Reaction Diffusion Systems 1

Anti-Symmetric Interaction Cancels Singularities

1. Reaction Diffusion Systems

$$\tau_j \frac{\partial u_j}{\partial t} - d_j \Delta u_j = f_j(u) \text{ in } Q_T$$
$$\left.\frac{\partial u_j}{\partial \nu}\right|_{\partial\Omega} = 0, \ \ u_j|_{t=0} = u_{j0}(x)$$

$\Omega \subset \mathbf{R}^n$ bounded domain, $\partial\Omega$ smooth

$Q_T = \Omega \times (0, T) \qquad 1 \leq j \leq N$

ν outer unit normal

$\tau = (\tau_j) > 0, \ d = (d_j) > 0$

$u_0 = (u_{j0}) \geq 0$ smooth

[local. Lipschitz cont.]

$f_j : \mathbf{R}^N \to \mathbf{R}, \ 1 \leq j \leq N$
loc. Lipschitz cont.

$\longrightarrow$

$\exists 1$ classical solution local-in-time

$T \in (0, +\infty]$ maximal existence time

[quasi-positive]

$$f_j(u_1, \cdots, u_{j-1}, 0, u_{j+1}, \cdots, u_n) \geq 0, \ \forall j$$

$0 \leq u_0 = (u_{j0}) \in \mathbf{R}^N \longrightarrow u = (u_j(\cdot, t)) \geq 0$

[mass dissipation]

$$\sum_{j=1}^N f_j(u) \leq 0, \ u = (u_j) \geq 0$$

$$\longrightarrow \frac{\partial}{\partial t}(\tau \cdot u) - \Delta(d \cdot u) \leq 0$$
$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\|\tau \cdot u(t)\|_1 \leq \|\tau \cdot u_0\|_1$$

[quadratic]

$$|\nabla f_j(u)| \leq C(1 + |u|), \ \forall j$$

Theorem (Fellner-Morgan-Tang 20, 21)

$$T = +\infty \qquad \|u(\cdot, t)\|_\infty \leq C$$

Examples

chemical reaction $A_1 + \cdots + A_m \rightleftharpoons A_{m+1} + \cdots + A_N$

$$\tau_j \frac{\partial u_j}{\partial t} - d_j \Delta u_j = \chi_j f(u), \quad \left. \frac{\partial u_j}{\partial \nu} \right|_{\partial\Omega} = 0$$

micro-canonical ensemble

$$f(u) = \prod_{j=1}^{m} u_j - \prod_{j=m+1}^{N} u_j, \ \chi_j = \begin{cases} -1, & 1 \leq j \leq m \\ 1, & m+1 \leq j \leq N \end{cases}$$

spatially homogeneous stationary state $\overline{w} = \frac{1}{|\Omega|}\int_\Omega w$

$$0 \leq \exists 1 z = (z_j) \in \mathbf{R}^N, \ f(z) = 0$$

$$z_i + z_k = \overline{u}_{i0} + \overline{u}_{k0}, \ 1 \leq i \leq m, \ m+1 \leq k \leq N$$

$\Longrightarrow$ $z = (z_j) > 0$

Theorem $m = 2, \ N = 4$ (quadratic)

$\Longrightarrow$ $T = +\infty$ $\quad \|u(\cdot, t) - z\|_\infty \leq Ce^{-\delta t}$

Devillettes-Fellner 08, Pierre-S.-Zou 17, Pierre-S.-Umakoshi 18

$$\Phi(s) = s(\log s - 1) + 1 \geq 0$$

relative entropy (diversity), entropy

$$E(w \mid v) = \int_\Omega v \ \Phi\left(\frac{w}{v}\right), \ E(w) = \int_\Omega \Phi(w)$$

$$E(u) = \sum_{j=1}^{N} \tau_j E(u_j), \ E(u \mid z) = \sum_{j=1}^{N} \tau_j E(u_j \mid z_j)$$

$\Longrightarrow$ $E(u|z) = E(u) - E(z)$

$$\boxed{\frac{d}{dt}E(u) = -D(u)}$$

$$D(u) = 4\sum_{j=1}^{N} d_j \|\nabla\sqrt{u_j}\|_2^2 + \int_\Omega f(u) \log \frac{\prod_{j=m+1}^{N} u_j}{\prod_{j=1}^{m} u_j}$$

[logarithmic Sobolev] $D(u) \geq 2\delta E(u|z)$

[Csiz'ar-Kullback] $\|v - \overline{v}\|_1^2 \leq 4\overline{v}E(v|\overline{v})$

Lotka-Volterra system

$$\tau_j \frac{\partial u_j}{\partial t} = d_j \Delta u_j + (e_j + \sum_k a_{jk} u_k) u_j$$

$$\left. \frac{\partial u_j}{\partial \nu} \right|_{\partial \Omega} = 0, \ \ u_j|_{t=0} = u_{j0}(x) \geq 0$$

$$(Au, u) \leq 0, \ \forall u \geq 0 \qquad A = (a_{jk})$$

$\longrightarrow$ $T = +\infty$

$e = (e_j) \leq 0$ $\longrightarrow$ $\|u(\cdot, t)\|_\infty \leq C$

Masuda-Takahashi 94 (n=1) S.-Yamada 15 (n=2)

scaling invariance (e=0)

$$u_j^\mu(x,t) = \mu^2 u_j(\mu x, \mu^2 t), \ \ \mu > 0$$

rigidness (n=2, quadratic growth by L^1 control)

$$\|u_0\|_1 \ll 1 \ \Rightarrow \ T = +\infty, \ \sup_{t \geq 0} \|u(\cdot, t)\|_\infty < +\infty$$

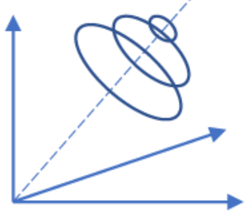
entropy $\longrightarrow$ asymptotic spatially homogenization

(S.-Yamada 15)

$E = L \cap \mathbf{R}_+^N, \ \exists L$ affine space of co-dimension 2

Any non-stationary solution is periodic-in-time with the orbit contractible to a stationary solution in $\mathcal{O} \cong S^1$ $\mathbf{R}_+^N \setminus E$

Any distinct two orbits $\mathcal{O}_1, \mathcal{O}_2 \cong S^1$ do not link in $\mathbf{R}_+^N$



spatially homogeneous part

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & \cdots & a_{1N} \\ a_{21} & 0 & a_{23} & a_{24} & \cdots & a_{2N} \\ a_{31} & a_{32} & 0 & a_{34} & \cdots & a_{3N} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & a_{N-1N} \\ a_{N1} & a_{N2} & \cdots & \cdots & a_{NN-1} & 0 \end{pmatrix}$$

free

2N-3 dimension

$$a_{kl} = \frac{a_{1k}a_{2l} - a_{1l}a_{2k}}{a_{12}}$$

$$3 \leq k < l \leq N$$

$$a_{12} \neq 0, \ e = (e_j) = 0$$

Kobayashi-S.-Yamada 19

Smoluchowski-Poisson equation – a model in statistical mechanics

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u \nabla v)$$

$$\left. \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0$$

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

canonical ensemble

1. total mass conservation $\quad \frac{d}{dt}\|u(t)\|_1 = 0$

2. free energy decreasing

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2} \int\int_{\Omega\times\Omega} G(x,x')u \otimes u$$

$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 \leq 0$$

$G(x,x') = G(x',x)$ Green's function

2D is critical for blowup of the solution to quadratic nonlinearity under the total mass control

self-similar transformation due to the quadratic growth

$$u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t), \ \mu > 0$$

$\|u\|_1 = \|u_\mu\|_1 \equiv \lambda \ \Leftrightarrow \ n = 2$ *critical dimension*

$$\mathcal{F}(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2}\langle \Gamma * u, u \rangle, \ \Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$\mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi}\right)\log\mu + \mathcal{F}(u)$ *critical mass* $\lambda = 8\pi$

quantized blowup mechanism with Hamiltonian control
1. stationary 2. finite time 3. infinite time

Former results (1) any space dimension

entropy dissipation

$$\sum_{j=1}^N f_j(u)\log u_j \leq 0$$

Capto-Goudon-Vasseur 09 $\Omega = \mathbf{R}^n$

loc. Lipschitz cont. quasi-positive
mass dissipation, entropy dissipation
quadratic growth

Souplet 18 $\Omega = \mathbf{R}^n$ or $\Omega \subset \mathbf{R}^n$

$$\sum_{j=1}^N f_j(u)(1+\log u_j) \leq C\sum_{j=1}^N u_j \log(1+u_j)$$

loc. Lipschitz cont. quasi-positive, quadratic growth

Fellner-Tang

1. Sobolev inequality in space-time
2. Parabolic Giorgi-Nash-Moser regularity
3. Regularity interpolation
4. Souplet's trick by semigroup estimate

Former results (2) without entropy dissipation

Pierre-Rolland 15

$$0 \leq \exists u = (u_j(\cdot,t)) \in C([0,+\infty), L^1(\Omega)^N)$$

global-in-time weak solution

Pierre-S.-Yamada 19

$\mathcal{O} = \{u(\cdot,t)\} \subset L^1(\Omega)^N$ pre-compact

1. Mechanism to protect the solution from the measure?
2. Why 2D is thought to be critical?

2. weak solutions

Pierre-Rolland 15 $0 \leq \exists u = (u_j(\cdot, t)) \in C([0, +\infty), L^1(\Omega)^N)$ global-in-time weak solution

Pierre-S.-Yamada 19 $\mathcal{O} = \{u(\cdot, t)\} \subset L^1(\Omega)^N$ pre-compact

weak solution to $0 \leq u = (u_j(\cdot, t)) \in L^\infty_{loc}([0, T), L^1(\Omega)^N)$

$$\tau_j \frac{\partial u_j}{\partial t} - d_j \Delta u_j = f_j(u), \quad \left. \frac{\partial u_j}{\partial \nu} \right|_{\partial\Omega} = 0, \quad u_j|_{t=0} = u_{j0}(x) \geq 0$$

$\Longleftrightarrow$ (def.)

$$f_j(u) \in L^1_{loc}(\overline{\Omega} \times (0, T))$$

as distributions

$$\frac{d}{dt} \int_\Omega u_j \varphi - d_j \int_\Omega u_j \Delta\varphi = \int_\Omega f_j(u)\varphi, \ \forall \varphi \in W^{2,\infty}(\Omega), \ \left. \frac{\partial \varphi}{\partial \nu} \right|_{\partial\Omega} = 0$$

$u_j|_{t=0} = u_{j0}(x)$ in the sense of measures

L2-L1 estimate

$$\tau_j \frac{\partial u_j}{\partial t} - d_j \Delta u_j = f_j(u)$$

$$\left.\frac{\partial u_j}{\partial \nu}\right|_{\partial\Omega} = 0, \quad u_j|_{t=0} = u_{j0}(x) \geq 0$$

$$\sum_{j=1}^{N} f_j(u) \leq 0 \quad \longrightarrow$$

$$\tau = (\tau_j), \ d = (d_j) > 0$$

$$\frac{\partial}{\partial t}(\tau \cdot u) - \Delta(d \cdot u) \leq 0, \ u = (u_j) \geq 0$$

$$\left.\frac{\partial}{\partial \nu}(d \cdot u)\right|_{\partial\Omega} \leq 0, \quad u|_{t=0} = u_0 = (u_{j0})$$

$$\downarrow$$

$$\frac{d}{dt}\int_\Omega \tau \cdot u \leq 0 \longrightarrow \boxed{\sup_{0\leq t<T} \|u(\cdot,t)\|_1 \leq C}$$

$$\swarrow$$

$$\tau \cdot u(\cdot,t) - \tau \cdot u_0 \leq \int_0^t \Delta(d \cdot u(\cdot,s))\ ds$$

$$\longrightarrow \quad (\tau \cdot u(\cdot,t), d \cdot u(\cdot,t)) - (\tau \cdot u_0, d \cdot u(\cdot,t)) \leq -(\nabla d \cdot u(\cdot,t), \nabla \int_0^t d \cdot u(\cdot,s)\ ds)$$

$$= -\frac{1}{2}\frac{d}{dt}\|\nabla \int_0^t d \cdot u(\cdot,s)\ ds\|_2^2$$

$$\longrightarrow$$

$$\int_0^T (\tau \cdot u(\cdot,t), d \cdot u(\cdot,t))\ dt \leq \|\tau \cdot u_0\|_2 \cdot \int_0^T \|d \cdot u(\cdot,t)\|_2\ dt$$

$$\leq CT^{\frac{1}{2}}\|\tau \cdot u_0\|_2 \cdot \{\int_0^T \|d \cdot u(\cdot,t)\|_2^2\ dt\}^{\frac{1}{2}} \longrightarrow \boxed{\|u\|_{L^2(Q_T)} \leq CT^{\frac{1}{2}}\|u_0\|_2}$$

L1 pre-compactness

1. semi-group reduction Baras-Pierre 84

$$\frac{\partial w}{\partial t} - \Delta w = H \in L^1(Q_T)$$

$$\left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega} = 0, \quad w|_{t=0} = w_0(x) \in L^1(\Omega)$$

$$w = w(\cdot, t) \in L^\infty(0,T;L^1(\Omega)) \bigcap L^1_{loc}(0,T;W^{1,1}(\Omega))$$

weak solution

i.e. as distributions

$$\frac{d}{dt}\int_\Omega w\varphi + \int_\Omega \nabla w \cdot \nabla\varphi = \int_\Omega H\varphi, \ \forall \varphi \in W^{1,\infty}(\Omega)$$

$w|_{t=0} = w_0$ in the sense of measures

$$\longrightarrow \quad w(\cdot,t) = e^{t\Delta}w_0 + \int_0^t e^{(t-s)\Delta}H(\cdot,s)\ ds$$

in particular $w \in C([0,T],L^1(\Omega))$

$$\mathcal{F} : (w_0, H) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in C([0,T],L^1(\Omega))$$

continuous

2. compactness c.f. Baras 78

$$\mathcal{F} : (w_0, H) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in L^1(Q_T)$$

compact

Proof $\mathcal{F}^* : L^\infty(Q_T) \to L^\infty(\Omega) \times L^\infty(Q_T)$

$$\mathcal{F}^*(h) = (\theta|_{t=0}, \theta)$$

$$\frac{\partial \theta}{\partial t} + \Delta\theta = h, \quad \left.\frac{\partial \theta}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \theta|_{t=T} = 0$$

compact from the parabolic regularity

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pre-compactness of the orbit in L1

$$0 \leq u_k(\cdot,t) = u(\cdot, t+t_k)$$
$$\leq \exists w_k(\cdot,t) \in L^2(\Omega \times (-1,1))$$

comparison theorem ($\leq$) — compact ($\exists w_k$) — dominated convergence theorem (u_k)

alternative argument applicable to other systems (S.-Yamada)

L2 estimate in space and time

quasi-positive
mass dissipation
quadratic growth → $$\sum_{j=1}^{N} f_j(u)\log u_j \leq C(1+|u|^2)$$ → $$\sup_{0\leq t<T}\int_\Omega \Phi(u_j(\cdot,t)) \leq C_T$$

singularity relaxation

$$\Phi(s)=s(\log s-1)+1\geq 0,\ \ s>0$$

global GN inequality

→ T=+∞ if n=1, 2

monotonicity formula $$\int_{-1}^{1}\left|\frac{d}{dt}\int_\Omega u_j(\cdot,t+t_k)\varphi\right| dt \leq C_\varphi$$

$$\forall\varphi\in C^2(\overline{\Omega}),\ \ \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega}=0$$

by L2 control in space-time

evokes the measure-valued continuation (very weak solution)

Smoluchowski-Poisson equation

2D case – time control

Corollary $n = 2 \;\Rightarrow\; T = +\infty,\ \|u(\cdot,t)\|_\infty \le C$

loc. Lipschitz cont.
quasi-positive
mass dissipation
quadratic growth

semi-group theory $T \in (0,+\infty],\ \limsup_{t\uparrow T} \|u(t)\|_2 < +\infty \;\Rightarrow\; \limsup_{t\uparrow T} \|u(t)\|_\infty < +\infty$

Gagliardo-Nirenberg $\frac{d}{dt}\|u\|_2^2 + \delta\|\nabla u\|_2^2 \le C\|u\|_3^3$

$$C\|u\|_3^3 \le C'\|u\|_2^{\frac{6-n}{2}}\|u\|_{H^1}^{\frac{n}{2}} \le \frac{\delta}{2}\|u\|_{H^1}^2 + C''\|u\|_2^{\frac{6-n}{2}\cdot\frac{4}{4-n}}$$

$$\frac{1}{\frac{4}{n}} + \frac{1}{\frac{4}{4-n}} = 1$$

$n \le 6$ $\qquad$ $n \le 3$

$\|u\|_1 \le C$ Poincare-Wirtinger $-\frac{2}{4-n} - 1 = -\frac{6-n}{4-n}$

$$\frac{d}{dt}\|u\|_2^2 \le C(\|u\|_2^2+1)^{\frac{6-n}{4-n}} \;\Longrightarrow\; -\frac{d}{dt}(\|u\|_2^2+1)^{-\frac{2}{4-n}} \le C$$

$$t_k \uparrow T \in (0,+\infty],\ u^k(t) = u(t+t_k)$$

$$-\frac{d}{dt}(\|u^k\|_2^2+1)^{-\frac{2}{4-n}} \le C$$

$$(\|u^k(-t)\|_2^2+1)^{-\frac{2}{4-n}} \le (\|u^k(0)\|_2^2+1)^{-\frac{2}{4-n}} + Ct, \quad 0<t<T$$

Pierre-S.-Yamada

assume $\displaystyle\lim_{k\to\infty} \|u^k(0)\|_2 = +\infty$ ⟹ subsequence

$$u^k \to \exists u^\infty \text{ in } C_{loc}((-\infty,0], L^1(\Omega)),\ L^2_{loc}(\overline{\Omega}\times(-\infty,0])$$

$$(\|u^\infty(-t)\|_2^2+1)^{-\frac{2}{4-n}} \le Ct$$

n=2 ⟹ $\|u^\infty(t)\|_2^2+1 \ge \delta(-t)^{-1},\ -T<t<0$

$$\|u^\infty(-t)\|_2^2+1 \ge \delta t^{-\frac{4-n}{2}}, \quad 0<t<T$$

$u^\infty \notin L^2_{loc}(\overline{\Omega}\times(-T,0])$ contradiction

Reaction Diffusion Systems 2

Beyond the Critical Dimension

1. Reaction Diffusion Systems (continued)

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[local. Lipschitz cont.]

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loc. Lipschitz cont.

$\longrightarrow$

$\exists 1$ classical solution local-in-time

$T \in (0, +\infty]$ maximal existence time

[quasi-positive]

$f_j(u_1, \cdots, u_{j-1}, 0, u_{j+1}, \cdots, u_n) \geq 0, \ \forall j$

$0 \leq u_0 = (u_{j0}) \in \mathbf{R}^N \longrightarrow$

$u = (u_j(\cdot, t)) \geq 0$

[mass dissipation]

$$\sum_{j=1}^N f_j(u) \leq 0, \ u = (u_j) \geq 0$$

$$\longrightarrow \frac{\partial}{\partial t}(\tau \cdot u) - \Delta(d \cdot u) \leq 0$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\|\tau \cdot u(t)\|_1 \leq \|\tau \cdot u_0\|_1$$

[quadratic]

$|\nabla f_j(u)| \leq C(1 + |u|), \ \forall j$

Theorem (Fellner-Morgan-Tang 20, 21)

$T = +\infty \qquad \|u(\cdot, t)\|_\infty \leq C$

Pierre-Rolland 15 $0 \le \exists u = (u_j(\cdot,t)) \in C([0,+\infty), L^1(\Omega)^N)$ global-in-time weak solution

Pierre-S.-Yamada 19 $\mathcal{O} = \{u(\cdot,t)\} \subset L^1(\Omega)^N$ pre-compact

alternative proof of $T = +\infty$ $\|u(\cdot,t)\|_\infty \le C$ for n=2 via space control $(L^{1,\infty}_{x,t})$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \sum_j \tau_j u_j^2 \ dx + \sum_j d_j \|\nabla u_j\|_2^2 \le C(1 + \|u\|_3^3)$$

Gagliardo-Nirenberg $\|u\|_3^3 \le C\|u\|_1 \|u\|_{H^1}^2 \ (n=2)$ semigroup estimate

→ $\|u_0\|_1 \ll 1 \ \Rightarrow \ T = +\infty, \ \|u(t)\|_\infty \le C$ (epsilon regularity)

Localization → $\lim_{R\downarrow 0} \limsup_{t\uparrow T} \|u(t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists \varepsilon_0 \quad \Rightarrow \quad x_0 \notin \mathcal{S}$ blowup set

while $\lim_{R\downarrow 0} \limsup_{t\uparrow T} \|u(t)\|_{L^1(\Omega\cap B(x_0,R))} \ge \varepsilon_0$ impossible

because pre-compactness of $\mathcal{O} = \{u(\cdot,t)\} \subset L^1(\Omega)^N$ c.f. Smoluchowski-Poisson equation

→ $\mathcal{S} = \emptyset$

2. polynomial growth rate

$$\tau_j \frac{\partial u_j}{\partial t} - d_j \Delta u_j = f_j(u) \text{ in } Q_T$$

$$\left.\frac{\partial u_j}{\partial \nu}\right|_{\partial\Omega} = 0, \ \ u_j|_{t=0} = u_{j0}(x)$$

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[quasi-positive]

$f_j(u_1, \cdots, u_{j-1}, 0, u_{j+1}, \cdots, u_n) \geq 0, \ \forall j$

$0 \leq u_0 = (u_{j0}) \in \mathbf{R}^N$ $\longrightarrow$

$u = (u_j(\cdot, t)) \geq 0$

[mass dissipation]

$$\sum_{j=1}^N f_j(u) \leq 0, \ u = (u_j) \geq 0$$

$\longrightarrow$ $\frac{\partial}{\partial t}(\tau \cdot u) - \Delta(d \cdot u) \leq 0$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\|\tau \cdot u(t)\|_1 \leq \|\tau \cdot u_0\|_1$$

[polynomial growth rate]

$$|\nabla f_j(u)| \leq C(1 + |u|^{q-1}), \ 1 \leq j \leq N$$

Theorem 1 (S. 20) $\forall n, \ \forall q > 1$

$$\exists \lim_{t \uparrow T} \left(\frac{d \cdot u}{\tau \cdot u}\right)(\cdot, t) \text{ in } C(\overline{\Omega}) \ \Rightarrow \ T = +\infty, \ \|u(t)\|_\infty \leq C$$

Remark 1 Pierre-Schmitt 97 N=2

$\exists$ nonlinearity (fifth-order polynomials) inhomogeneous boundary conditions $T < +\infty,\ n = 10$

$$\frac{d \cdot u}{\tau \cdot u} = \frac{d_1 + d_2 v}{\tau_1 + \tau_2 v},\ v \equiv u_2/u_1 = \frac{c + d|x|^2/(T-t)}{a + b|x|^2/(T-t)}$$

$\exists$ the other example even for n=1

Problem 1 classification of self-similar blowup to v

c.f. N=2 $T < +\infty \quad \Rightarrow \quad \limsup_{t \uparrow T} \|u_j(t)\|_\infty = +\infty,\ j = 1, 2$

$$\frac{d \cdot u}{\tau \cdot u} = \frac{d_1 u_2^{-1} + d_2 u_1^{-1}}{\tau_1 u_2^{-1} + \tau_2 u_1^{-1}} \in C(\overline{\Omega} \times [0, T])?$$

obstruction - collision of blowup points

blowup profile?

$$\frac{\partial u}{\partial t} - \Delta u = u^2$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\frac{du}{dt} = u^2,\ u|_{t=0} = T^{-1}$$

$$\Rightarrow\ u(t) = (T-t)^{-1}$$

$$v = u^{-2} \geq 0$$

viscosity solution?

$$\frac{\partial v}{\partial t} - \Delta v = -\frac{3}{2} v^{-1} |\nabla v|^2 - 2v^{1/2},\ \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

locally uniformly in backward parabolic region

$$n \leq 2\ \Rightarrow\ u(x,t) = (T-t)^{-1} + o(1)$$

$\longrightarrow$ $u(\cdot, T)^{-1} = v(\cdot, T)^{1/2} \in [0, +\infty)$ blowup pattern

3. blowup analysis

Proof of Theorem 1

uniqueness of the classical solution … OK

$$\sup_{0\le t<T} \|u(t)\|_\infty \le C$$ → existence of global in time uniformly bounded classical solution

assume the contrary (subsequence) $\exists x_k \to x_0 \in \overline{\Omega},\ \exists t_k \uparrow T,\ |u(x_k,t_k)| \to +\infty$

$$0<r\ll 1,\ \tilde{u}^k(x,t) = r^\alpha u(rx+x_k, r^2t+t_k),\ \alpha = \frac{2}{q-1}$$

→
$$\tau_j \frac{\partial \tilde{u}_j^k}{\partial t} - d_j \Delta \tilde{u}_j^k = \tilde{f}_j(\tilde{u}^k),\ \tilde{u}^k = (\tilde{u}_j^k) \ge 0 \text{ in } \Omega_k \times (T_k^1, T_k^2),\ \left.\frac{\partial \tilde{u}_j^k}{\partial \nu}\right|_{\partial\Omega_k} = 0$$

$$\tilde{f}_j(u) = r^{2+\alpha} f_j(r^{-\alpha}u),\ \Omega_k = r^{-1}(\Omega - \{x_k\}),\ T_k^1 = -t_k/r^2,\ T_k^2 = (T-t_k)/r^2$$

drop k, large $\exists \gamma \subset \mathbf{R}^n$ smooth hyper-plane, $B_2 \cap \gamma \neq \emptyset$ or $= \emptyset$ $\quad 0 \in \tilde{B}_2 =$ one-side of B_2 cut by γ

$$\tau_j \frac{\partial \tilde{u}_j}{\partial t} - d_j \Delta \tilde{u}_j = \tilde{f}_j(\tilde{u}),\ \tilde{u} = (\tilde{u}_j) \ge 0 \text{ in } \tilde{Q}_2,\ \left.\frac{\partial \tilde{u}_j}{\partial \nu}\right|_{\gamma\cap B_2} = 0 \qquad \tilde{Q}_2 = \tilde{B}_2 \times (-4,0),\ \tilde{Q}_1 = \tilde{B}_1 \times (-1,0)$$

→ derive uniform estimate in $0<r\ll 1$

Lemma 1 (c.f. Capto-Vasseur)

$$\forall p > (\frac{n}{2}+1)(q-1), \ \exists \varepsilon_0 > 0$$

Moser's iteration scheme

$$\|\tilde{u}\|_{L^p(\tilde{Q}_1)} < \varepsilon_0 \ \Rightarrow \ 0 \leq \tilde{u}_j(0,0) \leq 1, \ 1 \leq j \leq N$$

mass conservation by a suspend unknown

$$\sum_j f_j(u) = 0, \ u = (u_j) \geq 0$$

$$\tilde{M} = \tau \cdot \tilde{u}, \ v = \tilde{M}\zeta, \ \zeta(x,t) = \varphi(x)\eta(t) \ \text{cut-off} \qquad \varphi \in C_0^\infty(B_2), \ \left.\frac{\partial \varphi}{\partial \nu}\right|_\gamma = 0, \ \eta \in C_0^\infty(-4,0]$$

⟹

$$\frac{\partial v}{\partial t} - \Delta(\tilde{d}v) = f \ \text{in} \ \tilde{Q}_2, \ \left.\frac{\partial}{\partial \nu}(\tilde{d}v)\right|_{B_2\cap\gamma} = 0 \qquad \tilde{d} = \frac{d\cdot\tilde{u}}{\tau\cdot\tilde{u}}$$

$$f = M\zeta_t - 2\nabla\cdot(\tilde{d}M\nabla\zeta) + \tilde{d}M\Delta\zeta$$

$$0 < d_* = \frac{\min_j d_j}{\max_j \tau_j} \leq \tilde{d}(x,t) \equiv \frac{d\cdot\tilde{u}}{\tau\cdot\tilde{u}} \leq d^* = \frac{\max_j d_j}{\min_j \tau_j} < +\infty \qquad \tilde{u} = (\tilde{u}_j)$$

VMO

Apply parabolic L^p maximal regularity uniform on compact set of coefficients in VMO to the dual system

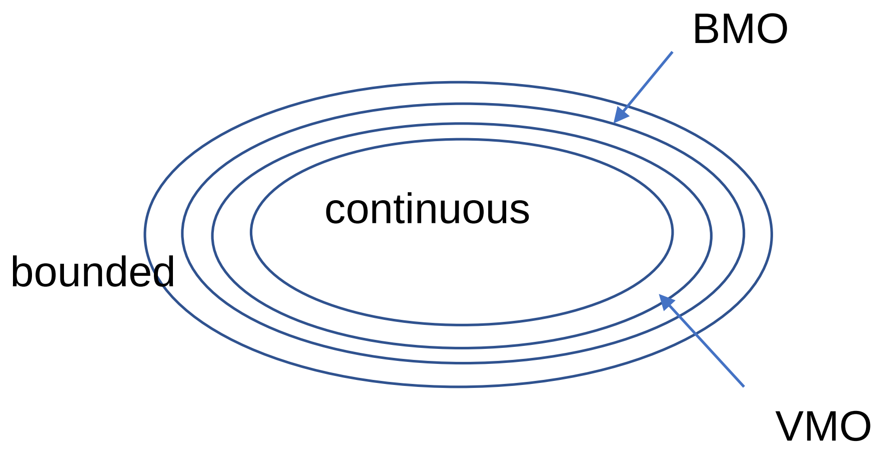
uniform estimate for coefficients in a compact set in VMO

Remark 5 $f \in \mathrm{VMO},\ \varepsilon > 0,\ x_0 \in \Omega\ \Rightarrow\ \exists g \in \mathrm{BMO},\ \exists r > 0$

$\|g\|_{BMO(B)} < \varepsilon,\ f = g \text{ in } B(x_0, r)$

VMO:BMO ~ continuous: bounded

local smallness of the BMO norm



Lemma 2 (maximal regularity, Weidemaier 05)

$$w_t + \tilde{d}\Delta w = -\theta \geq 0 \text{ in } \omega \times (-4, 0),\ \left.\frac{\partial w}{\partial \nu}\right|_{\partial\omega} = 0,\ w|_{t=0} = 0 \longrightarrow$$

$$\int_{-4}^{0} \|w(t)\|_{W^{2,p}(\tilde{Q}_1)}^p dt \leq C\|\theta\|_{L^p(\tilde{Q}_2)}^p,\ 1 < p < \infty,\ p \neq 3$$

$$\tilde{B}_1 \subset \mathrm{supp}\ \varphi \subset \omega \subset \tilde{B}_2,\ \partial\omega \text{ smooth}$$

duality argument between $v = \tilde{M}\zeta,\ \tilde{M} = \tau \cdot u$

Lemma 3 $\|\tilde{M}\|_{L^q(\tilde{Q}_2)} \leq C \sup_{-16<t<0} \|\tilde{M}(t)\|_{L^1(\tilde{B}_4)},\ 1 < q < n$

$$\longrightarrow \|\tilde{M}\|_{L^{p'}(\tilde{Q}_1)} \leq C \sup_{-16<t<0} \|\tilde{M}(t)\|_{L^1(\tilde{B}_4)},\ \frac{1}{p'} = \frac{1}{q} - \frac{1}{n},\ p' > \frac{n}{n-1},\ p' \neq \frac{3}{2}$$

Dual Alexandroff – Bakelman - Pucci estimate (Caputo-Goudon-Vasseiur)

Lemma 4 (FMT) $\|\tilde{M}\|_{L^{1+\frac{1}{n}}(\tilde{Q}_2)} \le C \sup_{-16<t<0} \|\tilde{M}(t)\|_{L^1(\tilde{B}_4)}$ ABP… $L^\infty - L^{n+1}$

Lemma 3+ Lemma 4 → $\forall p > 1,\ \exists \rho > 4,\ \|\tilde{M}\|_{L^p(\tilde{Q}_1)} \le C \sup_{-\rho^2<t<0} \|\tilde{M}(t)\|_{L^1(\tilde{B}_\rho)}$

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pre-scaled analysis → duality argument (CGV)

Lemma 5 $\sup_{-\rho^2<t<0} \|\tilde{M}(t)\|_{L^1(\tilde{B}_\rho)} \le Cr^\theta$

$$M = \tau \cdot u,\ \hat{d} = \frac{d \cdot u}{\tau \cdot u}$$

$$\exists \Phi,\ -\Delta\Phi = M,\ \left.\frac{\partial \Phi}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\frac{\partial \Phi}{\partial t} = \hat{d}\Delta\Phi,\ \left.\frac{\partial \Phi}{\partial \nu}\right|_{\partial\Omega} = 0$$ → (Krylov-Safonov)

$$[\Phi]_{C^\theta(\Omega\times(t_0,t_0+1))} \le C\|\Phi\|_{L^\infty(\Omega\times(t_0-1,t_0+1))}$$

Lemma 5+ Lemma 1 →

$$0 \le u_j(x_k,t_k) \le Cr^{-\alpha},\ 1 \le j \le N,\ 0 < r \ll 1$$

contradiction

consequences derived from this argument

Theorem 2

$$n = 3$$

$$1 < q < 9/5 \quad \Rightarrow \quad T = +\infty, \ \|u(t)\|_\infty \leq C$$

$q = \sigma$ ↓

Theorem 3 [entropy inequality]

$$\sum_j f_j(u) \log u_j \leq C(1 + |u|^\sigma), \ 1 < \sigma \leq 1 + \frac{2}{n}$$

$$n = 2, 3, \ 1 < q < 2 + \frac{2}{n}, \quad \sigma = 1 + \frac{2}{n}$$

n=2, critical dimension in this context

or $\quad n \geq 4, \ 1 < q < 2 + \frac{1}{n}, \ \sigma = 1 + \frac{1}{n} \quad \Rightarrow \quad T = +\infty, \ \|u(t)\|_\infty \leq C$

(especially, n=2, q=2) S.-Yamada 15

1. n>3 ⟶ dual ABP

   n=3 ⟶ L^2 duality argument is efficient

2. entropy inequality ⟶ local epsilon regularity in space-time

Reaction Diffusion Systems 3

Mathematics of Blowup Patterns

1. Semilinear Parabolic Equation

$$\frac{\partial u}{\partial t} - \Delta u = u^p \text{ in } Q_T$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x)$$

$\Omega \subset \mathbf{R}^n$ bounded domain

$\partial\Omega$ smooth boundary

$Q_T = \Omega \times (0,T)$

ν outer unit normal, $1 < p < \infty$

$0 < u_0 = u_0(x) \in C(\overline{\Omega})$

blowup of the solution

$$T < +\infty, \ \lim_{t\uparrow T} \|u(t)\|_\infty = +\infty$$

blowup set

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0, \ t_k \uparrow T, \ u(x_k,t_k) \to +\infty\}$$

Question 1 (profile) $\quad \exists \lim_{t\uparrow T} u(x,t) = u_*(x)$ locally uniformly in $x \in \overline{\Omega} \setminus \mathcal{S}$

Question 2 (continuation) $\quad \exists u = u(x,t) \in C(\overline{\Omega} \times [0,\tilde{T}), [-\infty,+\infty]), \ \tilde{T} > T$

Remark (S.-F. Takahashi)

$$u = u(x,t) \in C(\overline{\Omega} \times [0,T], [0,+\infty])$$

$$u_t - \Delta u \geq 0 \text{ in } \Omega \times [0,T] \setminus D$$

$$D = \bigcup_{0\leq t\leq T} D(t) \times \{t\}$$

$$D(t) = \{x \in \overline{\Omega} \mid u(x,t) = +\infty\} \subset \Omega, \ 0 \leq t \leq T$$

$$\Longrightarrow \quad \int_0^T \mathrm{Cap}_2((D(t))dt \leq \frac{1}{2} L^n(\Omega)$$

2. Blowup pattern

$$\frac{du}{dt} = u^p \qquad u(t) = \left(\frac{1}{p-1}\right)^{-\frac{1}{p-1}} (T-t)^{-\frac{1}{p-1}} \qquad \lim_{t\uparrow T} u(t) = +\infty$$

$$u|_{t=0} = u_0 > 0 \qquad T = \frac{1}{p-1} u_0^{-(p-1)} \qquad \lim_{t\downarrow T} u(t) = -\infty$$

Define $v > 0$ by $u^2 = v^{-\frac{1}{p-1}} > 0$

$$0 \leq \exists v = v(x,t) \in C(\overline{\Omega} \times [0,\infty))$$

$$v_t - \Delta v + \frac{2(2p-1)}{p-1} |\nabla \sqrt{v}|^2 + 2(p-1)\sqrt{v} = 0, \ \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \ v|_{t=0} = v_0(x) > 0$$

⟹

$$u_* = v_*^{-\frac{p-1}{2}} \in C(\overline{\Omega}), \ v_* = v(\cdot, T), \ T = \inf\{t > 0 \mid \inf_{\overline{\Omega}} v(\cdot, s) > 0, \ 0 < s < t\}$$ profile

$$u(x,t) = \zeta(x,t) v^{-\frac{p-1}{2}}(x,t), \ \zeta(x,t) = \begin{cases} +1, & t < T_x, \\ -1, & t > T_x, \end{cases} \quad T_x = \inf\{t > 0 \mid v(x,s) > 0, \ 0 < s < t\}$$ continuation

after stretching

given $\gamma > 0,\ 0 < v_0 = v_0(x) \in C^{2+\theta}(\overline{\Omega})$ find $0 \leq v = v(x,t) \in C(\overline{\Omega} \times [0,\infty))$

$$v_t - \Delta v + \gamma |\nabla \sqrt{v}|^2 + \sqrt{v} = 0, \ \left. \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0, \ v|_{t=0} = v_0(x) > 0$$

several exponents

$f(u) = u^p \Longrightarrow \gamma = \dfrac{2(2p-1)}{p-1} \in (4,\infty)$ $f(u) = \lambda e^u \Longrightarrow \gamma = 4$

elliptic singular solution $u(x) = C|x|^{-\frac{2}{p-1}},\ p > \dfrac{n}{n-2} \longrightarrow -\Delta u = u^p \text{ in } \mathbf{R}^n \setminus \{0\}$

$$\sqrt{v} = Cr^2 \in C^\infty$$

$\Updownarrow$

$$\gamma < n+2$$

3. Approach by functional analysis

scheme $\varepsilon > 0$ $\quad 0 \leq \exists 1 v_\varepsilon = v_\varepsilon(x,t) \in C^{2+\theta,1+\theta/2}(\overline{\Omega} \times [0,\infty))$

$$v_{\varepsilon t} - \Delta v_\varepsilon + \frac{\gamma}{4}\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon + \varepsilon} + \sqrt{v_\varepsilon} = 0, \ \left.\frac{\partial v_\varepsilon}{\partial \nu}\right|_{\partial\Omega} = 0, \ v_\varepsilon|_{t=0} = v_0(x) > 0$$

Uniqueness follows from the monotonicity of the nonlinearity $v \in [0,\infty) \mapsto \sqrt{v}$

$$\|v_\varepsilon(t)\|_\infty \leq \|v_0\|_\infty, \ \int_0^T \|\nabla\sqrt{v_\varepsilon + \varepsilon}\|_2^2 \ dt \leq C\|v_0\|_1$$

$$z_\varepsilon = \frac{\partial v_\varepsilon}{\partial \varepsilon} \geq 0, \ \exists \lim_{\varepsilon \downarrow 0} v_\varepsilon(x,t) = v(x,t) \geq 0, \ \forall (x,t)$$

$$w_\varepsilon = \sqrt{v_\varepsilon + \varepsilon} \quad \Longrightarrow \quad \|w_\varepsilon\|_{L^\infty(Q_T)} + \|\nabla w_\varepsilon\|_{L^2(Q_T)} \leq C$$

$$\exists \lim_{\varepsilon \downarrow 0} w_\varepsilon(x,t) = w(x,t) \equiv \sqrt{v(x,t)}, \ \forall (x,t)$$

$$w_\varepsilon \rightharpoonup w \text{ in } L^2(0,T;H^1(\Omega))$$

if $w_\varepsilon \to w \text{ in } L^2(0,T;H^1(\Omega))$ $\longrightarrow$ $0 \le v \in L^\infty(Q_T),\ \sqrt{v} \in L^2(0,T;H^1(\Omega))$ is a solution to

$$v_t - \Delta v + \gamma|\nabla\sqrt{v}|^2 + \sqrt{v} = 0,\ \ \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0,\ \ v|_{t=0} = v_0(x) > 0$$

i.e. $\forall \varphi \in C^2(\overline{\Omega}),\ \ \frac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} = 0$

$$\frac{d}{dt}\int_\Omega v\varphi + (\nabla v, \nabla\varphi) + \int_\Omega (\gamma|\nabla\sqrt{v}|^2 + \sqrt{v})\varphi = 0$$

in the sense of distributions

regard $v \in L^2(0,T;H^1(\Omega))$ by $\nabla v = 2\sqrt{v}\nabla\sqrt{v} \in L^2(Q_T)$

Theorem 1

$w \in C(\overline{Q_T})$
$\gamma \ge 2$
$\longrightarrow$ $w_\varepsilon \to w \text{ in } L^2(0,T;H^1(\Omega))$

$$v_{\varepsilon t} - \Delta v_\varepsilon + \frac{\gamma}{4}\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon + \varepsilon} + \sqrt{v_\varepsilon} = 0, \quad \left.\frac{\partial v_\varepsilon}{\partial \nu}\right|_{\partial\Omega} = 0, \quad v_\varepsilon|_{t=0} = v_0(x) > 0$$

$$w_\varepsilon = \sqrt{v_\varepsilon + \varepsilon} > 0$$

$$\exists \lim_{\varepsilon \downarrow 0} w_\varepsilon = w \equiv \sqrt{v}$$ pointwise monotone

$$w_{\varepsilon t} - \Delta w_\varepsilon = -g_\varepsilon, \quad \left.\frac{\partial w_\varepsilon}{\partial \nu}\right|_{\partial\Omega} = 0, \quad w_\varepsilon|_{t=0} = w_{\varepsilon 0}(x) \geq 0$$

$$g_\varepsilon = 2(\gamma - 2)|\nabla\sqrt{w_\varepsilon}|^2 + \frac{\sqrt{v_\varepsilon}}{2\sqrt{v_\varepsilon + \varepsilon}} \geq 0$$

$$w_\varepsilon \rightharpoonup w \text{ in } L^2(0,T;H^1(\Omega))$$

$$v_\varepsilon = w_\varepsilon^2 - \varepsilon$$

$$\frac{d}{dt}\int_\Omega w_\varepsilon = -\int_\Omega g_\varepsilon, \quad w_\varepsilon \geq 0$$

$$\longrightarrow \quad \|g_\varepsilon\|_{L^1(Q_T)} \leq C$$

subsequence $\quad g_\varepsilon \rightharpoonup \exists\mu \in \mathcal{M}(\overline{Q_T}) = C'(\overline{Q_T})$

L^1 compact property of the heat equation (c.f. Pierre-S. Yamada 15)

$$w \in C([0,\infty), L^1(\Omega))$$

Bothe-Pierre 10

Remark

$$w_\varepsilon \to w \text{ in } L^p(Q_T), \ 1 \leq p < \frac{n+2}{n}, \quad \nabla w_\varepsilon \to \nabla w \text{ in } L^q(Q_T), \ 1 \leq q < \frac{n+2}{n+1}$$

$$\nabla v_\varepsilon = 2w_\varepsilon \nabla w_\varepsilon \qquad \frac{1}{\frac{n+2}{n}} + \frac{1}{\frac{n+2}{n+1}} = \frac{2n+1}{n+2} \geq 1$$

$$w_\varepsilon = \sqrt{v_\varepsilon + \varepsilon} > 0 \qquad w_{\varepsilon t} - \Delta w_\varepsilon = -g_\varepsilon, \ \left.\frac{\partial w_\varepsilon}{\partial \nu}\right|_{\partial\Omega} = 0, \ w_\varepsilon|_{t=0} = w_{\varepsilon 0}(x) \geq 0 \qquad w_\varepsilon \downarrow w \equiv \sqrt{v}$$

pointwise monotone

Lemma $w \in C(\overline{Q_T})$ ⟶ $\displaystyle\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2 \, dt \leq \langle w_\varepsilon - w, \mu\rangle + \frac{1}{2}\|w_{\varepsilon 0} - w_0\|_2^2$

<u>Proof</u> $(w_\varepsilon - w_{\varepsilon'})_t - \Delta(w_\varepsilon - w_{\varepsilon'}) = -g_\varepsilon + g_{\varepsilon'}, \ \left.\frac{\partial}{\partial \nu}(w_\varepsilon - w_{\varepsilon'})\right|_{\partial\Omega} = 0, \ (w_\varepsilon - w_{\varepsilon'})|_{t=0} = 0$

$$\frac{1}{2}\frac{d}{dt}\|w_\varepsilon - w_{\varepsilon'}\|_2^2 + \|\nabla(w_\varepsilon - w_{\varepsilon'})\|_2^2 = (w_\varepsilon - w_{\varepsilon'}, -g_\varepsilon + g_{\varepsilon'}) \leq (w_\varepsilon - w_{\varepsilon'}, g_{\varepsilon'}) \qquad 0 < \varepsilon' < \varepsilon$$

$$\longrightarrow \int_0^T \|\nabla(w_\varepsilon - w_{\varepsilon'})\|_2^2 \, dt \leq \int_0^T (w_\varepsilon - w_{\varepsilon'}, g_{\varepsilon'}) \, dt + \frac{1}{2}\|w_{\varepsilon 0} - w_{\varepsilon' 0}\|_2^2$$

$$\liminf_{\varepsilon' \downarrow 0} \int_0^T \|\nabla(w_\varepsilon - w_{\varepsilon'})\|_2^2 \, dt \geq \int_0^T \|\nabla(w_\varepsilon - w)\|_2^2 \, dt \qquad w_\varepsilon \rightharpoonup w \text{ in } L^2(0,T;H^1(\Omega))$$

Dini's theorem

$w \in C(\overline{Q_T})$ ⟶ $w_{\varepsilon'} \downarrow w$ uniformly on $\overline{Q_T}$

$$\lim_{\varepsilon' \downarrow 0} \int_0^T (w_{\varepsilon'}, g_{\varepsilon'}) dt = \langle w, \mu\rangle \qquad g_\varepsilon \rightharpoonup \mu \in \mathcal{M}(\overline{Q_T}) = C'(\overline{Q_T})$$

Proof of Theorem 1 $$\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2 \, dt \le \langle w_\varepsilon - w, \mu \rangle + o(1)$$

$w \in C(\overline{Q_T})$ → (Dini's theorem) $w_\varepsilon \downarrow w$ uniformly on $\overline{Q_T}$

$$\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2 \, dt = o(1)$$

Theorem 2 $\gamma \ge 2$ $\{g_\varepsilon\} \subset L^1(Q_T)$ compact → $w_\varepsilon \to w \text{ in } L^2(0,T;H^1(\Omega))$

Proof
$$\int_0^T \|\nabla(w_\varepsilon - w_{\varepsilon'})\|_2^2 \, dt \le \int_0^T (w_\varepsilon - w_{\varepsilon'}, g_{\varepsilon'}) \, dt + \frac{1}{2}\|w_{\varepsilon 0} - w_{\varepsilon' 0}\|_2^2$$

$$\lim_{\varepsilon' \downarrow 0} \int_0^T (w_{\varepsilon'}, g_{\varepsilon'}) dt = \iint_{Q_T} w g \, dx dt$$

$w_\varepsilon \rightharpoonup w \text{ in } L^\infty(Q_T) = L^1(Q_T)'$
$g_\varepsilon \to g \in L^1(Q_T)$

$$\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2 \, dt \le \iint_{Q_T} (w_\varepsilon - w) g \, dx dt + o(1)$$

$$\limsup_{\varepsilon \downarrow 0} \int_0^T \|\nabla(w_\varepsilon - w)\|_2^2 \, dt \le 0$$ monotone convergence theorem

Theorem 3 $2 \leq \gamma < 4 \quad 0 \leq \exists z \in C(\overline{Q_T}) \quad w \geq z,\ \liminf_{\varepsilon \downarrow 0} \langle w_\varepsilon - z, \mu \rangle \leq 0 \ \Longrightarrow \ w_\varepsilon \to w \text{ in } L^2(0,T;H^1(\Omega))$

Lemma $$\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2\, dt \leq \langle w_\varepsilon, \mu \rangle - \iint_{Q_T} \frac{\gamma-2}{2}|\nabla w|^2 + \frac{w}{2}\, dxdt + \frac{1}{2}\|\sqrt{v_0+\varepsilon} - \sqrt{v_0}\|_2^2$$

Proof $$\int_0^T \|\nabla(w_\varepsilon - w_{\varepsilon'})\|_2^2\, dt \leq \int_0^T (w_\varepsilon - w_{\varepsilon'}, g_{\varepsilon'})\, dt + \frac{1}{2}\|w_{\varepsilon 0} - w_{\varepsilon' 0}\|_2^2$$

$$(w_{\varepsilon'}, g_{\varepsilon'}) = \int_\Omega \frac{\gamma-2}{2}|\nabla w_{\varepsilon'}|^2 + \frac{1}{2}\frac{\sqrt{v_{\varepsilon'}}}{\sqrt{v_{\varepsilon'}+\varepsilon'}} w_{\varepsilon'}\, dx \ \Longrightarrow \ \liminf_{\varepsilon' \downarrow 0} \int_0^T (w_{\varepsilon'}, g_{\varepsilon'})dt \geq \iint_{Q_T} \frac{\gamma-2}{2}|\nabla w|^2 + \frac{w}{2}\, dxdt$$

Proof of Theorem 3 $$\langle w_\varepsilon, \mu \rangle = \langle w_\varepsilon - z, \mu \rangle + \langle z, \mu \rangle \leq \langle z, \mu \rangle + o(1) = \langle z, g_\varepsilon \rangle + o(1)$$

$$\langle z, g_\varepsilon \rangle \leq \iint_{Q_T} \frac{\gamma-2}{2}\frac{|\nabla w_\varepsilon|^2}{w_\varepsilon} w + \frac{1}{2}\frac{\sqrt{v_\varepsilon}}{\sqrt{v_\varepsilon+\varepsilon}} w\, dxdt \leq \iint_{Q_T} \frac{\gamma-2}{2}|\nabla w_\varepsilon|^2 + w\, dxdt + o(1)$$

$$\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2\, dxdt \leq \frac{\gamma-2}{2}\iint_{Q_T} |\nabla w_\varepsilon|^2 - |\nabla w|^2\, dxdt + o(1) \qquad w_\varepsilon \rightharpoonup w \text{ in } L^2(0,T;H^1(\Omega))$$

$$= \frac{\gamma-2}{2}\int_0^T \|\nabla(w_\varepsilon - w)\|_2^2\, dt + o(1) \qquad \gamma < 4 \ \Longrightarrow \ \int_0^T \|\nabla(w_\varepsilon - w)\|_2^2\, dt = o(1)$$