Elliptic Theory 1

Interface Vanishing of Non-stationary Maxwell Equation

1. Introduction

Non-stationary Maxwell equation

E: electric field, B: magnetic field, J : current density, ρ : electric charge

$$\nabla \times B - \frac{\partial E}{\partial t} = J, \ \nabla \cdot E = \rho$$
$$\nabla \times E + \frac{\partial B}{\partial t} = 0, \ \nabla \cdot B = 0 \quad \text{in } \Omega$$

$$(x,t) \in \Omega \subset \mathbf{R}^4, \ \text{domain}$$
$$x = (x_1, x_2, x_3), \ \nabla = \nabla_x$$

Definition $\Omega \subset \mathbf{R}^n$ region with interface ⟺ $\exists \mathcal{M}, \Gamma \equiv \Omega \cap \mathcal{M} \neq \emptyset$

smooth non-compact hyper-surface without boundary

$$\longrightarrow \quad \Omega = \Omega_+ \cup \Gamma \cup \Omega_-, \quad \Gamma_\pm = \partial \Omega_\pm = \partial \Omega (= \Gamma)$$

Assumption discontinuity of permeability, electric conductivity ⟶ interface of J, ρ

(magnetoencephalography)

conclusion interface vanishing of some components of B, E

Theorem 1 $\Omega \subset \mathbf{R}^4$ region with interface $E, B \in H^1(\Omega)^3,\ J \in L^2(\Omega)^3,\ \rho \in L^2(\Omega)$

$$\nabla \times B - \frac{\partial E}{\partial t} = J,\ \nabla \cdot E = \rho$$
$$\nabla \times E + \frac{\partial B}{\partial t} = 0,\ \nabla \cdot B = 0 \quad \text{in } \Omega$$

$$\nabla \times J \in L^2(\Omega_\pm)^3$$
$$\frac{\partial J}{\partial t} + \nabla\rho \in L^2(\Omega_\pm)^3$$

$\Longrightarrow$

$$(-\partial_t^2 + \Delta_x)(\nu^0 B + \tilde{\nu} \times E) \in L^2(\Omega)^3$$
$$(-\partial_t^2 + \Delta_x)(\tilde{\nu} \cdot B) \in L^2(\Omega)$$

in the sense of distributions

$$\nu = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \\ \nu^0 \end{pmatrix}, \quad \tilde{\nu} = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix}$$

outer normal unit on Γ_-

H2 singularities of the above components of electric magnetic fields pass through the interface with light velocity

Remark

$$(-\partial_t^2 + \Delta_x)E \in L^2(\Omega_\pm)^3$$
$$(-\partial_t^2 + \Delta_x)B \in L^2(\Omega_\pm)^3$$

Interface vanishing does not occur to all components

Corollary $\Omega \subset \mathbf{R}^3$ region with interface ν outer unit normal on Γ_-

1. $B \in H^1(\Omega)^3,\ \nabla \times B = J,\ \nabla \cdot B = 0$ in Ω

$\nabla \times J \in L^2(\Omega_\pm)^3 \quad \Rightarrow \quad \Delta(\nu \cdot B) \in L^2(\Omega)$

Kobayashi-S.-Watanabe 03

2. $E \in H^1(\Omega)^3,\ \nabla \times E = 0,\ \nabla \cdot E = \rho$ in Ω

$\nabla \rho \in L^2(\Omega_\pm)^3 \quad \Rightarrow \quad \Delta(\nu \times E) \in L^2(\Omega)$

Kanou-Sato-Watanabe 13

layer potential, 3-D vector analysis, differential forms

$H^q(\Lambda^p(D))$ p-forms with Hq coefficients on $D = \Omega,\ \Omega_\pm$

$$B \in H^0(\Lambda^1(D)),\ B^\nu = (B,\nu)\nu,\ B^\tau = B - B^\nu,\ \nu = \sum_i \nu^i dx_i$$

Theorem 2 $\Omega \subset \mathbf{R}^n$ region with interface ν outer unit normal on Γ_- $B \in H^1(\Lambda^1(\Omega))$

1. $dB = J,\ \delta B = 0,\ J \in H^0(\Lambda^2(\Omega)),\ \delta J \in H^0(\Lambda^1(\Omega_\pm)) \ \Rightarrow\ \Delta B^\nu \in H^0(\Lambda^1(\Omega))$

2. $dB = 0,\ \delta B = g,\ g \in H^0(\Lambda^0(\Omega)),\ dg \in H^0(\Lambda^0(\Omega_\pm)) \ \Rightarrow\ \Delta B^\tau \in H^0(\Lambda^1(\Omega))$

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strategy

1. Maxell equation ⟶ 2-form equation on the Minkowski space
2. interface vanishing of 2-form on Riemann space
3. interface vanishing on Minkowski metric

obstruction

1. Normal and tangential components of 2-forms are not defined
2. Traces on interface are beyond functions

Theorem 3 $\Omega \subset \mathbf{R}^n$ region with interface ν outer unit normal on Γ_- $\omega \in H^1(\Lambda^2(\Omega))$

$$d\omega = \theta, \ \delta\omega = 0, \ \delta\theta \in H^0(\Lambda^2(\Omega_\pm)) \ \Rightarrow \ \Delta(\nu, \hat{\omega}^i) \in H^0(\Lambda^0(\Omega)), \ 1 \leq i \leq n$$

$$\hat{\omega}^i = \sum_\ell \tilde{\omega}^{\ell i} dx_i, \quad \tilde{\omega}^{ij} = \begin{cases} \omega^{ij}, & i < j \\ 0, & i = j \\ -\omega^{ij}, & i > j \end{cases}, \quad \omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j$$

1. T. Suzuki, Mean Field Theories and Dual Variation, 2nd edition, Mathematical Structures of the Mesoscopic Model, Atlantis Press, Paris, 2015.
2. 鈴木貴, 数理医学入門, 共立出版, 2015.

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2. Preliminaries

$D \subset \mathbf{R}^n$ open set

$H^q(\Lambda^p) = H^q(\Lambda^p(\Omega)) = \{p\text{-forms} \mid \text{coefficients are in } H^q\}$

$\Lambda^p = \Lambda^p(D) = \{p\text{-forms}\},\ \wedge$ wedge product, $d : \Lambda^p \to \Lambda^{p-1}$ outer derivative

$L^2(\Lambda^0) = L^2(D)$

$\alpha = \sum_\ell \alpha^\ell dx_\ell,\ \beta = \sum_\ell \beta^\ell dx_\ell$ 1-forms $\Longrightarrow$ $(\alpha, \beta) = \sum_\ell \alpha^\ell \beta^\ell$

$\lambda = \alpha_1 \wedge \cdots \wedge \alpha_p,\ \mu = \beta_1 \wedge \cdots \wedge \beta_p$ p-forms $\Longrightarrow$ $(\lambda, \mu) = \det\ ((\alpha_i, \beta_j))_{i,j}$

$* : \Lambda^p(D) \to \Lambda^{n-p}(D)$ Hodge operator $\omega \wedge \tau = (*\omega, \tau)\ dx_1 \wedge \cdots \wedge dx_n,\ \omega \in \Lambda^p(D),\ \tau \in \Lambda^{n-p}(\Lambda)$

$\Longrightarrow$ $*(dx_{j_1} \wedge \cdots \wedge dx_{j_p}) = \mathrm{sgn}\ \sigma \cdot dx_{j_{p+1}} \wedge \cdots \wedge dx_{j_n},\ \sigma : (1, \cdots, n) \mapsto (j_1, \cdots, j_n)$

co-derivative

$\delta = (-1)^p *^{-1} d* : \Lambda^p(D) \to \Lambda^{p-1}(D)$

$$B = \sum_i B^i dx_i$$

$$\Rightarrow\ \delta B = -\sum_i B^i_i$$

$$\omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j$$

$$\Rightarrow\ \delta\omega = -\sum_{i,\ell} \tilde{\omega}^{\ell i}_\ell dx_i$$

$$\tilde{\omega}^{ij} = \begin{cases} \omega^{ij}, & i < j \\ 0, & i = j \\ -\omega^{ij}, & i > j \end{cases}$$

Laplacian

$-\Delta = \delta d + d\delta : \Lambda^p \to \Lambda^p$

$D \subset \mathbf{R}^n$ Lipschitz domain $\quad \exists \gamma : H^1(D) \to H^{1/2}(\partial D)$ trace operator $\quad H^{1/2}(\partial D) \cong H^1(D)/H^1_0(\Omega)$

$\longrightarrow$ $C^\infty(\overline{D}) \subset H^1(D)$ dense $\quad$ write $\quad \varphi|_{\partial D} = \gamma\varphi, \quad \varphi \in H^1(D)$

ν outer unit normal vector

$\exists ds$ (area element) $\quad \nu^i ds = *dx_i, \ q \le i \le n$

Lemma 1 $\quad B \in \Lambda^1(D),\ C \in \Lambda^2(D) \longrightarrow *B = (B,\nu)\ ds, \ B \wedge *C = (\nu \wedge B, C)\ ds$

write $\quad \int_D \cdots\ dx_1 \wedge \cdots \wedge dx_n = \int_D, \quad \int_{\partial D} \cdots\ ds = \int_{\partial D},$

Lemma 2 $\quad \varphi \in H^1(\Lambda^0)$, $B \in H^1(\Lambda^1)$, $J \in H^1(\Lambda^2)$ $\longrightarrow$

$$\int_D (\delta B, \varphi) = \int_D (B, d\varphi) - \int_{\partial D} (B,\nu)\varphi \qquad \text{Gauss}$$

$$\int_D (dB, J) = \int_D (B, \delta J) + \int_{\partial D} (\nu \wedge B, J) \qquad \text{Stokes}$$

Lemma 3 $p \in H^1(\Lambda^0) \quad H^{-1/2}(\partial D) = H^{1/2}(\partial D)'$

1. $\Delta p \in H^1(D)' \;\Rightarrow\; (dp, \nu)|_{\partial D} \in H^{-1/2}(\partial D)$

$$\langle (dp,\nu), \varphi \rangle = \int_D (dp, d\varphi) + \langle \Delta p, \varphi \rangle, \;\forall \varphi \in H^1(D)$$

2. $\nu \wedge dp|_{\partial D} \in H^{-1/2}(\Lambda^2(\partial D))$

$$\langle \nu \wedge dp, J \rangle = -\int_D (dp, \delta J), \;\forall J \in H^1(\Lambda^2(D))$$

$\Omega \subset \mathbf{R}^n$ region with interface $\quad \nu$ outer unit normal on Γ_-

Notation f: 0- form $\quad f_i = \dfrac{\partial f}{\partial x_i}$

identify 1-form ⟷ vector field

$$(\nu, d)f = (\nu, df) = \sum_i \nu^i f_i = \frac{\partial f}{\partial \nu}$$

Lemma 4

$$p \in H^0(\Lambda^0(\Omega)) \;\Rightarrow\; [\nu \wedge dp]_-^+ = 0, \; H^{-1/2}(\Lambda^2(\Gamma))$$

$$[\nu \wedge dp]_-^+ = \nu \wedge dp|_{\Gamma_+} - \nu \wedge dp|_{\Gamma_-}$$

proof $J \in C_0^\infty(\Omega)$ 2-form

$$\pm \langle \nu \wedge dp, J \rangle = \int_{\Omega_\pm} (dp, \delta J)$$

$$\longrightarrow \quad [\langle \nu \wedge dp, J \rangle]_-^+ = \int_\Omega (dp, \delta J)$$

Stokes

$$\int_\Omega (dB, J) = \int_\Omega (B, \delta J) + \int_{\partial\Omega} (\nu \wedge B, J)$$

$$B = dp, \; p \in H^2(\Omega) \quad \longrightarrow \quad \int_\Omega (dp, \delta J) = 0$$

$C^\infty(\overline{D}) \subset H^1(D)$ dense

Lemma 5 $\omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j \in H^1(\Lambda^2(\Omega))$

$$\hat{\omega}^i = \sum_\ell \tilde{\omega}^{\ell i} dx_i, \quad \tilde{\omega}^{ij} = \begin{cases} \omega^{ij}, & i<j \\ 0, & i=j \\ -\omega^{ij}, & i>j \end{cases}$$

$\Rightarrow$

$$\left[\delta\omega + \sum_i [(\nu, d)(\nu, \hat{\omega}^i) dx_i\right]_-^+ = 0 \quad \text{in } H^{-1/2}(\Gamma)$$

Notation $A \sim B \Leftrightarrow A - B \in H^1(\Omega)$

$\rightarrow$ $[A-B]_-^+ = 0 \quad \text{in } H^{1/2}(\Gamma)$

Proof show;

$$\left[\sum_\ell \tilde{\omega}_\ell^{\ell i} - \frac{\partial}{\partial \nu}(\nu, \hat{\omega}^i)\right]_-^+ = 0 \ \text{ in } H^{-1/2}(\Gamma), \ 1 \le i \le n$$

i: fix $\quad B^\ell = \tilde{\omega}^{\ell i}, \ B = \sum_\ell B^\ell dx_\ell (= \hat{\omega}^i)$

$\rightarrow$ $$\sum_\ell \tilde{\omega}_\ell^{\ell i} - \frac{\partial}{\partial \nu}(\nu, \hat{\omega}^i) = \sum_\ell \{B_\ell^\ell - \nu^\ell (\nu, B)_\ell\}$$

$$(\nu, B) = \sum_k \nu^k B^k$$

$$\sum_\ell \tilde{\omega}_\ell^{\ell i} - \frac{\partial}{\partial \nu}(\nu, \hat{\omega}^i) \sim \sum_\ell B_\ell^\ell - \sum_{\ell, k} \nu^\ell \nu^k B_\ell^k$$

$$= \sum_\ell B_\ell^\ell - \sum_{k,\ell} \nu^k \nu^\ell B_k^\ell = \sum_\ell \{B_\ell^\ell - \nu^\ell (\nu, d) B^\ell\}$$

$p = B^\ell$ $\rightarrow$ $$B_\ell^\ell - \nu^\ell(\nu, d) B^\ell = p_\ell - \nu^\ell (\nu, d) p$$

$$= \sum_k \{(\nu^k)^2 p_\ell - \nu^\ell \nu^k p_k\} = \sum_k \nu^k (\nu^k p_\ell - \nu^\ell p_k)$$

Lemma 4

$$[B_\ell^\ell - \nu^\ell (\nu, d) B^\ell]_-^+ = \sum_k \nu^k \left[\nu^k p_\ell - \nu^\ell p_k\right]_-^+ = 0$$

on $H^{-1/2}(\Gamma)$

Proof of Theorem 3 $\omega \in H^1(\Lambda^2(\Omega))$

$$d\omega = \theta \in L^2(\Omega), \delta\theta \in H^0(\Lambda^2(\Omega_\pm))$$

$$\Rightarrow \ -\Delta\omega = (d\delta + \delta d)\omega = \delta\theta \in L^2(\Omega_\pm)$$

$$-\Delta(\nu, \hat{\omega}^i) = \exists h^i_\pm \text{ in } L^2(\Omega_\pm)$$
$$\exists h^i \in L^2(\Omega), \ h^i = h^i_\pm \text{ in } \Omega_\pm$$

Lemma 5

$$\delta\omega = 0 \ \Rightarrow \ \left[\frac{\partial}{\partial\nu}(\nu, \hat{\omega}^i)\right]_-^+ = 0 \text{ in } H^{-1/2}(\Gamma)$$

Gauss

$$\int_\Omega h^i\varphi = \int_\Omega (-\Delta\varphi)\cdot(\nu, \hat{\omega}^i), \ \forall\varphi \in C_0^\infty(\Omega)$$

$$\Rightarrow \quad -\Delta(\nu, \hat{\omega}^i) = h^i \in L^2(\Omega)$$

Proof of Theorem 1 $\mathbf{R}^4 \cong \mathbf{R}^{3,1}$ Minkowski space

$$\alpha = \sum_{i=1}^3 \alpha^i dx_i + \alpha^0 dx_0, \ \beta = \sum_{i=1}^3 \beta^i dx_i + \beta^0 dx_0 \quad \text{1-forms}$$

$$(\alpha, \beta) = -\sum_{i=1}^3 \alpha^i\beta^i + \alpha^0\beta^0 \qquad x = (x_1, x_2, x_3), \ t = x_0$$

$$d\delta + \delta d = -\frac{\partial^2}{\partial t^2} + \Delta_x : \Lambda^p(D) \to \Lambda^p(D)$$

Maxwell equation $d\omega = 0, \ d * \omega = -j \text{ in } \Omega$

$$\omega = E^2 dx_0 \wedge dx_1 + E^2 dx_0 \wedge dx_2 + E^3 dx_0 \wedge dx_3$$
$$-B^1 dx_2 \wedge dx_3 - B^2 dx_3 \wedge dx_1 - B^3 dx_1 \wedge dx_2$$
$$j = J^1 dx_0 \wedge dx_2 \wedge dx_3 + J^2 dx_0 \wedge dx_3 \wedge dx_1$$
$$+J^3 dx_0 \wedge dx_1 \wedge dx_2 + \rho dx_1 \wedge dx_2 \wedge dx_3$$

$$E = \begin{pmatrix} E^1 \\ E^2 \\ E^3 \end{pmatrix}, \quad B = \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}, \quad J = \begin{pmatrix} J^1 \\ J^2 \\ J^3 \end{pmatrix}.$$

Elliptic Theory 2

Recursive Hierarchy in Boltzmann-Poisson Equation

1. Point Vortices

2D Euler Equation
(simply connected domain)

$$v_t + (v \cdot \nabla)v = -\nabla p$$
$$\nabla \cdot v = 0$$
$$\nu \cdot v|_{\partial\Omega} = 0$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}$$ gradient

vorticity

$$\omega = \nabla^\perp v$$
$$\nabla^\perp = \begin{pmatrix} \frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_1} \end{pmatrix}$$
$$x = (x_1, x_2)$$

vortex equation

$$\omega_t + v \cdot \nabla\omega = 0$$
$$v = \nabla^\perp \psi$$
$$\Delta\psi = -\omega$$ stream function
$$\psi|_{\partial\Omega} = 0$$

point vortices $$\omega(dx,t) = \sum_{i=1}^N \alpha_i \delta_{x_i(t)}(dx)$$

Kirchhoff equation $$\alpha_i \frac{dx_i}{dt} = \nabla_i^\perp H, \ 1 \le i \le N$$

Hamiltonian $$H = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

Green's function
$$-\Delta G(x,x') = \delta_{x'}(dx)$$
$$G(x,x')|_{\partial\Omega} = 0$$
$$(x,x') \in \overline{\Omega} \times \Omega$$

Robin function $$R(x) = \left[G(x,x') + \frac{1}{2\pi} \log|x - x'| \right]_{x'=x}$$

Onsager 49

Hamiltonian $H = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$

$H = \hat{H}_N(x_1, \cdots, x_N)$ $N \gg 1$ total energy

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad 1 \leq i \leq N$$

$$p_i = p_i(t), \ q_i = q_i(t) \ \in \mathbf{R}^2$$

micro-canonical ensemble $\mathbf{R}^{4N}/\{H = E\}$

$$x = (q_1, \ldots, q_N, p_1, \ldots, p_N)$$

co-area formula $dx = dE \cdot \frac{d\Sigma(E)}{|\nabla H|}$

$$d\Sigma(E) \ \leftrightarrow \ \{x \in \mathbf{R}^{4N} \mid H(x) = E\}$$

→ canonical ensemble

thermal equilibrium

Gibbs measure $\mu^{E,N} = \frac{1}{W(E)} \cdot \frac{d\Sigma(E)}{|\nabla H|}$

weight factor $W(E) = \int_{H=E} \frac{d\Sigma(E)}{|\nabla H|}$

inverse temperature $\beta = \frac{\partial}{\partial E} \log W(E) = \frac{\Theta''(E)}{\Theta'(E)}$

$$\Theta(E) = \int_{H<E} dx = \int_{-\infty}^{E} W(E')dE'$$

↑ bounded monotone

$E \gg 1 \Rightarrow \beta < 0$ ordered structure in negative temperature

Joyce-Montgomery 73

micro-canonical statistics

$$\mathbf{R}^{4N}/\{H=E\}$$

$$x=(q_1,\dots,q_N,p_1,\dots,p_N)$$

$$dx = dE\cdot\frac{d\Sigma(E)}{|\nabla H|}$$

$$d\Sigma(E)\ \leftrightarrow\ \{x\in\mathbf{R}^{4N}\mid H(x)=E\}$$

micro-canonical measure

$$d\mu^{E,N}=\frac{1}{W(E)}\cdot\frac{d\Sigma(E)}{|\nabla H|}$$

weight factor

$$W(E)=\int_{\{H=E\}}\frac{d\Sigma(E)}{|\nabla H|}$$

canonical statistics

$$\mathbf{R}^{4N}/\{T\}$$

inverse temperature

$$\beta=1/(kT)$$

canonical measure

$$d\mu^{\beta,N}=\frac{e^{-\beta H}dx}{Z(\beta,N)}$$

weight factor

$$Z(\beta,N)=\int_{\mathbf{R}^{4N}}e^{-\beta H}dx$$

thermo-dynamical relation

$$\beta=\frac{\partial}{\partial E}\log W(E)$$

micro-canonical probability measure

$$\mu^n=\mu^n(dx_1,\dots,dx_n)$$

one point pdf

$$\rho_1^n(x_i)dx_i = \int_{\Omega^{n-1}}\mu^n(dx_1\dots dx_{i-1}dx_{i+1}dx_n)$$

↑

equal a priori probability (independent of i)

k-point reduced pdf

$$\rho_k^n(x_1,\dots,x_k)dx_1\dots dx_k = \int_{\Omega^{n-k}}\mu^n(dx_{k+1},\dots,dx_n)$$

stationary point vortices

$$\omega_N(x)dx=\sum_{i=1}^N\alpha\delta_{x_i}(dx)$$

⟶

$$\langle\omega_N(x)\rangle = \sum_{i=1}^N\int_{\Omega^N}\alpha\delta(x_i-x)\mu^N(dx_1\dots dx_N) = N\alpha\rho_1^N(x)$$

phase mean

high energy limit
(single intensity)

$$\alpha_i = \hat{\alpha}, \ N \uparrow +\infty, \ \hat{\alpha} N = 1$$
$$\hat{H}_N = H, \ \hat{\alpha}^2 N \hat{\beta} = \beta$$

→ two point pdf compatibility

$$\hat{H}_N(x_1, \ldots, x_N) = \sum_i \frac{\alpha_i^2}{2} R(x_j) + \sum_{i<j} \alpha_i \alpha_j G(x_i, x_j)$$

Boltzmann

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}$$

↕ duality

Poisson

$$\psi = \int_\Omega G(\cdot, x')\rho(x')dx'$$

energy $\tilde{E} = H$

inverse temperature $\tilde{\beta} = \frac{\partial}{\partial \tilde{E}} \log W(\tilde{E})$

weight factor $W(\tilde{E}) = \int_{H=\tilde{E}} \frac{d\Sigma_{\tilde{E}}}{|\nabla H|}$

mean field limit $\lim_{N\to\infty} \langle \omega_N(x) \rangle = \rho(x) = \lim_{N\to\infty} N\alpha\rho_1^N(x)$

propagation of chaos
(factorization property)

$$\rho_k^N \rightharpoonup \rho^{\otimes k} = \Pi_{i=1}^k \rho(x_i)$$

(Suzuki's uniqueness theorem)

rigorous derivation

Caglioti-Lions-Marchioro-Pulvirenti 92, 95. Kiessling 93

1. Bounded Boltzmann weight factors {z}
2. Uniqueness of the solution to the limit equation

→

1. convergence to the limit
2. canonical-micro canonical equivalence in the limit
3. propagation of chaos

OK if $\beta > -8\pi$

Theorem A [S. 92]

$0 < \lambda < 8\pi \Rightarrow \exists 1$ solution

Boltzmann Poisson Equation

$\Omega \subset \mathbf{R}^2$ bounded domain $\partial\Omega$ smooth

$\lambda > 0$ constant

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega, \ v = 0 \text{ on } \partial\Omega$$

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}, \ \lambda = -\beta$$

$$\psi = \int_\Omega G(\cdot, x')\rho(x')dx'$$

mean field equation in stream function

Impact to the Elliptic Theory

Theorem B [Nagasaki-S. 90a]

$\{(\lambda_k, v_k)\}$ solution sequence s.t.

$\lambda_k \to \lambda_0 \in [0, \infty)$, $\|v_k\|_\infty \to \infty$

quantized blowup mechanism

$\Rightarrow \lambda_0 = 8\pi N$, $N \in \mathbf{N}$

$\exists$sub-sequence, $\exists \mathcal{S} \subset \Omega$, $\sharp\mathcal{S} = N$, s.t.

$v_k \to v_0$ loc. unif. in $\overline{\Omega} \setminus \mathcal{S}$

$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0)$$

recursive hierarchy

$$\nabla_{x_i} H_N(x_1^*, \cdots, x_N^*) = 0, \ 1 \le i \le N$$

$G = G(x, x')$ the Green's function

$\mathcal{S} = \{x_1^*, \ldots, x_N^*\}$

$$H_N(x_1, \ldots, x_N) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$$

$$R(x) = \left[G(x, x') + \frac{1}{2\pi}\log|x - x'|\right]_{x=x'}$$

1. non-radial bifurcation on annulus (S.S. Lin 89 Nagasaki-S. 90b)

2. effective bound of blowup points for simply-connected domain (S.-Nagasaki 89 Grossi-F.Takahashi 10)

3. classification of singular limits (Nagasaki-S. 90a)

4. spherical mean value theorem (S. 90)

5. localization (Brezis-Merle 91)

6. entire solution (W. Chen-C. Li 91)

7. sup + inf inequality (Shafrir 92)

8. uniqueness (S. 92)

9. field-particle duality (S. 92 Wolansky 92)

10. singular perturbation (Weston 78 Moseley 83 S. 93 Baraket-Pacard 98 Esposito-Grossi-Pistoia 05 del Pino-Kowarzyk-Musso 05)

11. blowup analysis (Li-Shafrir 94)

12. Chern-Simons theory (Tarantello 96)

13. global bifurcation (S.-Nagasaki 89 Mizoguchi-S. 97 Chang-Chen-Lin 03)

14. min-max solution (Ding-Jost-Li Wang 99)

15. local uniform esitmate (Y.Y. Li 99)

16. variable coefficient (Ma-Wei 01)

17. refined asymptotics (Chen-Lin 02)

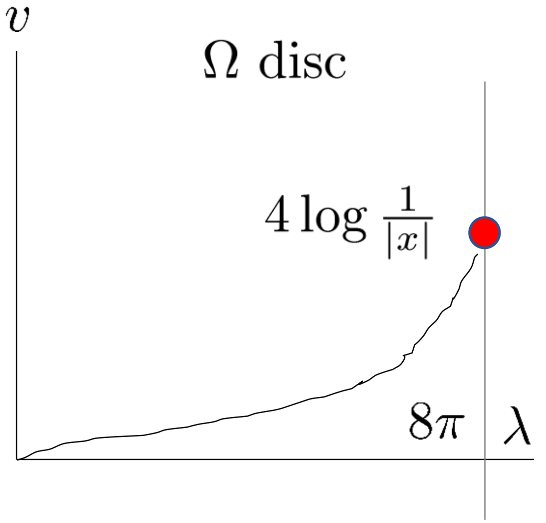
18. topological degree (Li 99 C.C. Chen-C.S. Lin 03 Malchiodi 08)

19. asymptotic non-degeneracy (Gladiali-Grossi 04 Grossi-Ohtsuka-S. 11)

20. isoperimetric profile (Lin-Lucia 06)

21. deformation lemma (Lucia 07)

22. Morse index (Gladiali-Grossi 09)



$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega \subset \mathbf{R}^2$$

$$v|_{\partial\Omega} = 0$$

2. Boltzmann-Poisson Equation

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \quad v|_{\partial\Omega} = 0$$

L. Onsager 49

point vortices
ordered structure in negative temperature

Poisson

$$-\Delta v = u$$

$$v|_{\partial\Omega} = 0$$

Boltzmann

$$u = \frac{\lambda e^v}{\int_\Omega e^v}$$

$$G(x,x') = G(x',x) \quad \text{Green}$$

$$R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x-x'| \right]_{x'=x} \quad \text{Robin}$$

Theorem 1 (Nagasaki-S. 90)

$$\{(\lambda_k, v_k)\}, \ \lambda_k \to \lambda_0 \in (0,\infty), \ \|v_k\|_\infty \to \infty$$

$$\Rightarrow \ \lambda_0 = 8\pi\ell, \ \ell \in \mathbf{N}, \ \exists \mathcal{S} \subset \Omega, \ \sharp\mathcal{S} = \ell$$

$$v_k \to v_0 \ \text{loc. unif. in} \ \overline{\Omega} \setminus \mathcal{S} \quad \text{(sub-sequence)}$$

$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x,x_0), \ \mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$$

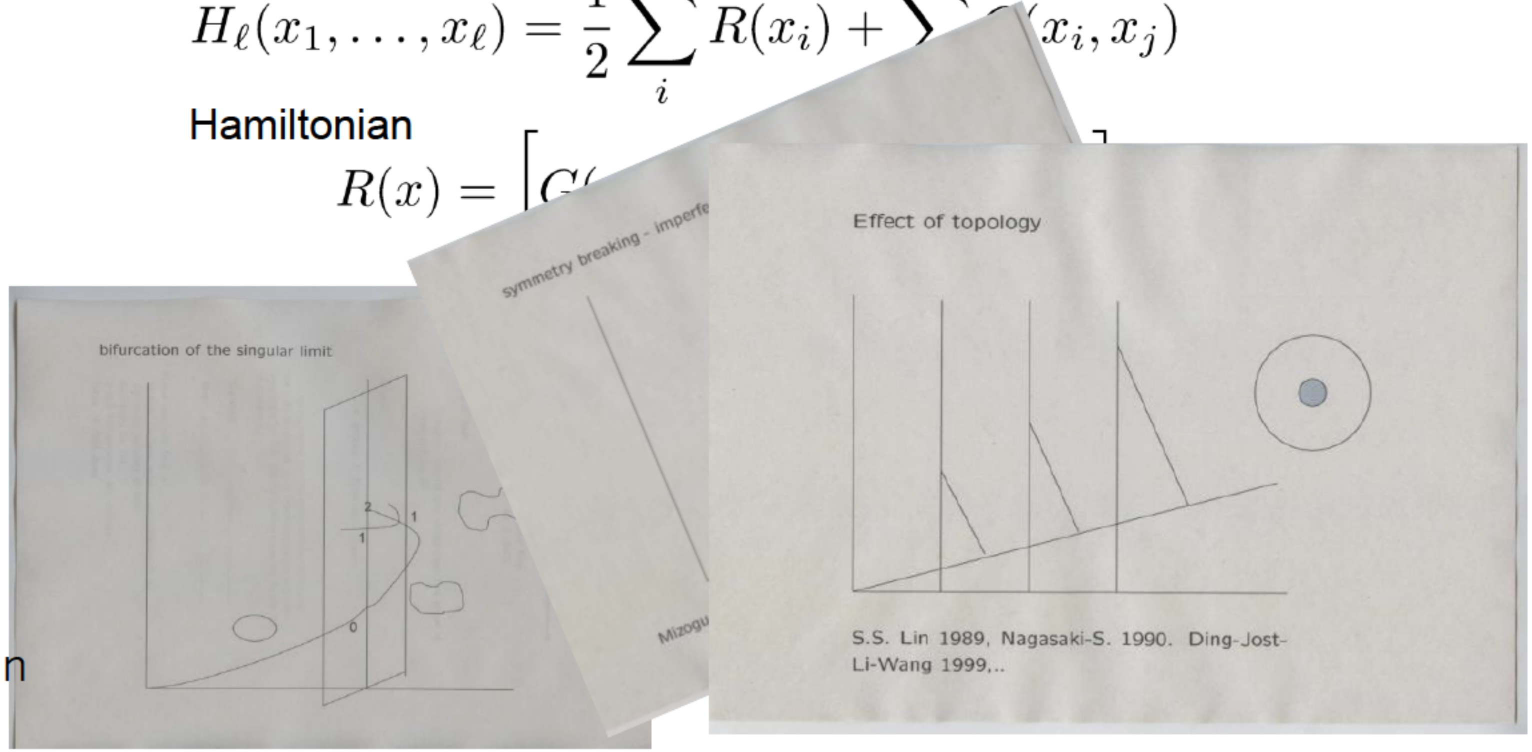
singular limit — blowup set

$$\nabla H_\ell|_{(x_1,\cdots,x_\ell)=(x_1^*,\cdots,x_\ell^*)} = 0$$

$$H_\ell(x_1,\ldots,x_\ell) = \frac{1}{2}\sum_i R(x_i) + \sum \ldots (x_i,x_j)$$

Hamiltonian

$$R(x) = \Big[G(\ldots$$



complex structure (Liouville integral)

$$-\Delta v = \sigma e^v$$
$$\Leftrightarrow \quad \exists F = F(z), \ z \in \Omega \subset \mathbf{R}^2 \cong \mathbf{C}$$ meromorphic

$$\rho(F) = \left(\frac{\sigma}{8}\right)^{1/2} e^{v/2} = \frac{|F'|}{1+|F|^2}$$ spherical derivative

$$-\Delta v = \sigma e^v, \ v|_{\partial\Omega} = 0 \ \Leftrightarrow \ \rho(F)|_{\partial\Omega} = \left(\frac{\sigma}{8}\right)^{1/2}$$

$$\hat{F} = \sqrt{8} \circ F : \Omega \to S^2$$ conformal

$$\left.\frac{d\Sigma}{ds}\right|_{\partial\Omega} = \sigma^{1/2}$$

$(S^2, d\Sigma)$ round sphere

$|S^2| = 8\pi$

$$\int_{\partial\Omega} \frac{d\Sigma}{ds} ds = |\partial\Omega|\,\sigma^{1/2}$$

immersed length of $\hat{F}(\partial\Omega)$

Proof of Theorem (90)

1. Liouville integral
2. boundary reflection
3. elliptic regularity

4. complex function theory
4-1. maximum principle
4-2. Montel's theorem
4-3. theorem of coincidence
4-4. residue analysis

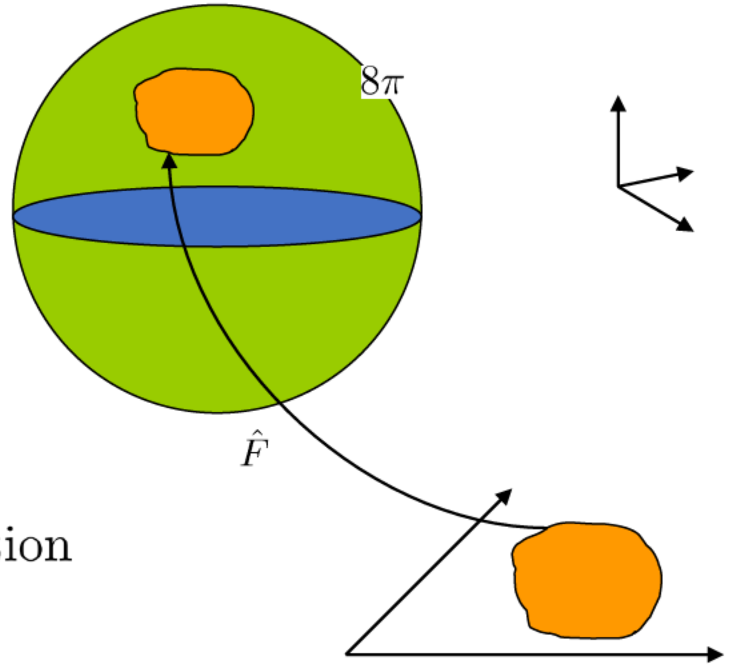
$$\int_\Omega \left(\frac{d\Sigma}{ds}\right)^2 dx = 8\int_\Omega \rho(F)^2 dx = \int_\Omega \sigma e^v$$

immersed area of $\hat{F}(\Omega)$

$$\lambda = \int_\Omega \sigma e^v \to 8\pi\ell$$

$\Leftrightarrow$ total mass quantization

due to ℓ-covering



Blowup analysis

$\Omega \subset \mathbf{R}^2$: open set, $V \in C(\overline{\Omega})$

$-\Delta v = V(x)e^v, \ \ 0 \leq V(x) \leq b \quad \text{in } \Omega$

$\int_\Omega e^v \leq C$

Theorem 2 [Li-Shafrir 94]

$\{(V_k, v_k)\}$ solution sequence

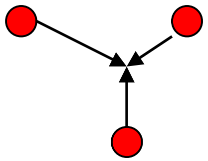
$V_k \to V$ loc. unif. in Ω

$\Rightarrow {}^{\exists}$ sub-sequence with the alternatives;

1. $\{v_k\}$: loc. unif. bdd in Ω

2. $\exists \mathcal{S} \subset \Omega$, $\sharp \mathcal{S} < +\infty$
$v_k \to -\infty$ loc. unif. in $\Omega \setminus \mathcal{S}$
$\mathcal{S} = \{x_0 \in \Omega \mid \exists x_k \to x_0, \ v_k(x_k) \to +\infty$
$$V_k(x)e^{v_k}dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) \text{ in } \mathcal{M}(\Omega)$$
$m(x_0) \in 8\pi\mathbf{N}$

3. $v_k \to -\infty$ loc. unif. in Ω



Comments

1. mass quantization for variable coefficients without boundary condition
2. possible collapse collision
3. many applications together with the proof

prescaled analysis ...Brezis-Merle 91

linear theory $\Rightarrow$

1, 2 with $m(x_0) \geq 4\pi$ (rough estimate), 3

2... localized to $B = B(0, R)$

$-\Delta v_k = V_k(x)e^{v_k}$, $V_k(x) \geq 0$ in B

$V_k \to V$ unif. in $\overline{B}$, $\max_{\overline{B}} v_k \to +\infty$

$\max_{\overline{B}\setminus B_r} v_k \to -\infty$, ${}^{\forall}r \in (0, R)$

$$\lim_k \int_B V_k e^{v_k} = \alpha, \ \int_B e^{v_k} \leq C$$

$\Rightarrow \alpha \in 8\pi\mathbf{N}$

Boltzmann-Poisson-Gel'fand equation

$$-\Delta v = \lambda e^v, \ \ v|_{\partial\Omega} = 0$$

$$\{(\lambda_k, v_k)\}, \ \lambda_k \to 0 \ \Rightarrow \ \text{(sub-sequence)}$$

$$\lambda_k \int_\Omega e^{v_k} \to 8\pi\ell, \ \ell = 0, 1, 2, \cdots, +\infty$$

$$0 < \ell < +\infty \ \Rightarrow \ \exists \mathcal{S} \subset \Omega, \ \sharp\mathcal{S} = \ell$$

$$v_k \to v_0 \text{ loc. unif. in } \overline{\Omega} \setminus \mathcal{S} \qquad \mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$$

$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0) \qquad x_* = (x_1^*, \cdots, x_\ell^*)$$

$$\nabla H_\ell(x_*) = 0, \ H_\ell(x_1, \ldots, x_\ell) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$$

Theorem 3 (Gladiali-Grossi-Ohtsuka-S. 14) $k \gg 1$

(augmented) Morse indices

$$\ell + \mathrm{ind}_M\{-H_\ell(x_*)\} \leq \mathrm{ind}_M(v_k)$$

$$\mathrm{ind}_M^*(v_k) \leq \ell + \mathrm{ind}_M^*\{-H_\ell(x_*)\}$$

Corollary (Gladiali-Grossi 09) x_* non-degenerate

➡ $v_k, \ k \gg 1$ non-degenerate

Theorem 2 (Baraket-Pacard 98)

$(x_1^*, \ldots, x_\ell^*) \in \Omega \times \ldots \times \Omega$

non-degenerate critical point of $H_\ell(x_1, \ldots, x_\ell)$

$\exists$ sequence of ℓ point blow up solutions

Remark

1. only one point blowup and $\exists 1$ blowup spot for convex domain
2. effective bound of the number of blowup points for simply connected domain
3. domain homology and Hamiltonian (Cao 10)
4. inhomogeneous coefficients, equations on manifold, etc. (Ohtsuka-Sato-S.)
5. one-point blowup case
6. refined asymptotics with Morse index correspondence
7. asymptotic non-degeneracy in multi-blowup

3. Asymptotic non-degeneracy

$$-\Delta v = \lambda e^v \text{ in } \Omega,\ \ v|_{\partial\Omega} = 0$$

$$\lambda_k \to 0,\ \lambda_k \int_\Omega e^{v_k} \to 8\pi$$

$v_k(x) \to 8\pi G(x, x_0),\ x \in \overline{\Omega} \setminus \{x_0\}$ locally uniformly

$$\nabla R(x_0) = 0$$

Theorem (corollary of Theorem 3)

$x_0 \in \Omega$ non-degenerate critical point of $R(x)$

$\Longrightarrow$ $-\Delta_D - \lambda_k e^{v_k},\ 0 < \sigma_k \ll 1$ non-degenerate

Proof. otherwise

$$\exists \lambda_k \downarrow 0,\ v_k,\ w_k,\ \ -\Delta v_k = \lambda_k e^{v_k} \text{ in } \Omega,\ \ v_k|_{\partial\Omega} = 0$$

$$-\Delta w_k = \lambda_k e^{v_k} w_k \text{ in } \Omega,\ \ w_k|_{\partial\Omega} = 0,\ \ \|w_k\|_\infty = 1$$

$$v_k(x_k) = \|v_k\|_\infty,\ x_k \to x_0$$

drop k

Green $$\int_{\partial\Omega} w\frac{\partial v_i}{\partial \nu} - v_i \frac{\partial w}{\partial \nu} ds = 0,\ \ v_i = \frac{\partial v}{\partial x_i}$$

scaling $$\delta_k^2 \lambda_k e^{v_k(x_k)} = 1$$

sub-sequence ~ locally uniformly in $\mathbf{R}^2$

$$\tilde{v}_k(x) = v_k(\delta_k x + x_k) - v(x_k) \to v_0(x)$$

$$\tilde{w}_k(x) = w_k(\delta_k x + x_k) \to w_0(x)$$

$$-\Delta v_0 = e^{v_0} \text{ in } \mathbf{R}^2,\ \ \int_{\mathbf{R}^2} e^{v_0} < +\infty$$

$$-\Delta w_0 = e^{v_0} w_0 \text{ in } \mathbf{R}^2,\ \ \|w_0\|_\infty \le 1$$

Liouville property – Baraket-Pacard 98

$$w_0(x) = a \cdot \frac{x}{1+|x|^2} + b\frac{8-|x|^2}{8+|x|^2},\ a \in \mathbf{R}^2,\ b \in \mathbf{R}$$

Lemma 1 (Nagasaki-S.)

$$v_{ki} \to 8\pi \frac{\partial G}{\partial x_i}(\cdot, x_0)$$ locally uniformly $(\text{except for } x_0)$

Lemma 2 (Gladiali-Grossi 09)

$$\delta_k^{-1} w_k \to 2\pi a \cdot \nabla_{x'} G(\cdot, x_0)$$ locally uniformly

Step 1

$$w_k = \gamma_k \{G(\cdot, x_0) + o(1)\} + 2\pi\delta_k a \cdot \nabla_{x'} G(\cdot, x_0) + o(\delta_k)$$

$$\gamma_k = \int_{\Omega \cap B(x_0, R)} \lambda_k e^{v_k} w_k dx'$$

1. removable singularity theory

$$w_k \to 0$$ locally uniformly

2. Green's formula

$$w_k(x) = \int_\Omega G(x, x') \lambda_k e^{v_k(x')} w_k(x') dx'$$

3. localization around $x' = x_0$

4. Y.Y. Li's estimate $|x - x_0| \geq \delta^k,\ 0 < k < 1/4$

5. Taylor's expansion $G(x, x'),\ x' = x_0,\ |x' - x_0| < \delta^k$

Step 2

$$\overline{w_k}(x = (x - x_0) \cdot \nabla v_k + 2,\ -\Delta \overline{w_k} = \lambda_k e^{v_k} \overline{w_k}$$

$$\int_{\partial B_R(x_0)} \frac{\partial \overline{w_k}}{\partial \nu} w_k - \overline{w_k} \frac{\partial w_k}{\partial \nu} d\sigma = 0 \quad \Longrightarrow \quad \gamma_k = o(\delta_k)$$

completion of the proof

non-degeneracy + Green+ $$\int_{\partial\Omega} \frac{\partial G}{\partial x_i}(x, y) \frac{\partial}{\partial \nu_x} \frac{\partial}{\partial y_j} G(x, y) ds_x = -\frac{1}{2} \frac{\partial^2 R}{\partial y_i \partial y_j}(y)$$

$\Longrightarrow$ a=0, b=0 $\Longrightarrow$ $|\exists \tilde{x}_k| \to +\infty,\ w_k(\tilde{x}_k) = 1$

exclude by

1. Kelvin transformation
2. Y.Y. Li's estimate
3. maximum principle

Open questions

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \quad v|_{\partial\Omega} = 0$$

$$\{(\lambda_k, v_k)\}, \ \lambda_k \to 8\pi, \ \|v_k\|_\infty \to +\infty$$

$$v_k \to v_0 \text{ loc. unif. in } \overline{\Omega} \setminus \mathcal{S}$$

$$v_0(x) = 8\pi G(x, x_0), \ \nabla R(x_0) = 0$$

$$g : B = B(0,1) \to \Omega \qquad \text{conformal}$$

$$g(z) = x_0 + \sum_{k=1}^\infty a_k z^k \qquad \nabla R(x_0) = 0 \ \Leftrightarrow \ a_2 = 0$$

$$\exists \nabla^2 R(x_0)^{-1} \ \Leftrightarrow \ |a_3/a_1| \neq 1/3$$

$$\lambda = 8\pi + C\sigma_k + o(\sigma_k), \ \sigma_k = \frac{\lambda_k}{\int_\Omega e^{v_k}} \to 0$$

$$\frac{C}{\pi} = -|a_1|^2 + \sum_{k=3}^\infty \frac{k^2}{k-2}|a_k|^2$$

$|a_3/a_1| \neq 1/3, \ C \neq 0$ **Conjecture**

$\longrightarrow \ v_k, \ k \gg 1$ $\quad \mathcal{L}$ non-degenerate

Variation functional $\quad J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v, v \in H_0^1(\Omega)$

Quadratic form

$\varphi \in H_0^1(\Omega)$

$p = \frac{\lambda e^v}{\int_\Omega e^v}$]

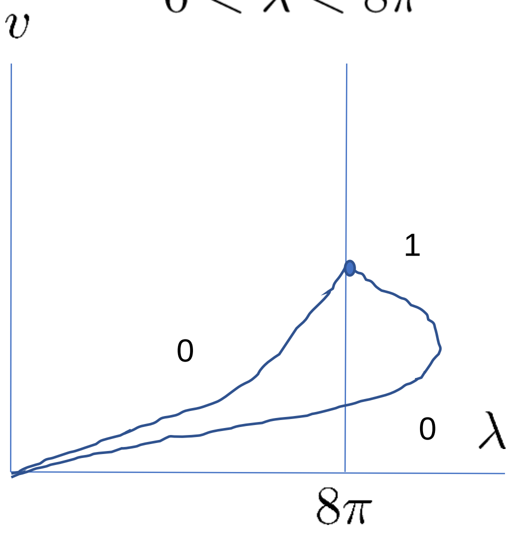
$$Q(\varphi, \varphi) = \frac{d^2}{ds^2} J_\lambda(v + s\varphi)\Big|_{s=0} = (\nabla\varphi, \nabla\varphi) - \int_\Omega p\varphi^2 + \frac{1}{\lambda}\left(\int_\Omega p\varphi\right)^2$$

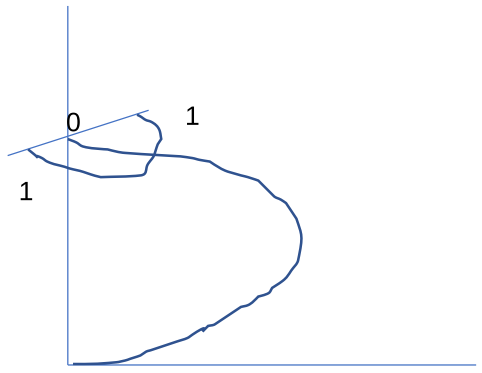
Linearized operator $\quad \mathcal{L}\psi = -\Delta\psi - p\psi + \frac{1}{\lambda}\left(\int_\Omega p\psi\right) p$

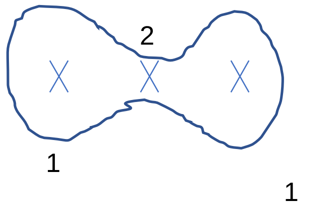
$$D(\mathcal{L}) = H_0^1(\Omega) \cap H^2(\Omega)$$

Theorem 3 (S. 92, Bartoulucci-Lin 15)

$0 < \lambda < 8\pi$ $\longrightarrow$ non-degenerate







Gladial-Grossi 04
Sato-S. 07
Grossi-Ohtsuka-S. 11
Ohtsuka-Sato-S. 13

Elliptic Theory 3

Local Behavior of the Solution Derives Recursive Hierarchy

1. Multi-Intensity Model

1. Stochastic Case (Neri 04)

1-species relative intensity $\alpha \in [-1,1]$ is a random variable subject to the distribution function $P(d\alpha)$

$$-\Delta v = \lambda \frac{\int_{[-1,1]} \alpha e^{\alpha v} P(d\alpha)}{\int_{]-1,1]} \int_\Omega e^{\alpha v} P(d\alpha)}, \ v|_{\partial\Omega} = 0$$

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_{[-1,1]} [\int_\Omega e^{\alpha v}] P(d\alpha)$$

$v \in H_0^1(\Omega)$

2. Deterministic Case (Onsager's note, Sawada-S. 08)

ℓ-species

$n^i N$-particles take the intensity $\alpha^i \hat{\alpha}$

$0 < n^i < 1, \ -1 \leq \alpha^i \leq 1, \ \sum_i n_i = 1$

$1 \leq i \leq \ell, \ N\hat{\alpha} = 1, \ N \uparrow +\infty$

$$-\Delta v = \lambda \int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}} P(d\alpha), \ v|_{\partial\Omega} = 0$$

$$P(d\alpha) = \sum_{i=1}^{\ell} n^i \delta_{\alpha_i}, \text{ may } \ell = \infty$$

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \int_{[-1,1]} [\log \int_\Omega e^{\alpha v}] P(d\alpha)$$

$v \in H_0^1(\Omega)$

Ω closed Riemann surface $E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$

1. stochastic intensity (Neri)

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_{[-1,1]} [\int_\Omega e^{\alpha v}] P(d\alpha)$$

$$-\Delta v = \lambda \left(\frac{\int_{[-1,1]} \alpha e^{\alpha v} P(d\alpha)}{\int_{[-1,1]} \int_\Omega e^{\alpha v} P(d\alpha)} - \frac{1}{|\Omega|} \right)$$

2. deterministic intensity (Sawada-S.)

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \int_{[-1,1]} [\log \int_\Omega e^{\alpha v}] P(d\alpha)$$

$$-\Delta v = \lambda \int_{[-1,1]} \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}} - \frac{1}{|\Omega|} \right) P(d\alpha)$$

$P(d\alpha)$ probability measure on $[-1,1]$

c.f. $P = \frac{1}{2}(\delta_{-1} + \delta_1)$

probability for vorticities to take renormalized intensity α

$$-\Delta v = \frac{\lambda(e^v - e^{-v})}{\int_\Omega e^v + e^{-v} \ dx}$$

sinh-Gordon equation
constant mean curvature
quaternion

distribution of vortices with renormalized intensity α

$$-\Delta v = \frac{\lambda}{2} \left(\frac{e^v}{\int_\Omega e^v dx} - \frac{e^{-v}}{\int_\Omega e^{-v} dx} \right)$$

neutral vortex

2. Deterministic intensities

Sawada-S. functional $E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$

$$J_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \lambda \int_{[-1,1]} [\log \int_\Omega e^{\alpha v}]\, P(d\alpha)$$

$$\overline{\lambda} = \sup\{\lambda \mid \inf_E J_\lambda > -\infty\}$$

approach by blowup analysis (ORS 10)

$$\overline{\lambda} \geq \lambda_* = \inf \frac{8\pi}{\int_{I_\pm} \alpha^2 P(d\alpha)} \qquad \inf_E J_{\lambda_*} > -\infty$$

approach by duality (RS14) $\overline{\lambda} = \lambda^* \geq \lambda_*,$

$$\lambda^* = \inf \left\{ \frac{8\pi P(K_\pm)}{\left[\int_{K_\pm} \alpha P(d\alpha)\right]^2} \middle| K_\pm \subset I_\pm \cap \mathrm{supp}\, P \right\}$$

Theorem 1 (Ohtsuka-Ricciardi-S. 10) $\{v_k\} \subset E$

$$-\Delta v_k = \lambda_k \int_{[-1,1]} \alpha\left(\frac{e^{\alpha v_k}}{\int_\Omega e^{\alpha v_k}} - \frac{1}{|\Omega|}\right) P(d\alpha)$$

non-compact ⟹ sub-sequence

$$\frac{\lambda_k e^{\alpha v_k}}{\int_\Omega e^{\alpha v_k}} dx P(d\alpha) \rightharpoonup [\sum_{x_0 \in \mathcal{S}} m(x_0,\alpha)\delta_{x_0} + r(x,\alpha)] dx P(d\alpha)$$

$$m(x_0,\alpha) \geq 0, \ \sharp\mathcal{S} < +\infty$$

$$0 \leq r = r(x,\alpha) \in L^1(\Omega \times [-1,1], dxdP)$$

$$8\pi \int_{[-1,1]} m(x_0,\alpha) P(d\alpha) = \left\{\int_{[-1,1]} \alpha m(x_0,\alpha) P(d\alpha)\right\}^2, \ \forall x_0 \in \mathcal{S}$$

$$I_- = [-1,0], \ I_+ = [0,1]$$

approach by duality

X Banach space/$\mathbf{R}$

$F : X \to (-\infty, +\infty]$ prop. c'x l.s.c.

$\Rightarrow$

Legendre transformation

$F^* : X^* \to (-\infty, +\infty]$ prop. c'x l.s.c.

$$F^*(p) = \sup_{x \in X} \{\langle x, p\rangle - F(x)\}$$

Fenchel-Moreau duality

$$F^{**} = F$$

$$F^{**}(x) = \sup_{p \in X^*} \{\langle x, p\rangle - F^*(p)\}$$

Toland duality 78, 79

$F, G : X \to (-\infty, +\infty]$ prop. c'x l.s.c.

$$J(x) = G(x) - F(x)$$

$$J^*(p) = F^*(p) - G^*(p)$$

$$L(x,p) = F^*(p) + G(x) - \langle x, p\rangle$$

... Lagrange function

$$\inf_{X \times X^*} L = \inf_X J = \inf_{X^*} J^*$$

Smoluchowski-Poisson equation

point vortex mean field equation (single intensity)

free energy $\longleftrightarrow$ field functional

duality

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle (-\Delta)^{-1} u, u\rangle \qquad J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v + \lambda(\log \lambda - 1)$$

$u \geq 0,\ \|u\|_1 = \lambda$

particle distribution

potential density

Theorem 2 (Ricciardi-S. 14)

$$\overline{\lambda} = \lambda^*$$

$$\lambda^* = \inf \left\{ \frac{8\pi P(K_\pm)}{\left[\int_{K_\pm} \alpha P(d\alpha)\right]^2} \middle| K_\pm \subset I_\pm \cap \operatorname{supp} P \right\}$$

$$I_- = [-1, 0]$$
$$I_+ = [0, 1]$$

$$J_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \lambda \int_{[-1,1]} (\log \int_\Omega e^{\alpha v})\, P(d\alpha)$$

$$E = \{v \in H^1 \mid \int_\Omega v = 0\}$$

field functional

$$L(\oplus\rho_\alpha, v) = \int_{[-1,1]} \left[\int_\Omega \rho_\alpha(\log \rho_\alpha - 1)\right] P(d\alpha) + \frac{1}{2}\|\nabla v\|_2^2 - \int_{[-1,1]} \left[\int_\Omega \alpha \rho_\alpha v\right] P(d\alpha)$$

Lagrangian

$$\inf_{\oplus\Gamma_\lambda \times E} L = \inf_E J_\lambda + \lambda(\log\lambda - 1) = \inf_{\oplus\Gamma_\lambda} \mathcal{F}$$

unfolding-minimality

$$\mathcal{F}(\oplus\rho_\alpha) = \int_{[-1,1]} \left[\int_\Omega \rho_\alpha(\log\rho_\alpha - 1)\right] P(d\alpha) - \frac{1}{2}\int_{[-1,1]^2} \alpha\beta \left\langle \rho_\alpha, (-\Delta)^{-1}\rho_\beta \right\rangle P\otimes P(d\alpha d\beta)$$

free energy

$$\oplus\rho_\alpha \in \oplus\Gamma_\lambda,\ \Gamma_\lambda = \{\rho \geq 0 \mid \int_\Omega \rho = \lambda \quad \oplus\Gamma_\lambda = \{\oplus\rho_\alpha \mid \rho_\alpha \in \Gamma_\lambda, P\text{-a.e. } \alpha\}$$

discrete measure

$$P = \sum_i n^i \delta_{\alpha_i}, \ \rho_i \geq 0, \ \int_\Omega \rho_i = \lambda, \ \sharp\{i\} < +\infty$$

$$\mathcal{F}(\oplus \rho_i) = \sum_i n^i \int_\Omega \rho_i(\log \rho_i - 1) + \sum_{i,j} n^i n^j \alpha_i \alpha_j \langle \rho_i, (-\Delta)^{-1} \rho_j \rangle$$

Shafrir-Wolansky 05 a,b (positive case)

Ω m-dimensional compact Riemann manifold

$$\tilde{\mathcal{F}}(\oplus \rho_i) = \sum_i \int_\Omega \rho_i \log \rho_i - \sum_{i,j} a_{ij} \int_{\Omega^2} \rho_i(x) \log d(x,y) \rho_j(y) \ dxdy, \ a_{ji} = a_{ij} \geq 0, \ \rho_i \geq 0, \ \int_\Omega \rho_i = M_i$$

bounded $\Leftrightarrow$

1. $\Lambda_J(M) \geq 0, \ \forall J \subset \{i\}$
2. $\Lambda_J(M) = 0 \Rightarrow a_{ii} + \Lambda_{J \setminus \{i\}}(M) > 0, \ \forall i \in J$

$$\Lambda_J(M) = m \sum_{i \in J} M_i - \sum_{i,j \in J} a_{ij} M_i M_j$$

$$M = (M_i)$$

$$m \int_{\Omega^2} F_1(x) \log \frac{1}{d(x,y)} F_2(y) dxdy \leq (1-\alpha) \int_\Omega F_1 \log F_1 + \alpha \int_\Omega F_2 \log F_2 + C_\alpha$$

$$0 < \alpha < 1, \ F_i \geq 0, \ \int_\Omega F_i = 1$$

+linear programing

$\alpha = \frac{1}{2}$ ... Carlen-Loss 92 Beckner 93

$P(d\alpha)$ discrete

$$\inf_{\oplus\Gamma_\lambda} \mathcal{F} > -\infty, \ \forall \lambda < \lambda^*$$

$$\inf_{\oplus\Gamma_\lambda} \mathcal{F} = -\infty, \ \forall \lambda > \lambda^*$$

$$\Gamma_\lambda = \{\rho \geq 0 \mid \int_\Omega \rho = \lambda\}$$

$$\oplus\Gamma_\lambda = \{\oplus\rho_\alpha \mid \rho_\alpha \in \Gamma_\lambda, P\text{-a.e. } \alpha\}$$

$$\mathcal{F}(\oplus\rho_\alpha) = \int_{[-1,1]} \left[\int_\Omega \rho_\alpha(\log\rho_\alpha - 1)\right] P(d\alpha) - \frac{1}{2}\int_{[-1,1]^2} \alpha\beta \left\langle \rho_\alpha, (-\Delta)^{-1}\rho_\beta \right\rangle P\otimes P(d\alpha d\beta)$$

approximation $\Longrightarrow$ $\overline{\lambda} = \lambda^*$ for non-discrete $P(d\alpha)$

(duality) $\Rightarrow \overline{\lambda} = \lambda^*$ for $J_\lambda(v)$, $v \in E$

$$J_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \lambda \int_{[-1,1]} (\log \int_\Omega e^{\alpha v})\, P(d\alpha)$$

$$E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$$

open questions

1. $\inf_E J_{\lambda^*} > -\infty$ for non-discrete $P(d\alpha)$
2. $\nexists$ minimizer

discrete critical bound: OK

strategy for non-discrete measure

$$\overline{\lambda} = \lambda^*$$

$$\forall \lambda < \lambda^* \ \exists \text{ minimizer of } \inf_E J_\lambda$$

$$\lambda_k \uparrow \lambda^*, \ v_k \text{ mimizer of } \inf_E J_{\lambda_k}$$

$$\Rightarrow$$

$$\{(\lambda_k, v_k)\} \text{ solution sequence to}$$

$$-\Delta v = \lambda \int_{[-1,1]} \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}} - \frac{1}{|\Omega|} \right) P(d\alpha)$$

$$v \in E$$

$$\{v_k\} \subset E \text{ compact} \Rightarrow \inf_E J_{\lambda^*} \text{ attained}$$

$$\Rightarrow \inf_E J_{\lambda^*} > -\infty$$

otherwise (subsequence)

$$\frac{\lambda_k e^{\alpha v_k}}{\int_\Omega e^{\alpha v_k}} dx P(d\alpha) \rightharpoonup \mu(dx d\alpha)$$

$$\mu(dx d\alpha) = [\sum_{x_0 \in \mathcal{S}} m(x_0, \alpha) \delta_{x_0}(dx) + r(x, \alpha) dx] P(d\alpha)$$

$$m(x_0, \alpha) \geq 0$$

$$0 \leq r = r(x, \alpha) \in L^1(\Omega \times [-1,1], dx dP)$$

$$\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-, \ \sharp \mathcal{S} < +\infty$$

$$\mathcal{S}_\pm = \{x_0 \mid \exists x_k \to x_0 \text{ s.t. } v_k(x_k) \to \pm\infty\}$$

$$\forall x_0 \in \mathcal{S}, \ 8\pi \int_{[-1,1]} m(x_0, \alpha) P(d\alpha) = \left\{ \int_{[-1,1]} \alpha m(x_0, \alpha) P(d\alpha) \right\}^2$$

Key ingredient ~ Y.Y. Li estimate

c.f. S. –Toyota 19

Interior blowup control

$$u = v + \log \lambda, \ V(x) = e^{-v} f(v)$$

Brezis-Merle 91

$$-\Delta u_k = V_k(x) e^{u_k} \text{ in } B = B_R, \ 0 \le V_k(x) \le b, \ \int_B e^{u_k} \le C$$

classification of the behavior, rough estimate of concentration ⟶ $\lambda_k \downarrow 0$

Li-Shafrir 94

$V_k \to \exists V \text{ in } C(\overline{B})$ mass quantization and formation of bubble for the concentration case

possible collision of bubbles

bubble center can be local minimum

Y.Y. Li 99

$\|\nabla V_k\|_\infty \le C$ ⟶ local uniform estimate ⟶ non-degeneracy
Morse index calculation

3. Point Vortices Limit of Neri type in the Gel'fand form

$$-\Delta v = \lambda \int_I \alpha e^{\alpha v} P(\alpha) \text{ in } \Omega$$
$$v = 0 \text{ on } \partial\Omega$$
$$\lambda \int\int_{I\times\Omega} \alpha e^{\alpha v} P(d\alpha) dx \leq C$$

$\Omega \subset \mathbf{R}^2$ bounded domain

$\partial\Omega$ smooth boundary

$P(d\alpha)$ Borel measure on $I = [0,1]$

$1 \in \text{supp } P(d\alpha)$

Ricciardi-Zecca 16a,b
deMarchis-Ricchiardi 17

$P(d\alpha) = \delta_1(d\alpha)$ single intensity

1. blowup analysis
2. asymptotic non-degeneracy, Morse index calculation
3. deformation theory, topological degree calculation

$P(\{1\}) = \tau > 0$ non-degenerate case $\lambda\tau \mapsto \lambda$

$$-\Delta v = \lambda f(v), \; v|_{\partial\Omega} = 0, \quad 0 \leq f(v) = e^v + o(e^v), \; v \uparrow +\infty$$

Ye 97

otherwise

$$f(v) \equiv \int_I \alpha e^{\alpha v} P(d\alpha) = o(e^{\beta v}), \; v \uparrow +\infty, \forall \beta > 1$$
$$\lim_{v\uparrow+\infty} e^{-\beta v} f(v) = +\infty, \; \forall \beta < 1$$

$$-\Delta v = \lambda f(v), \ v|_{\partial\Omega} = 0$$

$$0 \leq f(v) = \int_I \alpha e^{\alpha v} dP(\alpha), \ f''(v) \geq 0, \ 0 < \lambda \leq \exists \overline{\lambda}$$

Blowup analysis λ_k, v_k solution $\|v_k\| \to +\infty, \ \Sigma_k = \lambda_k \int_\Omega f(v_k) \leq C$

Exclusion of boundary blowup

[Gidas-Ni-Nirenberg 79] $\Omega \subset \mathbf{R}^2, \ 0 \leq f = f(v) \in C^1$

⟶ uniform decreasing property of the solution near the boundary

[deFigueiredo-Lions-Nussbaum82] $\forall K \text{ compact } \subset \Omega, \ \exists C_K > 0, \ \|v\|_{L^1(K)} \leq C_K$

⟶ $\partial\Omega \subset \exists\omega$ open. independent of $f(v)$

$\|v\|_{L^\infty(\Omega\cap\omega)} \leq \exists C$ for the solution under the above property

[Brezis-Strauss 73] $\|\Delta u\|_1 \leq C \ \Rightarrow \ \|v\|_{W^{1,q}} \leq C_q, \ 1 \leq q < 2 = \frac{n}{n-1}$

c.f. S. (preprint)