma \frac{e^{-c(\sigma)-\beta\sigma\overline{\psi}}}{\int_I e^{-c(\sigma')-\beta\sigma'\overline{\psi}} d\sigma'} d\sigma,\quad \overline{\psi}|_{\partial\Omega} = 0$$

maximal entropy production principle

$$D = D(x,t) > 0$$

diffusion coefficient

β inverse temperature

$c(\sigma)$ chemical potential

$M(\sigma)$ patch area

kinetic

Robert-Sommeria 92

$$\frac{\partial p}{\partial t} + \nabla \cdot p\overline{u} = \nabla \cdot D\left(\nabla p + \beta\left(\sigma - \overline{\omega}\right) p \nabla\overline{\psi}\right)$$

$$\overline{\omega} = \int_I \sigma p\ d\sigma = -\nabla^\perp \overline{u} = -\Delta\overline{\psi},\quad \overline{\psi}|_{\partial\Omega} = 0$$

$$\beta = -\frac{\int_\Omega D\nabla\overline{\omega}\cdot\nabla\overline{\psi}\ dx}{\int_\Omega D\left(\int_I \sigma^2 p\ d\sigma - \overline{\omega}^2\right)|\nabla\overline{\psi}|^2\ dx}$$

$$\Delta = \nabla^\perp \cdot \nabla^\perp$$

$$\overline{u} = \nabla^\perp \psi$$

point vortex model

$$\omega(dx,t)=\sum_{i=1}^{N}\alpha_i\delta_{x_i(t)}(dx)$$

$$\alpha_i=\tilde{\alpha}_i\alpha,\ \tilde{\alpha}_i\in I=[-1,1]$$

$$\alpha N=1,\ N\to\infty,\ H_N=\text{constant},\ \alpha^2 N\beta_N=\beta$$

$$\rho^{\tilde{\alpha}}(x)=\lim_{N\to\infty}\int_{\Omega^{N-1}}\mu_N^{\beta_N}(dx,dx_2,\cdots dx_N)$$

$$\mu_N^{\beta_N}(dx_1,\cdots dx_N)=\frac{1}{Z(N,\beta_N)}e^{-\beta_N H_N}dx_1\cdots dx_N$$

stationary

$$\overline{\omega}(x)=\int_I\tilde{\rho}^{\tilde{\alpha}}(x)P(d\tilde{\alpha}),\ x\in\Omega$$

$$\int_\Omega\rho^{\tilde{\alpha}}(x)\ dx=1,\quad\forall\tilde{\alpha}\in I$$

→ Sawada-S. 08

$$\overline{\omega}=-\Delta\overline{\psi}$$

$$-\Delta\overline{\psi}=\int_I\tilde{\alpha}\frac{e^{-\beta\tilde{\alpha}\overline{\psi}}}{\int_\Omega e^{-\beta\tilde{\alpha}\overline{\psi}}dx}P(d\tilde{\alpha}),\quad\overline{\psi}|_{\partial\Omega}=0$$

one component case $P(d\tilde{\alpha}) = \delta_1(d\tilde{\alpha}),\ D(x,t) = 1$

$$\omega_t + \nabla\cdot\omega\nabla^\perp\psi = \nabla\cdot(\nabla\omega + \beta\omega\nabla\psi)$$

$$\frac{\partial\omega}{\partial\nu} + \beta\omega\frac{\partial\psi}{\partial\nu}\Big|_{\partial\Omega} = 0,\ \omega|_{t=0} = \omega_0(x) \geq 0$$

$$-\Delta\psi = \omega, \quad \psi|_{\partial\Omega} = 0$$

micro-canonical → $\beta = -\frac{\int_\Omega \nabla\omega\cdot\nabla\psi}{\int_\Omega \omega|\nabla\psi|^2}$

$\|\omega(\cdot,t)\|_1 = \lambda$ mass

$\frac{d}{dt}(\omega,\psi) = 0$ energy

$$\frac{d}{dt}\int_\Omega \Phi(\omega) = -\int_\Omega \omega|\nabla(\log\omega + \beta\psi)|^2 \leq 0$$

$\Phi(s) = s(\log s - 1) + 1$ entropy

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canonical → β constant

2D Brownian vortices
Chavanis 08

Euler-Smolchowski-Poisson

$\beta = -8\pi/\lambda$ blowup threshold
S. 14

$\|\omega(\cdot,t)\|_1 = \lambda$ mass

$$\frac{d\mathcal{F}}{dt} = -\int_\Omega \omega|\nabla(\log\omega + \beta\psi)|^2 \leq 0$$

$\mathcal{F}(\omega) = \int_\Omega \Phi(\omega) - \frac{1}{2}((-\Delta)^{-1}\omega,\omega)$ free energy

K. Sawada (2015)

point vortex model ← patch model

kinetic

$$\frac{\partial \rho^{\tilde{\alpha}}}{\partial t} + \nabla \cdot \rho^{\tilde{\alpha}} \overline{u} = \nabla \cdot D \left(\nabla \rho^{\tilde{\alpha}} + \beta \tilde{\alpha} \rho^{\tilde{\alpha}} \nabla \overline{\psi} \right)$$

$$\overline{\omega} = \int_I \tilde{\alpha} \rho^{\tilde{\alpha}} P(d\tilde{\alpha}) = -\nabla^{\perp} \overline{u} = -\Delta \overline{\psi}, \quad \overline{\psi}\big|_{\partial\Omega} = 0$$

$$\beta = -\frac{\int_\Omega D \nabla \overline{\omega} \cdot \nabla \overline{\psi} \; dx}{\int_\Omega D \int_I \tilde{\alpha}^2 \rho^{\tilde{\alpha}}(x) P(d\tilde{\alpha}) |\nabla \overline{\psi}|^2 \; dx}$$

← Robert-Sommeria 92

static

Joyce-Montgomery 73 ･･･Sawada-S. 08 ← Robert-Sommeria 91

3. Methods of Mathematical Modeling

Coarsening key factors

1. supply-consumption

$$u_t = \alpha,\ v_t = -\beta$$

2. production-annihilation

$$u_t = \alpha u,\ v_t = -\beta v$$

3. transport

flux

$$u_t = -\nabla \cdot j,\ j = \text{mass} \times \text{velocity}$$

4. gradient

$$j = -d_u \nabla u$$ diffusion

$$j = d_v u \nabla v$$ chemotaxis

5. chemical reaction

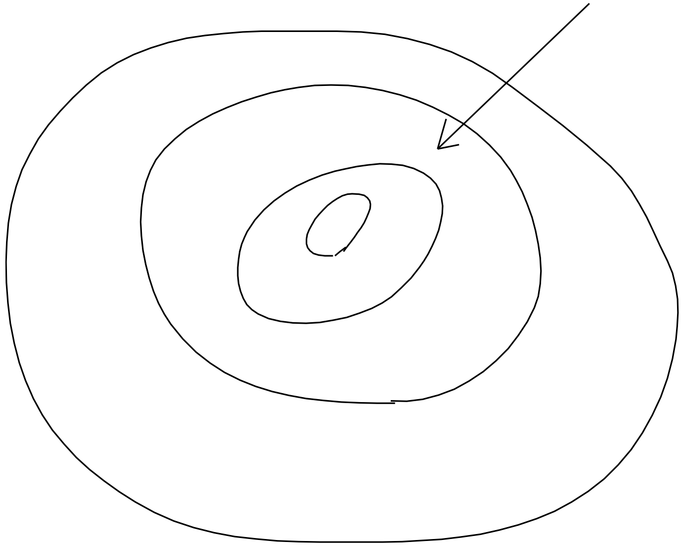
$$A + B \to C\ (k)$$

(mass action)

$$\frac{d[A]}{dt} = -k[A][B]$$

$$\frac{d[B]}{dt} = -k[A][B]$$

$$\frac{d[C]}{dt} = k[A][B]$$



$$u_t = D\Delta u$$

$$u_t = \nabla \cdot (D \nabla u)$$

$$u_t = \Delta(Du)$$?

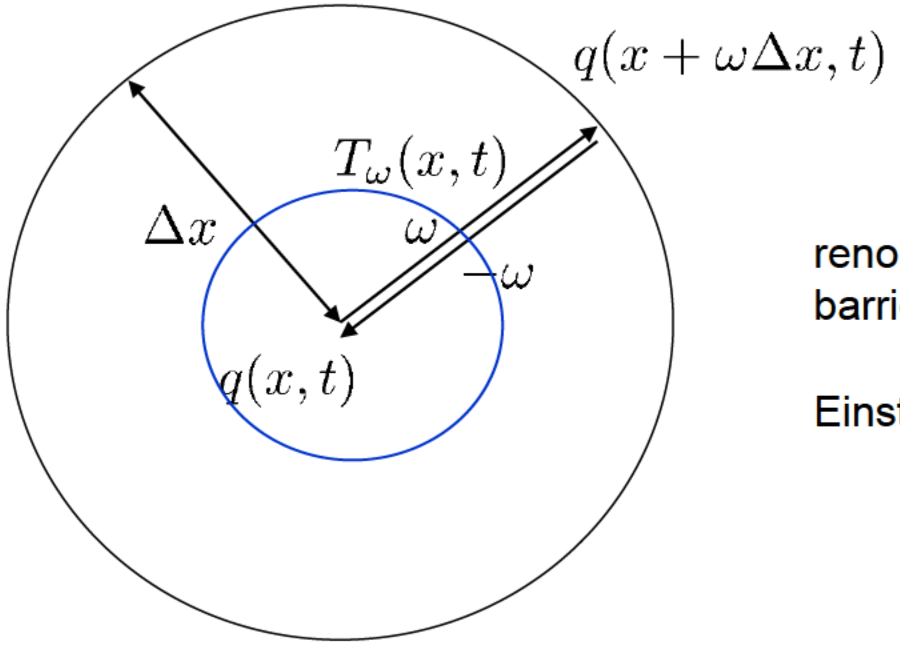
Averaging particle movements

$$S^{N-1} = \{\omega \in \mathbf{R}^N \mid |\omega| = 1\}$$

$q = q(x,t)$ particle density $\qquad T = T_\omega(x,t)$ transient probability

master equation

$$\frac{q(x,t+\Delta t) - q(x,t)}{\Delta t} = \int_{S^{N-1}} T_{-\omega}(x+\omega\Delta x, t) q(x+\omega\Delta x, t) d\omega - \int_{S^{N-1}} T_\omega(x,t) d\omega \cdot q(x,t)$$



$$\int_{S^{N-1}} T_\omega(x,t) d\omega = \tau^{-1}$$ τ mean waiting time

renormalization barrier

$$\tau T_\omega(x,t) = \frac{T(x+\omega\frac{\Delta x}{2}, t)}{\int_{S^{N-1}} T(x+\omega'\frac{\Delta x}{2}, t) d\omega'}, \quad T = T(x,t)$$

Einstein formula

$$\tau^{-1}(\Delta x)^2 = 2nD$$

space dimension $\qquad$ diffusion coefficient

→ Smoluchowski equation

$$\frac{\partial q}{\partial t} = D\nabla \cdot (\nabla q - q\nabla \log T)$$

spatially-temporally continuous distribution function
Ichikawa-Rouzi-S. 12

S. Mean Field Theories and Dual Variation, 2nd edition, Atlantis Press, 2015

system	consistency	dynamics	ensemble
isolated	energy	entropy	micro-canonical
closed	temperature	Helmholtz free energy	canonical
open	pressure	Gibbs free energy	grand-canonical

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particle density **duality** $\longleftrightarrow$ **field potential**

Smoluchowski

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega} = 0$$

Poisson

$$v = (-\Delta)^{-1}u, \ \int_\Omega G(\cdot, x')u(x')dx'$$

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \ \frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega} = 0, \ \int_\Omega v = 0$$

symmetry

Helmholtz fee energy

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle$$

$$\delta\mathcal{F}(u) = \log u - (-\Delta)^{-1}u$$

Model (B) equation

$$u_t = \nabla u \cdot \nabla \delta\mathcal{F}(u), \ \frac{\partial}{\partial \nu}\delta\mathcal{F}(u)\bigg|_{\partial\Omega} = 0$$

total mass conservation
free energy decreaseing

Duality of Field and Particles

X Banach space/$\mathbf{R}$

$F : X \to (-\infty, +\infty]$ prop. c'x l.s.c.

$\Rightarrow$

Legendre transformation

$F^* : X^* \to (-\infty, +\infty]$ prop. c'x l.s.c.

$$F^*(p) = \sup_{x \in X} \{\langle x, p \rangle - F(x)\}$$

Fenchel-Moreau duality

$$F^{**} = F$$

$$F^{**}(x) = \sup_{p \in X^*} \{\langle x, p \rangle - F^*(p)\}$$

Toland duality 78, 79

$F, G : X \to (-\infty, +\infty]$ prop. c'x l.s.c.

$$J(x) = G(x) - F(x)$$

$$J^*(p) = F^*(p) - G^*(p)$$

$$L(x, p) = F^*(p) + G(x) - \langle x, p \rangle$$

... Lagrange function

$$\inf_{X \times X^*} L = \inf_X J = \inf_{X^*} J^*$$

free energy ⟵ Legendre transformation / duality ⟶ field functional

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle (-\Delta)^{-1} u, u \rangle$$

$u \geq 0, \ \|u\|_1 = \lambda$

particle distribution

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v + \lambda(\log \lambda - 1)$$

potential density

$v \in H^1(\Omega), \ \int_\Omega v = 0, \ \|u\|_1 = \lambda$

楕円型統一理論

領域の形状

近平衡力学系標準理論

孤立系の均質化

ネットワークからの創発

歪対称性が導く特異性の解消

場と粒子の双対性

Hamiltonian　無限時間爆発　有限時間爆発

凝縮
臨界
爆発

量子化する爆発機構

単純性

エントロピーの排出
（創発性）

重複
衝突

Boltzmann-Poisson方程式　→　Smoluchowski Poisson 方程式

閉鎖系
正準集団

非線形スペクトル理論
（自己組織化のポテンシャル）

質量保存
自由エネルギー

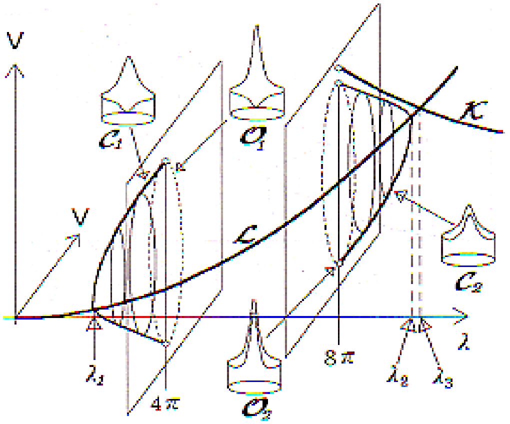
遠隔作用
作用反作用

多数粒子
平均場極限
カオスの伝播

循環的階層

点渦系

孤立系
小正準集団



S. Liouville’s Theory in Linear and Nonlinear PDEs (Springer)

Smoluchwoski Poisson Equation 2

Potentials of Self-Organization

1. multi-scale model

Keller-Segel 70

$$u_t = \nabla \cdot (d_1(u,v)\nabla u) - \nabla \cdot (d_2(u,v)\nabla v)$$
$$v_t = d_v \Delta v - k_1 vw + k_{-1} p + f(v)u$$
$$w_t = d_w \Delta w - k_1 vw + (k_{-1} + k_2)p + g(v,w)u$$
$$p_t = d_p \Delta p + k_1 vw - (k_{-1} + k_2)p$$

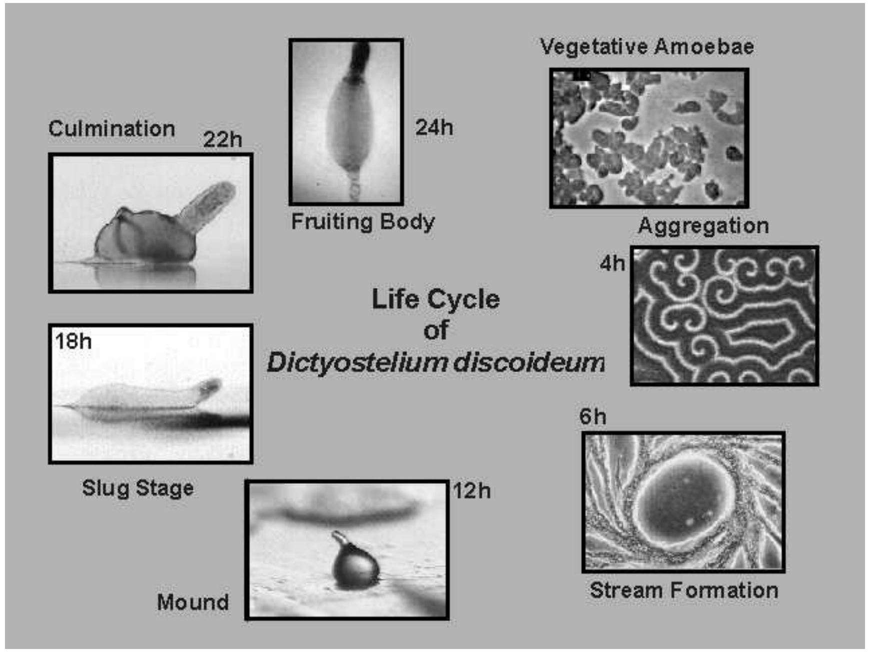
$u = u(x,t)$	cellular slime molds
$v = v(x,t)$	chemical substances
$w = w(x,t)$	enzymes
$p = p(x,t)$	complices

1. transport, gradient

   (a) diffusion u, v, w, p

   (b) chemotaxis $v \to u$

2. production $u \to (v, w)$



moving clustered cells

aggregating cells

3. chemical reaction v, w, p

$$V + W \underset{\leftarrow (k_{-1})}{\overset{(k_1) \rightarrow}{}} P \overset{(k_2)}{\rightarrow} W + A$$

$$v_t = -k_1 vw + k_{-1} p$$
$$w_t = -k_1 vw + (k_{-1} + k_2)p$$
$$p_t = k_1 vw - (k_{-1} + k_2)p$$

Michaelis-Menten reduction

$$v_t = -k_1 vw + k_{-1} p$$
$$w_t = -k_1 vw + (k_{-1} + k_2) p$$
$$p_t = k_1 vw - (k_{-1} + k_2) p$$

$k_1 vw - (k_{-1} + k_2) p = 0$ quasi-static

$w + p = c$ mass conservation

$\Longrightarrow$

$$u_t = \nabla \cdot (d_1(u,v)\nabla u) - \nabla \cdot (d_2(u,v)\nabla v)$$
$$v_t = d_v \Delta v - k(v)v + f(v)u$$

$$k(v) = \frac{ck_1k_2}{(k_{-1} + k_2) + k_1 v}$$

Nanjundiah 73

$d_1(u,v),\ k(v),\ f(v)$ constants

$d_2(u,v) = u\chi'(v)$ flux=mass $\times$ velocity

sensitivity

$$u_t = d_u \Delta u - \nabla \cdot (u \nabla \chi(v))$$
$$v_t = d_v \Delta v - b_1 v + b_2 u$$

Childress-Percus 81, Jager-Luckhaus 92

$$u_t = \nabla \cdot (\nabla u - u\nabla v), \quad \left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$
$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \quad \int_\Omega v = 0, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

Smoluchowski-Poisson equation

Competitive System

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

$$\tau_1 \frac{\partial u_1}{\partial t} = d_1 \Delta u_1 - \chi_1 \nabla \cdot u_1 \nabla v, \quad d_1 \frac{\partial u_1}{\partial \nu} - \chi_1 u_1 \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad u_1|_{t=0} = u_{10}(x) > 0$$

$$\tau_2 \frac{\partial u_2}{\partial t} = d_2 \Delta u_2 - \chi_2 \nabla \cdot u_2 \nabla v, \quad d_2 \frac{\partial u_2}{\partial \nu} - \chi_2 u_2 \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad u_2|_{t=0} = u_{20}(x) > 0$$

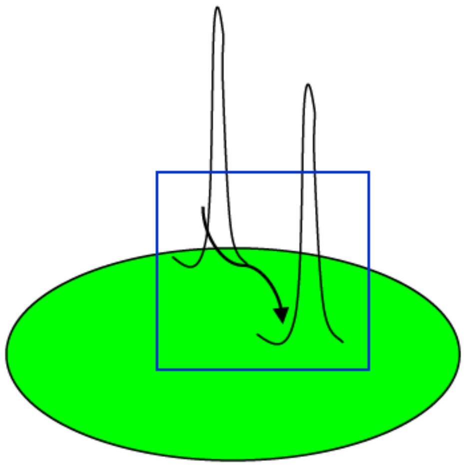
$$-\Delta v = u_1 + u_2, \quad v|_{\partial\Omega} = 0$$

other cells u_1 ⇄ chemical v (production)

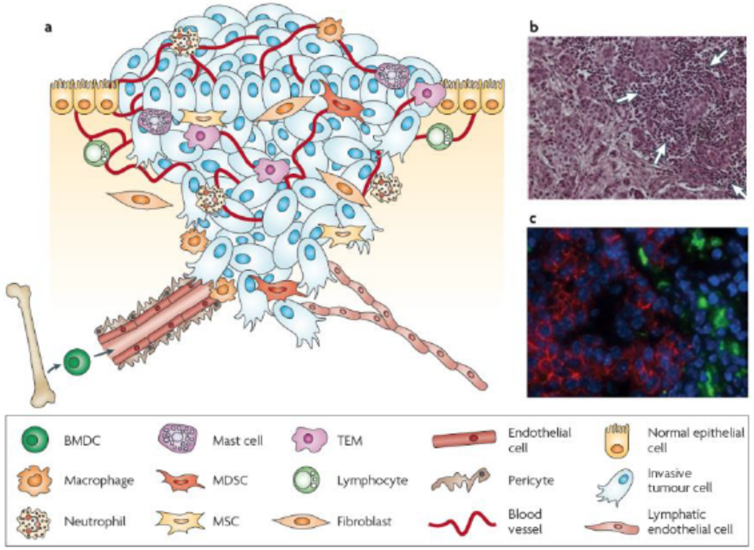
cancer cell u_2 ⇄ chemical v (chemotaxis)

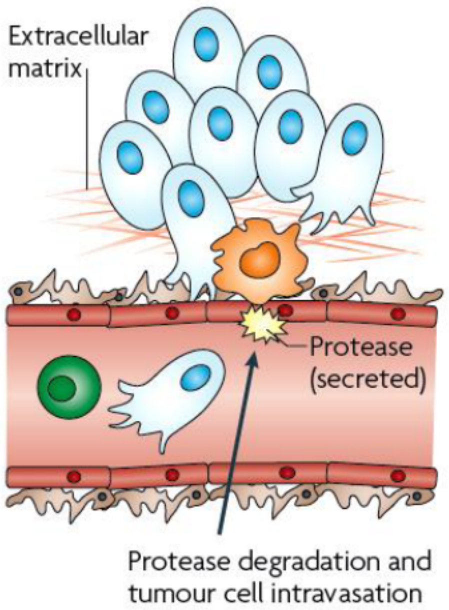
Espejo-Stevens-Velazquez 09
Espejo-Stevens-S. 12

cell sorting



tumor-associated micro environment





J. Joyce, and J. Pollard. *Nat Rev Cancer* 9: 239-252 (2009)

2. Thermo-dynamical structure

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u \nabla v)$$
$$\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \ u|_{t=0} = u_0(x) > 0$$

transport

closed system ↕

potential

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

Sire-Chavanis 02
motion of the mean field of many self-gravitating Brownian particles
kinetic equation + maximum entropy production

Chavanis 08
relaxation to the equilibrium in the point vortices BBGKY hierarchy
+ factorization

other Poisson parts

a) Debye system (DD model)

$$\Delta v = u, \quad v|_{\partial\Omega} = 0$$

global-in-time existence with compact orbit
Biler-Hebisch-Nadzieja 94

$$\|u\nabla u \cdot \nabla v\|_2 \leq C\|u\|_2 \|\nabla u\|_2 \|\nabla v\|_6$$

b) Childress-Percus-Jager-Luckhaus
model (chemotaxis)

$$-\Delta v = u - \frac{1}{|\Omega|}\int u$$
$$\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \int_\Omega v = 0$$

blowup threshold
a. Biler 98, Gajewski-Zacharias 98, Nagai-Senba-Yoshida 97
b. Nagai 01, Senba-S. 01b

SP equation $u_t = \nabla \cdot (\nabla u - u\nabla v),\ -\Delta v = u$

$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, v\right)\Big|_{\partial\Omega} = 0$$

$u = u(x,t) \geq 0$ density

$j = -\nabla u + u\nabla v$ flux (diffusion + chemotaxis)

$u_t + \nabla \cdot j = 0$ conservation law

$v = (-\Delta)^{-1}u$ potential

1. total mass conservation $\frac{d}{dt}\|u(t)\|_1 = 0$

attractive (chemotaxis, gravitation)
action at a distance (long range potential)
symmetry (action-reaction)

$G(x,x') = G(x',x)$ Green's function

2. free energy decreasing

$$\frac{d}{dt}\{\int_\Omega u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u\otimes u\} = -\int_\Omega u|\nabla(\log u - v)|^2 \leq 0$$

$u \otimes u = u(x,t)u(x',t)\ dxdx'$

3. weak form $\varphi \in C^2(\overline{\Omega}),\ \frac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} = 0$

$$\begin{aligned}\rho_\varphi(x,x') &= \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x') \\ &\in L^\infty(\Omega\times\Omega)\end{aligned}$$

$$\frac{d}{dt}\int_\Omega \varphi u(\cdot,t) = \int_\Omega \Delta\varphi\cdot u(\cdot,t) + \frac{1}{2}\int\int_{\Omega\times\Omega}\rho_\varphi(x,x')u\otimes u$$

Scaling and Variation

$$u_t = \nabla \cdot (\nabla u - u\nabla\Gamma * u)$$

$$(x,t) \in \mathbf{R}^n \times (0,T)$$

$$-\Delta\Gamma = \delta$$

self-similar transformation $u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t)$

$$\mu > 0$$

critical dimension $u_\mu(x) = \mu^2 u(\mu x), \ \mu > 0$

$$\|u\|_1 = \|u_\mu\|_1 \equiv \lambda \ \Leftrightarrow \ n = 2$$

Critical mass

$$\mathcal{F}(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2}\langle \Gamma * u, u\rangle$$

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

⟶

$$\mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi}\right)\log\mu + \mathcal{F}(u) \qquad \lambda = 8\pi$$

Trudinger-Moser inequality

$$\inf\{\mathcal{F}(u) \mid u \geq 0, \ \|u\|_1 = 8\pi\} > -\infty$$

$$u_t = \nabla \cdot (\nabla u - u\nabla v), \ -\Delta v = u$$

$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, v\right)\Big|_{\partial\Omega} = 0$$

blowup threshold

$$\lambda = \|u_0\|_1 < 8\pi \quad \Rightarrow \quad T = +\infty$$

$$\|\exists u_0\|_1 > 8\pi, \ T < +\infty$$

3. blowup of the solution

$$\frac{du}{dt} = u^2,\ u(0) = T^{-1} > 0$$

→

$$u(t) = (T-t)^{-1},\ \lim_{t\uparrow T} u(t) = +\infty$$

quantity distributed in space - time

$\Omega \subset \mathbf{R}^n$ bounded open set, $T > 0$

$u = u(x,t) : \Omega \times [0,T] \to (-\infty, +\infty]$ continuous

$$D(t) = \overline{\{x \in \Omega \mid u(x,t) = +\infty\}}$$

$$D = \bigcup_{0 \leq t \leq T} D(t) \times \{t\} \subset \Omega \times [0,T]$$

$u_t - \Delta u \geq 0$ on $\Omega \times (0,T) \setminus D$

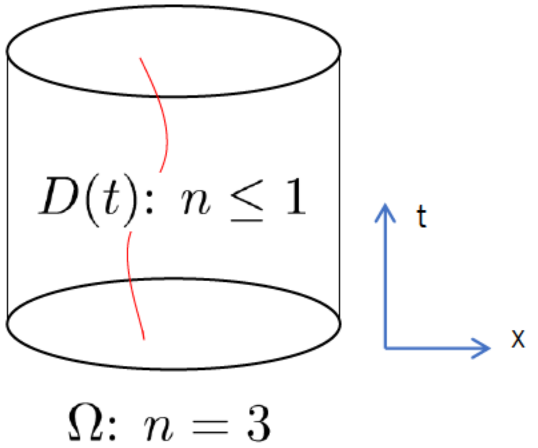
$u = u(x,t)$, Lip. conti. near in $\partial\Omega$ unif. in $t \in [0,T]$

Theorem A [F. Takahashi-S. 08]

$$\int_0^T \mathrm{Cap}_2(D(t))dt \leq \frac{L^n(\Omega)}{2}$$

~ (n-2) dimensional Hausdorff measure

Temperature infinite region enclosed in a bounded domain in a positive time interval takes a dimension lower than 2



quantized blowup mechanism – spectral level (Boltzmann-Poisson equation)

Theorem B [Nagasaki-S. 90]

$\{(\lambda_k, v_k)\}$ solution sequence

$\lambda_k \to \lambda_0 \in (0,\infty)$, $\|v_k\|_\infty \to \infty$

$\Rightarrow$

$\lambda_0 = 8\pi\ell$, $\exists \ell \in \mathbf{N}$

$\exists$sub-sequence, $\exists \mathcal{S} \subset \Omega$, $\sharp\mathcal{S} = \ell$

$v_k \to v_0$ locally uniform in $\overline{\Omega} \setminus \mathcal{S}$

$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0) \text{ singular limit}$$

$\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$ blowup set

$$\nabla_{x_i} H_\ell|_{(x_1,\cdots,x_\ell)=(x_1^*,\cdots,x_\ell^*)} = 0,\ 1 \le i \le \ell$$

$$H_\ell(x_1, \ldots, x_\ell) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$$

point vortex Hamiltonian

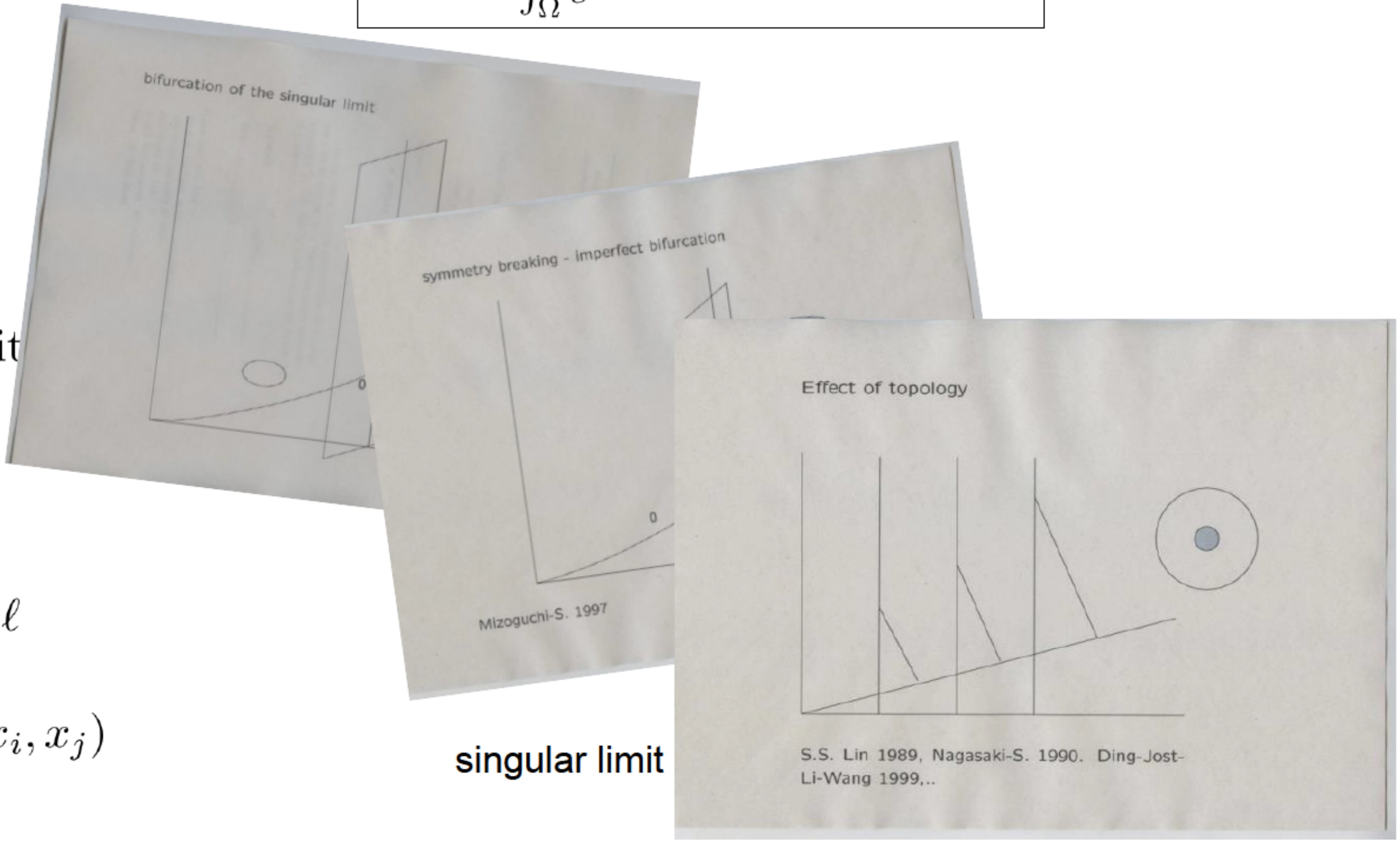
$\lambda = \|u\|_1$

duality

u $\longleftrightarrow$ v

$\Omega \subset \mathbf{R}^2$ bounded domain $\partial\Omega$ smooth
$\lambda > 0$ constant

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega,\ v = 0 \text{ on } \partial\Omega$$



singular limit

a recursive hierarchy

thermally closed system → total mass conservation
free energy decreasing

$$u \geq 0, \ \frac{d}{dt}\|u(\cdot,t)\|_1 = 0$$

$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 dx$$

stationary state →

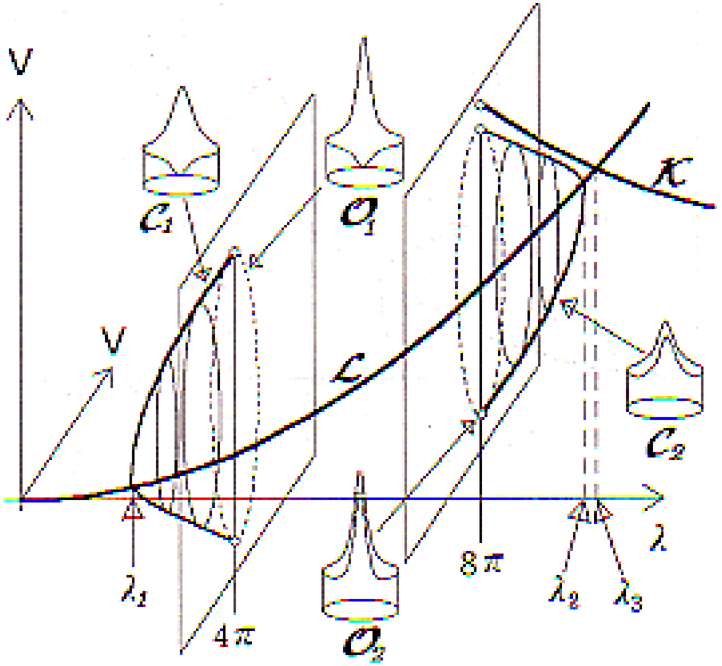
$$\log u - v = \text{constant}, \ \|u\|_1 = \lambda$$

→ $$u = \frac{\lambda e^v}{\int_\Omega e^v}$$

Poisson $$-\Delta v = u, \ v|_{\partial\Omega} = 0$$

→ $$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \ v|_{\partial\Omega} = 0$$

Boltzmann-Poisson equation



$$-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right), \ \int_\Omega v = 0$$

Senba-S. 00

blowup threshold

→

$$\lambda = 8\pi, \ 4\pi$$

interior boundary

Theorem B1 (blowup in infinite time)

$$T=+\infty,\ \limsup_{t\uparrow+\infty}\|u(\cdot,t)\|_\infty=+\infty$$

$\longrightarrow$

$$\lambda\equiv\|u_0\|_1=8\pi\ell,\ \exists\ell\in\mathbf{N}$$

$$\exists x_*\in\Omega^\ell,\ \nabla H_\ell(x_*)=0$$ recursive hierarchy

Corollary 1 $T<+\infty$ if

(1) $\lambda\notin 8\pi\mathbf{N},\ \not\exists$ stationary solution or $\mathcal{F}(u_0)\ll -1$

(2) $\lambda\in 8\pi\mathbf{N},\ \not\exists$ singular limit

Corollary 2 Ω convex $\lambda\neq 8\pi$

$\Rightarrow\ T<+\infty$ or $T=+\infty$ compact orbit

$\Downarrow$

$\exists$ stationary solution

c.f. Grossi-F. Takahashi 2010

Theorem B2 (blowup in finite time) $T<+\infty$

$$u(x,t)dx\rightharpoonup\sum_{x_0\in\mathcal{S}}m(x_0)\delta_{x_0}(dx)+f(x)dx$$

$$m(x_0)\in 8\pi\mathbf{N}$$

blowup set

$$\begin{aligned}\mathcal{S}\ &=\ \{x_0\in\overline{\Omega}\mid\exists x_k\to x_0,\ t_k\uparrow T,\ u(x_k,t_k)\to+\infty\}\\ &\subset\ \Omega\end{aligned}$$

$$0<f=f(x)\in L^1(\Omega)\cap C(\overline{\Omega}\setminus\mathcal{S})$$

S. Liouville's Theory in Linear and Nonlinear PDEs, Springer, 2021. to be published

quantized blowup mechanism in dynamical level

$G=G(x,x')$ Green's function

$R=R(x)$ Robin function

point vortex Hamiltonian

$$H_\ell(x_1,\cdots,x_\ell)=\frac{1}{2}\sum_{j=1}^{\ell}R(x_j)+\sum_{1\le i<j\le\ell}G(x_i,x_j)$$

Senba-S. 01	weak formulation monotonicity formula	formation of collapse weak solution generation
Senba-S. 02a	weak solution	instant blowup for over mass concentrated initial data
Kurokiba-Ogawa 03	scaling invariance	non-existence of over mass entire solution without concentration
Senba-S. 04	backward self-similar transformation scaling limit	
S. 05	parabolic envelope (1) scaling invariance of the scaling limit a local second moment	sub-collapse quantization collapse mass quantization
Senba-Ohtsuka-S. 07	defect measure	radially symmetric dynamics
Senba 07, Naito-S. 08	parabolic envelope (2)	type II blowup rate
S. 08	scaling back	limit equation simplification
Senba-S. 11	translation limit	concentration-cancelation simplification
S. 13a	limit equation classification	boundary blowup exclusion
S. 13b	improved regularity concentration compactness	cloud formation
S. 14	tightness	residual vanishing
S. 18	Lioville’s formula	quantization of BUIT
S. 21	outer second moment	residual vanishing in finite time

Smoluchowski-Poisson Equation 3

Hamiltonian Control in Three Phases of Time Evolution

1. The model – statistical mechanics

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

$u = u(x,t) \geq 0$ density

$j = -\nabla u + u\nabla v$ flux (diffusion v.s. chemotaxis)

$u_t + \nabla \cdot j = 0$ conservation law

$v = (-\Delta)^{-1} u$ potential

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$\left. \frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0, \ \ u|_{t=0} = u_0(x) > 0$$

2. Poisson Part

$$-\Delta v = u, \ \ v|_{\partial\Omega} = 0$$

Green's function

$$G(x,x') = G(x',x)$$

attractive (chemotaxis, gravitation)
action at a distance (long range potential)
symmetry (action-reaction)

Chavanis 08 relaxation to the equilibrium in the point vortices, kinetic equation + maximum entropy production

Sire-Chavanis 02 motion of the mean field of many self-gravitating Brownian particles, BBGKY hierarchy + factorization

canonical ensemble

1. total mass conservation $\frac{d}{dt}\|u(t)\|_1 = 0$

2. free energy decreasing

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u \otimes u$$

$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 \leq 0$$

self-similar transformation

$$u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t), \ \mu > 0$$

$$u_\mu(x) = \mu^2 u(\mu x), \ \ \mu > 0$$

$\|u\|_1 = \|u_\mu\|_1 \equiv \lambda \ \Leftrightarrow \ n = 2$ *critical dimension*

$$\mathcal{F}(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2}\langle \Gamma * u, u\rangle, \ \Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$\mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi}\right)\log\mu + \mathcal{F}(u)$ *critical mass* $\lambda = 8\pi$

Theorem A (blowup in infinite time)

$$T=+\infty,\ \limsup_{t\uparrow+\infty}\|u(\cdot,t)\|_\infty=+\infty$$

$\longrightarrow$ $\lambda\equiv\|u_0\|_1=8\pi\ell,\ \exists\ell\in\mathbf{N}$ initial mass quantization

$\exists x_*\in\Omega^\ell\setminus D,\ \nabla H_\ell(x_*)=0$ recursive hierarchy

point vortex Hamiltonian — Robin function — Green function

$$H_\ell(x_1,\cdots,x_\ell)=\frac{1}{2}\sum_j R(x_j)+\sum_{i<j}G(x_i,x_j)$$

Theorem B (blowup in finite time) $T<+\infty$

$$u(x,t)dx\rightharpoonup\sum_{x_0\in\mathcal{S}}m(x_0)\delta_{x_0}(dx)+f(x)dx\ \text{in}\ \mathcal{M}=(\overline{\Omega})C(\overline{\Omega})'$$

$m(x_0)\in 8\pi\mathbf{N}$ collapse mass quantization possibly with sub-collapse collision

$$\mathcal{S}=\{x_0\in\overline{\Omega}\mid \exists x_k\to x_0,\ t_k\uparrow T,\ u(x_k,t_k)\to+\infty\}\subset\Omega$$

blowup set — exclusion of boundary blowup

$0<f=f(x)\in L^1(\Omega)\cap C(\overline{\Omega}\setminus\mathcal{S})$ measure theoretic regular part

Corollary 1 $T<+\infty$ if

(1) $\lambda\notin 8\pi\mathbf{N},\ \nexists$ stationary solution or $\mathcal{F}(u_0)\ll-1$

(2) $\lambda\in 8\pi\ell,\ \ell\in\mathbf{N},\ \nexists$ critical point of H_ℓ

Corollary 2 Ω convex $\lambda\neq 8\pi$

$\Rightarrow$ $T<+\infty$ or $T=+\infty$ compact orbit

c.f. Grossi-F. Takahashi $\exists$ stationary solution

Poisson $v=\int_\Omega G(\cdot,x')u(x')dx'\ \Leftrightarrow\ -\Delta v=u,\ v|_{\partial\Omega}=0$

diagonal $D=\{(x_i)\in\Omega^\ell\mid \exists i\neq j,\ x_i=x_j\}$

Robin function $R(x)=\left[G(x,x')+\frac{1}{2\pi}\log|x-x'|\right]_{x'=x}$

S. Chemotaxis, Reaction, Network, World Scientific, Singapore, 2018

2. Blowup in Finite Time (Proof of Theorem B)

- Rescaled Hamiltonian induces residual vanishing

symmetry of the Green function → weak form (symmetrization)

$$\varphi \in C^2(\overline{\Omega}), \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\frac{d}{dt}\int_\Omega \varphi u(\cdot,t) = \int_\Omega \Delta\varphi \cdot u(\cdot,t) + \frac{1}{2}\int\int_{\Omega\times\Omega} \rho_\varphi(x,x')u\otimes u$$

$$\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x')$$

$$\|\rho_\varphi\|_\infty \leq C\|\nabla\varphi\|_{C^1}$$

boundary behavior of the Green function
singularity cancellation by the symmetry

monotonicity formula $\lambda = \|u(\cdot,t)\|_1$

$$\left|\frac{d}{dt}\int_\Omega u\varphi\right| \leq C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1}$$

→ **weak continuation**

$$0 \leq \exists\mu(dx,t) \in C_*([0,T],\mathcal{M}(\overline{\Omega}))$$

$$u(x,t)dx = \mu(dx,t), \ 0 \leq t < T$$

epsilon regularity via Gagliard-Nirenberg inequality

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists\varepsilon_0 \Rightarrow x_0 \notin \mathcal{S}$$

→ **formation of collapse**

$$\mu(\cdot,T) = \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0} + f(x)$$

$$m(x_0) \geq \varepsilon_0, \ 0 \leq f \in L^1(\Omega), \ \sharp\mathcal{S} < +\infty$$

some technicalities

1. nice cut-off function

$$0 \leq \varphi \leq 1, \ \varphi = \begin{cases} 1, & x \in B(x_0, \frac{R}{2}) \\ 0, & x \in \mathbf{R}^2 \setminus B(x_0, R) \end{cases}$$

$$x_0 \in \overline{\Omega}, \ 0 < R \ll 1, \varphi = \varphi_{x_0,R} \in \mathcal{Y}$$

$$|\nabla \varphi| \leq C R^{-1} \varphi^{\frac{5}{6}}$$

$$|\nabla^2 \varphi| \leq C R^{-2} \varphi^{\frac{2}{3}} \qquad \mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0\}$$

2. Green's function

$$x' \in \Omega, \ -\Delta G(\cdot, x') = \delta_{x'}, \ \ G(\cdot, x')|_{\partial\Omega} = 0$$

fundamental solution $\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$

$$G = G(x,x') \in C^{2+\theta}(\overline{\Omega} \times \overline{\Omega} \setminus D)$$

$$D = \overline{\{(x,x) \mid x \in \Omega\}}, \ 0 < \theta < 1$$

2.1. interior regularity

$$G(x,x') = \Gamma(x-x') + K(x,x')$$

$$K \in C^{2+\theta,1}(\overline{\Omega} \times \Omega) \cap C^{1,2+\theta}(\Omega \times \overline{\Omega})$$

$$\varphi \in C^2(\overline{\Omega}), \ \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0 \Rightarrow \rho_\varphi \in L^\infty(\Omega \times \Omega)$$

discontinuity at the diagonal

2.2. boundary regularity

$$x_0 \in \partial\Omega, \ X : \overline{\Omega \cap B(x_0, 2R)} \to \overline{\mathbf{R}^2_+}$$

conformal diffeo.

$$X(x_0) = 0, \ \mathbf{R}^2_+ = \{(X_1, X_2) \in \mathbf{R}^2 \mid X_2 > 0\}$$

$$G(x,x') = E(x,x') + K(x,x')$$

$$K \in C^{2+\theta,1} \cap C^{1,2+\theta}(\overline{\Omega \cap B(x_0,R)} \times \overline{\Omega \cap B(x_0,R)})$$

$$E(x,x') = \Gamma(X - X') - \Gamma(X - X'_*)$$

$$X_* = (X_1, -X_2), \ X = (X_1, X_2)$$

weak scaling limit ➡ exclusion of boundary blowup

$$\rho_\varphi(x,x') = \nabla\varphi(x) \cdot \nabla_x G(x,x') + \nabla\varphi(x') \cdot \nabla_{x'} G(x,x')$$

related notions

$$0 \le \mu = \mu(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$$ weak solution

$$\longleftrightarrow \quad 0 \le \exists \mathcal{N} = \mathcal{N}(\cdot,t) \in L^\infty_*([0,T], \mathcal{X}')$$

1. $t \in [0,T] \mapsto \langle \varphi, \mu(dx,t)\rangle, \ \varphi \in \mathcal{Y}$ a.c.
2. $\frac{d}{dt}\langle \varphi, \mu\rangle = \langle \Delta\varphi, \mu\rangle + \frac{1}{2}\langle \rho_\varphi, \mathcal{N}(\cdot,t)\rangle$ a.e. $t \in [0,T]$
3. $\mathcal{N}|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu \otimes \mu$

Theorem $\mu_k(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$

$\mathcal{N}_k \in L^\infty_*([0,T], \mathcal{X}')$ weak solutions

$$0 \le \mu_k(\overline{\Omega}, t) \le C$$
$$\|\mathcal{N}_k(\cdot,t)\|_{\mathcal{X}'} \le C$$
$\Rightarrow$ sub-sequence

$$\mu_k(dx,t) \rightharpoonup \mu(dx,t) \quad \text{in } C_*([0,T], \mathcal{M}(\overline{\Omega}))$$
$$\mathcal{N}_k(\cdot,t) \rightharpoonup \mathcal{N}(\cdot,t) \quad \text{in } L^\infty_*([0,T], \mathcal{X}')$$ weal solution

$$\mathcal{Y} = \{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0\} \qquad \mathcal{X} = [\mathcal{X}_0]^{L^\infty(\Omega\times\Omega)}$$
$$\mathcal{X}_0 = \{\rho_\varphi + \psi \mid \varphi \in \mathcal{Y}, \ \psi \in C(\overline{\Omega}\times\overline{\Omega})\}$$

$\Rightarrow$
$$\mu(\overline{\Omega}, t) = \mu(\overline{\Omega}, 0) \equiv \lambda, \ 0 \le t \le T$$
$$\left|\frac{d}{dt}\langle \varphi, \mu(dx,t)\rangle\right| \le C(\lambda + \lambda^2)\|\nabla\varphi\|_{C^1}$$

$u = u(x,t)$ classical solution

$\Rightarrow$
$$\mathcal{N}(\cdot,t) = u(x,t) \otimes u(x',t) \ dxdx'$$
$$\|\mathcal{N}(\cdot,t)\|_{\mathcal{X}'} = \lambda^2, \ \lambda = \|u_0\|_1$$

Theorem $\exists \varepsilon_0, \ \sigma_0, \ C$ via improved epsilon regularity

$$u_t = \Delta u - \nabla\cdot(u\nabla\Gamma * u) \text{ in } \mathbf{R}^2 \times (-T,T)$$
$$u_0 = u|_{t=0}$$

$$\|u_0\|_{L^1(B(x_0,2R))} < \varepsilon_0, \ u_0 = u|_{t=0} \ \Rightarrow$$
$$\sup_{t\in[-\sigma_0 R^2, \sigma_0 R^2]\cap(-T,T)} \|u(\cdot,t)\|_{L^\infty(B(x_0,R))} \le CR^{-2}$$

Proof of Theorem B (continued) $x_0 \in \mathcal{S}$

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

backward self-similar transformation

$$y=(x-x_0)/(T-t)^{1/2},\ s=-\log(T-t)$$
$$z(y,s)=(T-t)u(x,t)$$

weak limit $s_k \uparrow +\infty$ subsequence

$$z(y,s+s_k)dy \rightharpoonup \exists\zeta(dy,s) \text{ in } C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$

limit equation exclusion of boundary blowup $x_0 \in \Omega$

$$\zeta_s = \nabla\cdot(\nabla\zeta - \zeta\nabla(\Gamma*\zeta + |y|^2/4)) \ \text{ in } \mathbf{R}^2\times(-\infty,+\infty)$$

scaling back

$$\zeta(dy,s)=e^{-s}A(dy',s'),\ y'=e^{-s/2}y,\ s'=-e^{-s}$$

$$A_s=\nabla\cdot(\nabla A - A\nabla\Gamma*A) \text{ in } \mathbf{R}^2\times(-\infty,0)$$
$$A=A(dy,s)\geq 0,\ A(\mathbf{R}^2,s)=m(x_0)$$

parabolic envelope

$$m(x_0)=\zeta(\mathbf{R}^2,s) \qquad \langle |y|^2, \zeta(dy,s)\rangle \leq C$$

weak Liouville property

$$a_s=\nabla\cdot(\nabla a - a\nabla\Gamma*a) \text{ in } \mathbf{R}^2\times(-\infty,+\infty)$$
$$\Rightarrow\ a(\mathbf{R}^2,s)=0 \text{ or } 8\pi$$

translation limit $\zeta^s(dy,s)=\sum_{j=1}^{\ell} 8\pi\delta_{y_j(s)}(dy)$

scaling invariant regularity (scaling back)

$$\zeta(B(y_0,2r),s)<\varepsilon_0 \ \Rightarrow\ \|\zeta(\cdot,s)\|_{L^\infty(B(y_0,r))}\leq Cr^{-2}$$

$\Longrightarrow$ $|y_j(s)|\leq C$

residual vanishing

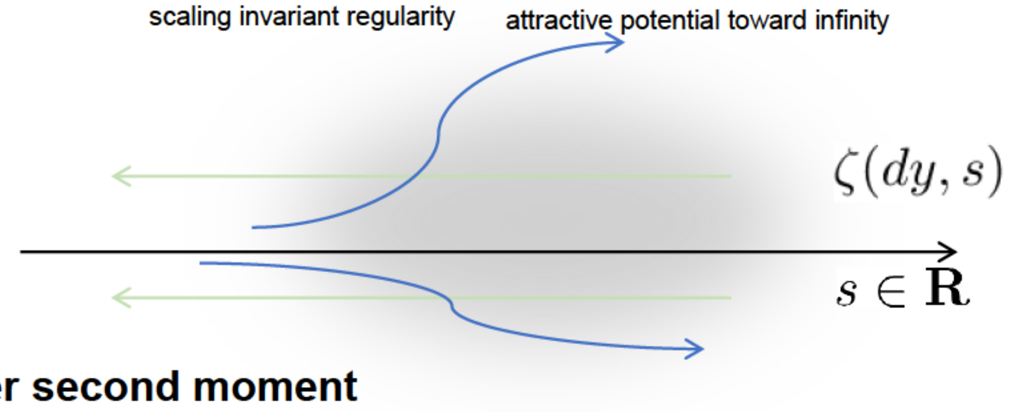
1st envelope

$$m(x_0) = \zeta(\mathbf{R}^2, s)$$

2nd envelope

$$\langle |y|^2, \zeta(dy, s)\rangle \le C$$

$$\zeta_s = \nabla \cdot (\nabla \zeta - \zeta \nabla(\Gamma * \zeta + |y|^2/4))$$



outer second moment

$$\frac{d}{ds}\langle \varphi, \zeta\rangle \ge \langle \Delta\varphi - C\varphi_r + \frac{1}{2} r\varphi_r, \zeta\rangle, \ \varphi = \varphi(r)$$

$$\varphi(r) = \xi(r/R), \ \xi(r) = r^2 - 1$$

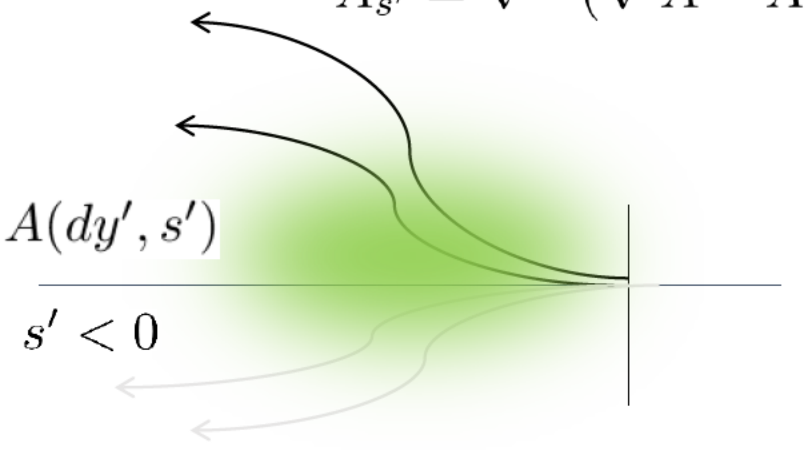
$$R \gg 1 \ \Rightarrow \ \Delta\varphi + \frac{1}{2} r\varphi_r \ge C\varphi_r, \ r \ge R$$

$$\frac{d}{ds}\langle (\frac{|y|^2}{R^2} - 1)_+, \zeta(dy, s)\rangle \ge 0 \quad \longrightarrow \quad \zeta(dy, s) = \zeta^s(dy, s)$$

collapse mass quantization

$$\zeta(\mathbf{R}^2, s) = m(x_0) \in 8\pi\mathbf{N}$$

$$A_{s'} = \nabla' \cdot (\nabla' A - A\nabla'\Gamma * A)$$



simple blowup point

$$\ell = 1 \ \Rightarrow \ \zeta(dy, s) = 8\pi\delta_0(dy)$$

recursive hierarchy $\ell \ge 2$

$$\frac{dy'_j}{ds'} = 8\pi\nabla_j H^0_\ell(y'_1, \cdots, y'_\ell)$$

$$A(dy', s') = \sum_{j=1}^{\ell} 8\pi\delta_{y'_j(s')}(dy')$$

$$H^0_\ell(y'_1, \cdots, y'_\ell) = \sum_{1\le j<k\le \ell} \Gamma(y'_j - y'_k), \ \Gamma(y') = \frac{1}{2\pi}\log\frac{1}{|y'|}$$

3. Blowup in infinite time (Proof of Theorem A)

assume $T=+\infty,\ t_k\uparrow+\infty,\ \lim_{k\to\infty}\|u(\cdot,t_k)\|_\infty=+\infty$

subsequence $u(\cdot,t+t_k)dx \rightharpoonup \mu(dx,t)\in C_*(-\infty,+\infty;\mathcal{M}(\overline{\Omega}))$ weak solution

$$\mu(dx,t)=\sum_{x_0\in\mathcal{S}_t} m(x_0)\delta_{x_0}(dx)+f(x,t)dx$$

improved regularity
formation of collapse in infinite time

$$m(x_0)\geq\varepsilon_0,\ 0\leq f=f(\cdot,t)\in L^1(\Omega)$$

blowup set

exclusion of boundary blowup

$$\mathcal{S}_t=\{x_0\in\overline{\Omega}\mid \exists x_k\to x_0,\ \lim_k u(x_k,t+t_k)=+\infty\}\subset\Omega$$

dilation $x_0=0\in\mathcal{S}_0,\ \beta>0$

$$\mu_\beta(dx',t')=\beta^2\mu(dx,t),\ x'=\beta x,\ t'=\beta^2 t$$

$\beta_k\downarrow 0$ subsequence

$$\mu_{\beta_k}(dx,t)\rightharpoonup\tilde{\mu}(dx,t)\in C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$

scaling limit

$$m(x_0)=\tilde{\mu}(\mathbf{R}^2,0)=8\pi\geq\varepsilon_0$$

full orbit of weak solutions on the whole space

Liouville property

collapse mass quantization

local second moment traces the collapse dynamics

$$\sharp\,\mathcal{S}_t\equiv\ell,\ \mu^s(dx,t)=\sum_{i=1}^{\ell}8\pi\delta_{x_i(t)}(dx)$$

$$\frac{dx_i}{dt}=8\pi\nabla_{x_i}H_\ell(x_1,\cdots,x_\ell),\ 1\leq i\leq\ell$$

recursive hierarchy

anti-gradient system

a blowup criterion excludes the collapse collision in infinite time

$$x(t)=(x_i(t))\in\Omega^\ell\setminus D$$

pre-compact

$$D=\{(x_i)\mid\exists i\neq j,\ x_i=x_j\}$$

⟹

$$\exists x^*\in\Omega^\ell\setminus D,\ \nabla_{x_i}H_\ell(x^*)=0,\ 1\leq i\leq\ell$$

residual vanishing

$$x_i = x_i(t),\ u_k(x,t) = u(x,t+t_k),\ v_k(x,t) = v(x,t+t_k),\ 0 < r \ll 1$$

$$\frac{d}{dt}\int_{B(x_i,r)} |x-x_i|^2 u_k = \int_{B(x_i,r)} \frac{\partial}{\partial t}(|x-x_i|^2 u_k) + \dot{x}_i \cdot \nabla(|x-x_i|^2 u_k)\ dx$$

Liouville's formula

$$= \int_{B(x_i,r)} |x-x_i|^2 u_{kt} + \dot{x}_i \cdot |x-x_i|^2 \nabla u_k\ dx$$

$$\int_{B(x_i,r)} |x-x_i|^2 u_{kt}$$

$$= \int_{B(x_i,r)} |x-x_i|^2 \nabla \cdot (\nabla u_k - u_k \nabla v_k)\ dx$$

$$\le r^2 \int_{\partial B(x_i,r)} \frac{\partial u_k}{\partial \nu} - u_k \frac{\partial v_k}{\partial \nu}\ dS$$

$$+ \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k \nabla v_k\ dx$$

$$= \int_{B(x_i,r)} r^2 u_{kt} + 4u_k + 2(x-x_i)\cdot u_k \nabla v_k\ dx$$

$$\int_{B(x_i,r)} \dot{x}_i \cdot |x-x_i|^2 \nabla u_k$$

$$= \int_{\partial B(x_i,r)} (\dot{x}_i \cdot \nu)|x-x_i|^2 u_k\ dS$$

$$- \int_{B(x_i,r)} 2(x-x_i)\cdot \dot{x}_i u_k$$

$$= \int_{B(x_i,r)} r^2 \dot{x}_i \cdot \nabla u_k$$

$$-2(x-x_i)\cdot \dot{x}_i u_k\ dx$$

$$\frac{d}{dt}\int_{B(x_i,r)} u_k = \int_{B(x_i,r)} u_{kt}$$

$$+\dot{x}_i \cdot \nabla u_k\ dx$$

$$\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2) u_k$$

$$\le \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k \nabla v_k - 2(x-x_i)\cdot \dot{x}_i u_k\ dx$$

$$x_i = x_i(t)$$
$$u_k(x,t) = u(x, t+t_k)$$
$$v_k(x,t) = v(x, t+t_k)$$
$$0 < r \ll 1$$

$$\longrightarrow$$

$$\frac{d}{dt}\int_{B(x_i,r)} (|x-x_i|^2 - r^2)u_k$$
$$\leq \int_{B(x_i,r)} 4u_k + 2(x-x_i)\cdot u_k \nabla v_k$$
$$-2(x-x_i)\cdot \dot{x}_i u_k \ dx$$

$$v_k(x,t) = \sum_{i=0}^{3} v_k^i(x,t)$$

$$v_k^0(x,t) = \int_{B(x_i,r)} \Gamma(x-x')u_k(x',t)dx'$$

$$v_k^1(x,t) = \int_{B(x_i,r)} K(x,x')u_k(x',t)\ dx'$$

$$v_k^2(x,t) = \int_{\Omega\setminus \mathcal{S}_t^{2r}} G(x,x')u_k(x',t)dx'$$

$$v_k^3(x,t) = \int_{\mathcal{S}_t^{2r}\setminus B(x_i,r)} G(x,x')u_k(x',t)dx'$$

$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$
$$G(x,x') = \Gamma(x-x') + K(x,x')$$

$$2\int_{B(x_i,r)} (x-x_i)\cdot u_k\nabla v_k^0\ dx$$
$$= -\frac{1}{2\pi}\left(\int_{B(x_i,r)} u_k\ dx\right)^2$$

$$\|u_k(\cdot,t)\|_1 = \lambda,\ \ K(x,x') \in C^1(\Omega\times\Omega)$$

$$\sup_x \int_\Omega |\nabla_x G(x,x')|\ dx' \leq C$$

$$\longrightarrow$$

$$\|\nabla v^i(\cdot,t)\|_{L^\infty(B(x_i,r))} \leq C,\ \ 1\leq i\leq 3$$

$\mathcal{O} = \{x(t)\}$ compact $\longrightarrow$ $|\dot{x}_i| \leq C$

away from diagonal and boundary

$$\frac{d}{dt}\int_{B(x_i,r)}(|x-x_i|^2-r^2)u_k$$

$$\leq 4\int_{B(x_i,r)} u_k - \frac{1}{2\pi}\left(\int_{B(x_i,r)} u_k\right)^2$$

$$+C\int_{B(x_i,r)}|x-x_i|u_k$$

$k \to \infty$

as distributions in time

→

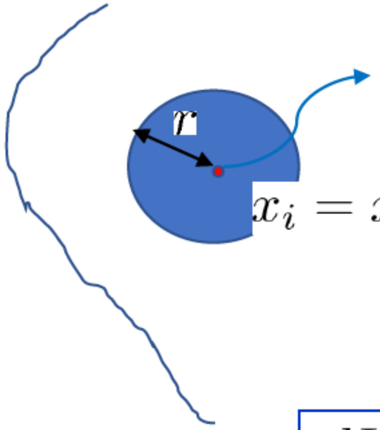
defect measure

$$\frac{d}{dt}\int_{B(x_i,r)}(|x-x_i|^2-r^2)f$$

$$\leq 4\left(8\pi+\int_{B(x_i,r)} f\right)$$

$$-\frac{1}{2\pi}\left(8\pi+\int_{B(x_i,r)} f\right)^2$$

$$+C\int_{B(x_i,r)}|x-x_i|f$$



$x_i = x_i(t)$

bounded free energy →

1. stationary collapse formation in infinite time
2. simple collapse formation in finite time

$\lambda = 8\pi$

↓ $0 < r \ll 1$

$$\frac{dI}{dt} \leq \int_{B(x_i,r)} -4f + C|x-x_i|f\ dx \leq \frac{2I}{r^2}$$

$$I(t) \equiv \int_{B(x_i,r)}(|x-x_i|^2-r^2)f \leq 0$$

→

$I(t) \equiv 0$

$f = 0 \text{ in } B(x_i, r)$

$f \equiv 0$

higher-dimensional analogue – a challenge

1. plasma confinement
2. mean filed limit of self-interacting particles associated with Tsallis entropy
3. incompressible Euler flow with self-gravitation