Diffusion Geometry 1

Analytic Approach to the 2D-NRF

1. Normalized Ricci Flow

Hamilton 82 Ricci flow approach to the Poincare-Thurston conjecture

bridge initial and expected ultimate metrics
geometric process of surgery at the formation of singularities

2D normalized Ricci flow
$$\frac{\partial g}{\partial t} = (r - R)g \qquad \text{on } \Omega \times (0, T)$$

Ω	compact Riemann surface
$g = g(t)$	metric
$R = R(t)$	scalar curvature
$r(t) = \frac{\int_\Omega R(t) d\mu_t}{\int_\Omega d\mu_t}$	average volume
$\mu = \mu_t$	volume element

Theorem (Hamilton 88)

$$g(t) \to g_\infty \text{ in } C^\infty \text{ as } t \uparrow \infty$$

metric with constant scalar curvature

Remark

1. No singularity in 2D-NRF

2. $$\frac{\partial R}{\partial t} = \Delta R + R(R - r), \ \Delta = \Delta_g$$

Henceforth, assume R>0

geometric part

$$\Delta f = R - r$$ curvature potential

$$M_g = \nabla\nabla f - \frac{1}{2}\Delta f \cdot g$$ trace free part

co-variant derivative

Lie derivate

→ $$2M_g = (r - R)g + \mathcal{L}_{\nabla f} g$$

→ modified Ricci flow

$$\frac{\partial \tilde{g}}{\partial t} = 2M_{\tilde{g}}, \ \tilde{g} = \tilde{g}(t) = T_t^* g(t), \ \{T_t\} \ \leftrightarrow \ \nabla f$$

$$|M_g|^2 = |\nabla\nabla f|^2 - \frac{1}{2}(\Delta f)^2$$

$$|M_g|^2 = |M_{\tilde{g}}|^2$$ invariant under the semigroup

analytic part

c.f. Smoluchowski-Poisson equation

$$\frac{d}{dt}\int_\Omega R \log R \ d\mu \leq 0 \ \Rightarrow \ \sup_{t\in(0,\infty)} \|R_t\|_\infty < +\infty$$

→ $$\inf_{0<t<\infty} \min_\Omega R_t > 0$$ Harnack inequality of Li-Yau type

$$\frac{\partial}{\partial t}|M_g|^2 = \Delta |M_g|^2 - 2|\nabla M_g|^2 - 2R|M_g|^2$$

→ comparison theorem

$$|M_g|^2 \leq Ce^{-\gamma t}, \ \gamma > 0$$

conclusion

$$\tilde{g}(t) = T_t^* g(t) \to \tilde{g}_\infty \text{ in } C^\infty \text{ as } t \uparrow \infty, \ M_{\tilde{g}_\infty} = 0$$

$$g(t) \to g_\infty \text{ in } C^\infty \text{ as } t \uparrow \infty, \ R_{g_\infty} = \text{constant}$$

Summary

1. Increase of the surface entropy provides an a priori estimate
2. Harnack inequality implies reverse inequality
3. Convergence of the transformed metric to the trivial state in infinite time
4. Convergence of the original metric
5. Geometric structure guarantees these transformations and a priori estimates

Achievement in the theory of dynamical systems

1. global in time existence of the solution
2. pre-compactness of the orbit
3. uniqueness of the omega-limit set

Analytic proof

1. Trudinger-Moser-Fontana inequality
2. Benilan-Crandall inequality
3. Concentration compactness
4. Gradient inequality

Bartz-Struwe-Ye 94

1. Modified flow by covariant and Lie derivative
2. Moving spheres based on the symmetry
3. Bochner-Weitzenbock, Harnack, etc,

→

Analytic formulation

Gauss-Bonnet

genus

$$\int_\Omega R_g d\mu_g = 4\pi\chi(\Omega), \ \chi(\Omega) = 2 - 2g(\Omega)$$

$$R_g > 0 \ \Rightarrow \ g(\Omega) = 0$$

standard metric

uniformization theorem $\Omega = S^2, \ g = e^w g_0$

$$\Delta = \Delta_{g_0}, \ dx = d\mu_{g_0}, \ R_0 = R_{g_0}$$

⟹

$$R_g = e^{-w}(-\Delta w + R_0), \ |\Omega| R_0 = 8\pi$$

$$\int_\Omega R_g d\mu_g = 8\pi, \ r = \frac{\int_\Omega R_g d\mu_g}{\int_\Omega d\mu_g} = \frac{8\pi}{\int_\Omega e^w dx}$$

$$\frac{\partial g}{\partial t} = (r - R)g \ \Longrightarrow$$

$$\boxed{\frac{\partial e^w}{\partial t} = \Delta w + 8\pi\left(\frac{e^w}{\int_\Omega e^w dx} - \frac{1}{|\Omega|}\right) \text{ in } \Omega \times (0,T)}$$

2. Normalized Ricci flow-like equation

constant — compact Riemann surface without boundary

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda\left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|}\right) \text{ in } \Omega \times (0,T)$$

1. expansion of a thermalized electron cloud
2. central limit approximation of Carleson's model of the Boltzmann equation
3. thin liquid film

full system of chemotaxis

$$\varepsilon u_t = \nabla \cdot (\nabla u - u \nabla v)$$

$$\tau v_t = \Delta v + u - \frac{1}{|\Omega|}\int_\Omega u$$

$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, \frac{\partial v}{\partial \nu}\right)\Big|_{\partial\Omega} = 0$$

$$\int_\Omega v = 0$$

$\tau = 0$ Smoluchowski-Poisson (Jager-Luckhaus)

$$u_t = \nabla \cdot (\nabla u - u \nabla v), \ -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$

$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, \frac{\partial v}{\partial \nu}\right)\Big|_{\partial\Omega} = 0, \ \int_\Omega v = 0$$

$\varepsilon = 0$ non-local parabolic equation

$$v_t = \Delta v + \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right)$$

$$\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \ \int_\Omega v = 0$$

$\Omega \subset \mathbf{R}^2$ bounded domain with smooth boundary quantized blowup mechanism

$$\varepsilon u_t = \nabla\cdot(\nabla u - u\nabla v), \ -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$

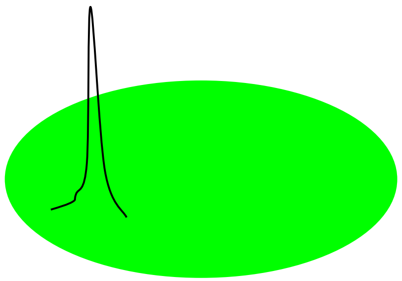
$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, \frac{\partial v}{\partial \nu}\right)\Big|_{\partial\Omega} = 0, \ \int_\Omega v = 0$$

$$n=2, \ T = T_{\max} < +\infty$$

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$$m(x_0) \in m_*(x_0)\mathbf{N}, \ m_*(x_0) = \begin{cases} 8\pi, & x_0\in\Omega \\ 4\pi, & x_0\in\partial\Omega \end{cases}$$

$$0 \le f = f(x) \in L^1(\Omega)\cap C(\overline{\Omega}\setminus\mathcal{S})$$



dis-quantized blowup mechanism

Wolansky 97

$$\Omega = B(0,1)\subset \mathbf{R}^2, \ v = v(|x|,t), \ \lambda \ge 8\pi$$

$$v_t = \Delta v + \frac{\lambda e^v}{\int_\Omega e^v}, \ v|_{\partial\Omega} = 0$$

$\Longrightarrow$

$$\frac{\lambda e^v}{\int_\Omega e^v}dx \rightharpoonup \lambda\delta_0, \ t\uparrow T = T_{\max}\in(0,+\infty]$$

Kavallaris-S. 07

$$\lambda > 8\pi \ \Rightarrow \ T = T_{\max} < +\infty$$

3. logarithmic diffusion

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda \left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|} \right) \longrightarrow \frac{d}{dt}\int_\Omega e^w = 0$$

$$u = re^w, \ t \mapsto t' = r^{-1}t, \ r = \frac{\lambda}{\int_\Omega e^w}$$

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u \quad \|u_0\|_1 = \lambda$$

c.f. logarithmic diffusion

$$u_t = \Delta \log u \ \text{in} \ \mathbf{R}^2 \times (0,T)$$

ODE part $\quad u_t = u - \frac{1}{|\Omega|}\int_\Omega u \quad \Omega \times (0,T)$

$$\longrightarrow \quad \frac{d}{dt}\int_\Omega u = 0$$

instability from the average $\quad \overline{u} = \frac{1}{|\Omega|}\int_\Omega u$

$\{u > \overline{u}\}, \ \{u < \overline{u}\}$ invariant in time

extinction in finite time before blowup

Rascle's Smolchowski-ODE quation 78

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$
$$\left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$
$$v_t = u - \frac{1}{|\Omega|}\int_\Omega u$$

Full system of chemotaxis (Keller-Segel 70)

$$\varepsilon u_t = \nabla \cdot (\nabla u - u\nabla v)$$
$$\tau v_t - \mu\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T)$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial u} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)$$

$\mu = 0$

$\varepsilon = 0$

$\tau = 0$

Nonlocal ODE equation

$$v_t = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right) \qquad \text{in } \Omega \times (0,T)$$

Smoluchowski-Poisson equation (Jäger-Luckhaus 91)

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$
$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T)$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)$$
$$\int_\Omega v = 0$$

mean field equation
as stationary states

Nonlocal parabolic equation (Wolansky 97)

$$v_t = \Delta v + \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right)$$

Hamilton's normalized Ricci flow 88

$$u_t = \Delta \log u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0,T)$$

Diffusion Geometry 2

Existence of Pre-Compact Global in Time Orbit

1. thermo-dynamical structure

Ω compact Riemann surface without boundary

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u \text{ in } \Omega \times (0,T)$$

$$u|_{t=0} = u_0(x) > 0, \ \|u_0\|_1 = \lambda$$

Hermholtz free energy

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle (-\Delta)^{-1}u, u\rangle$$

⟹

$$\delta\mathcal{F}(u) = \log u - (-\Delta)^{-1}u$$

Model (B) equation

$$u_t = \Delta(\log u - v), \quad v = (-\Delta)^{-1}u$$

⟺ $u_t = \Delta\delta\mathcal{F}(u)$

$$\frac{d}{dt}\int_\Omega u = 0, \quad \frac{d}{dt}\mathcal{F}(u) = -\int_\Omega |\nabla\delta\mathcal{F}(u)|^2 \le 0$$

Theorem $0 < \lambda \le 8\pi \Longrightarrow T = +\infty$

$$\|u(\cdot,t)\|_\infty + \|u(\cdot,t)^{-1}\|_\infty \le C$$

c.f. Smoluchowski-Poisson

$$u_t = \nabla\cdot(\nabla u - u\nabla v), \ -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \ \int_\Omega v = 0$$

⟺ $u_t = \nabla\cdot u\nabla\delta\mathcal{F}(u) \ \text{ in } \Omega\times(0,T)$

possible blowup in infinite time for $\lambda = 8\pi$

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u$$

$$u|_{t=0} = u_0(x) > 0$$

$w = \log u$ (↔)

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda\left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|}\right)$$

Jensen $\quad \overline{w} = \frac{1}{|\Omega|}\int_\Omega w \le \log\left(\frac{\lambda}{|\Omega|}\right), \ \int_\Omega e^w = \lambda$

⟶ $$\int_\Omega u = \lambda \equiv \|u_0\|_1$$

$$\begin{aligned} J_\lambda(w) &= \frac{1}{2}\|\nabla w\|_2^2 - \lambda \log \int_\Omega e^{w-\overline{w}} \\ &= \frac{1}{2}\|\nabla w\|_2^2 - \lambda\left(\log \int_\Omega e^{w} - \overline{w}\right) \end{aligned}$$

$$\begin{aligned} u_t &= \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u \\ &\le \Delta \log u + u \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}J_\lambda(w) &= \int_\Omega \nabla w \cdot \nabla w_t - \lambda\left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|}\right) w_t \ dx \\ &= -\int_\Omega e^w w_t^2 \le 0 \end{aligned}$$

⟶ maximum principle

$$T < +\infty \ \Rightarrow \ \lim_{t \uparrow T} \min_{\Omega} u(\cdot, t) = 0$$

Trudinger-Moser-Fontana inequality (94)

$$\inf_E J_{8\pi} > -\infty$$

$$E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$$

2. sub-critical case $\lambda < 8\pi$

TMF inequality

$$\frac{d}{dt}J_\lambda(w) \leq 0 \quad \longrightarrow \quad \|\nabla w\|_2 \leq C, \ \overline{w} \geq -C$$

Jensen $\overline{w} \leq C$

(Poincare-Wirtinger)

$\longrightarrow$

$$\|w\|_{H^1} \leq C$$

TMF inequality

$\longrightarrow$

$$\|u(\cdot,t)\|_p + \|u(\cdot,t)^{-1}\|_p \leq C_p, \ 1 \leq p < \infty$$

Moser's iteration scheme

$\longrightarrow$

$$\|u(\cdot,t)\|_\infty + \|u(\cdot,t)^{-1}\|_\infty \leq C$$

$T = +\infty$ pre-compactness of the orbit

3. critical case $\lambda = 8\pi$

$$\int_\Omega e^w = 8\pi \quad \longrightarrow$$

$$\begin{aligned} J_{8\pi}(w) &= \frac{1}{2}\|\nabla w\|_2^2 - 8\pi \log(\int_\Omega e^w - \overline{w}) \\ &= \frac{1}{2}\|\nabla w\|_2^2 + 8\pi\overline{w} - 8\pi\log(8\pi) \end{aligned}$$

$$-C \leq \frac{1}{2}\|\nabla w\|_2^2 + 8\pi\overline{w} \leq C$$

Jensen $\overline{w} \leq C$

$T < +\infty$ → $\liminf_{t\uparrow T} \overline{w} = -\infty$

Benilan Crandall inequality

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u$$

→ $$u_{tt} = \Delta(\frac{u_t}{u}) + u_t$$

→ $$p_t = e^{-w}\Delta p + p - p^2,\ p = \frac{u_t}{u}$$

$$\overline{p}_t = \overline{p} - \overline{p}^2,\ \overline{p} = \frac{e^t}{e^t - 1}$$

→ $$\frac{u_t}{u} \leq \frac{e^t}{e^t - 1}$$

→ $$\frac{\partial}{\partial t}\left(\frac{u}{e^t - 1}\right) \leq 0$$ → pointwise $$\exists \lim_{t\uparrow T} u(x,t) = u(x,T)$$

TMF inequality to $w/2$

$$\frac{1}{2}\int_\Omega |\nabla w|^2 \geq 4 \cdot 8\pi \log \int_\Omega e^{w/2} - 16\pi\overline{w} - C$$

recall $$-C \leq \frac{1}{2}\|\nabla w\|_2^2 + 8\pi\overline{w} \leq C$$

→ $$\overline{w} \geq 4\log\int_\Omega e^{w/2} - C$$

→ $$\liminf_{t\uparrow T}\int_\Omega e^{w/2} = \liminf_{t\uparrow T}\int_\Omega u(\cdot,t)^{1/2}$$

monotone convergence theorem $$= \int_\Omega u(\cdot,T)^{1/2} = 0$$

$$u(\cdot,T) = 0 \text{ a.e. in } \Omega$$

$$8\pi = \lim_{t\uparrow T}\int_\Omega u(\cdot,t) = \int_\Omega u(\cdot,T) = 0$$ a contradiction

proof of the pre-compactness of the orbit

show $\overline{w} \geq -C$ assume $\liminf_{t\uparrow+\infty} \overline{w} = -\infty$

$$H = \int_\Omega e^w w \longrightarrow$$

$$\begin{aligned} \frac{dH}{dt} &= -\|\nabla w\|_2^2 + H - 8\pi\overline{w} \\ &\leq H + 8\pi\overline{w} + C \end{aligned}$$

TMF inequality

$$\exists t_k \uparrow +\infty, \ \exists \delta > 0$$

Benilan-Crandall inequality

$$8\pi\overline{w}(t) + C \leq -k, \ t_k - \delta < t < t_k$$

$$H \geq -e|\Omega| \longrightarrow$$

$$\lim_{k\to\infty} \inf_{t_k-\delta<t<t_k-\delta/2} H(t) = +\infty$$

$$\sum_{k=1}^\infty \int_{t_k-\delta}^{t_k-\delta/2} dt \int_\Omega e^w w_t^2 \leq \int_0^\infty dt \int_\Omega e^w w_t^2 < +\infty$$

$$\|e^w w_t\|_1^2 \leq \int_\Omega e^w \cdot \int_\Omega e^w w_t^2 = 8\pi \int_\Omega e^w w_t^2$$

$$\longrightarrow \lim_{k\to\infty} \int_{t_k-\delta}^{t_k-\delta/2} \|e^w w_t(\cdot,t)\|_1^2 \ dt = 0,$$

$$t_k - \delta < \exists t_k' < t_k - \delta/2$$

$$\lim_{k\to\infty} \left\| \frac{\partial e^w}{\partial t}(\cdot, t_k') \right\|_1 = 0$$

$$\longrightarrow \lim_{k\to\infty} \int_\Omega e^w w(\cdot, t_k') = +\infty$$

$$u_k = u(\cdot, t_k)$$

$$\lim_{k\to\infty} \int_\Omega e^w w(\cdot, t'_k) = +\infty \quad \longrightarrow \quad \lim_{k\to\infty} \int_\Omega u_k \log u_k = +\infty$$

$$\mathcal{F}(u_k) \leq \mathcal{F}(u_0) \quad \longrightarrow \quad \lim_{k\to\infty} \langle (-\Delta)^{-1} u_k, u_k \rangle = +\infty$$

subsequence $$\lim_{k\to\infty} \int_\Omega x u_k = x_\infty \in \mathbf{R}^N$$

Lemma

$$\|u_k\|_1 = 8\pi$$

$$\mathcal{F}(u_k) \leq C$$

$$\lim_{k\to\infty} \langle (-\Delta)^{-1} u_k, u_k \rangle = +\infty$$

$$\lim_{k\to\infty} \int_\Omega x u_k(x) dx = x_\infty \in \mathbf{R}^N$$

$$\longrightarrow \quad x_\infty \in \Omega, \; u_k \rightharpoonup 8\pi \delta_{x_\infty}$$

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|})$$

$$\lim_{k\to\infty} \left\| \frac{\partial e^w}{\partial t}(\cdot, t'_k) \right\|_1 = 0$$

$$e^{w(\cdot, t'_k)} = u_k \rightharpoonup 8\,\pi\delta_{x_\infty} \quad \text{in } \mathcal{M}(\Omega)$$

elliptic estimate (Brezis-Strauss)

$$w(\cdot, t'_k) \to 8\pi G(\cdot, x_\infty) \text{ in } W^{1,q}(\Omega),\ 1 \le q < 2$$

$$\lim_{k\to\infty} \int_\Omega e^{w(\cdot, t'_k)} = +\infty$$ Fatou

$$G(x, x') \approx \frac{1}{2\pi} \log \frac{1}{\text{dist }(x, x')}$$

$$\int_\Omega e^w = 8\pi$$ a contradiction

Diffusion Geometry 3

Convergence of the Normalized Ricci Flow

1. Summary

Ω compact Riemann surface $\partial\Omega = \emptyset$

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u \quad \Longleftrightarrow \quad g_t = (r-R)g, \quad r = \frac{\int_\Omega R\, d\mu}{\int_\Omega d\mu}$$

R: scalar curvature

$$u|_{t=0} = u_0(x) > 0$$

$$\lambda = 8\pi,\ \Omega = S^2$$

Hamilton's normalized Ricci flow

$$\int_\Omega u_0 = \lambda \ \Rightarrow\ \int_\Omega u(t) = \lambda$$

stationary state $u_* = \frac{\lambda}{|\Omega|}$ uniformization theory

$$u(t) \to u_* \equiv \frac{\lambda}{|\Omega|} \quad \text{in } C^\infty \text{ topology}$$

$$\exists T > 0$$

maximal existence time of the classical solution

Kavallaris-S. 10 $0 \le \lambda < 8\pi \longrightarrow T = +\infty,\ \|u(t)\|_\infty + \|u(t)^{-1}\|_\infty \le C$ Trudinger-Moser-Fontana

$\lambda = 8\pi \longrightarrow T = +\infty$ Benilan-Crandall

Kavallaris-S. 15 $\lambda = 8\pi \longrightarrow \|u(t)\|_\infty + \|u(t)^{-1}\|_\infty \le C$ P.L. Lions variational structure

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1)\, dx + \frac{1}{2}\int_\Omega u \cdot \Delta^{-1} u\, dx$$ Helmholtz free energy

$$v = -\Delta^{-1} u \quad \Longleftrightarrow \quad -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u\ dx, \quad \int_\Omega v\ dx = 0$$ Poisson equation

$u_t = \Delta \delta \mathcal{F}(u)$ model (B) equation $\Longrightarrow$

$$\frac{d}{dt}\int_\Omega u = 0$$ total mass conservation

$$\frac{d}{dt}\mathcal{F}(u) = \langle u_t, \delta\mathcal{F}(u)\rangle = -\|\nabla \delta \mathcal{F}(u)\|_2^2 \leq 0$$ free energy decreasing

pre-compact orbit in C^∞ topology $\Longrightarrow$ $\emptyset \neq \omega(u_0)$ contained in the set of stationary solutions F_λ

theory of dynamical systems

connected, compact

stationary solutions

$$\delta\mathcal{F}(u^*) = \log u^* + \Delta^{-1} u^* = \text{constant}\ , \int_\Omega u^*\, dx = \lambda,\ u^* = u^*(x) > 0 \quad \Longleftrightarrow \quad u_* \in F_\lambda$$

stationary problem ⟷

$$-\Delta \log u_* = u_* - \frac{1}{|\Omega|}\int_\Omega u_*\, dx, \quad x \in \Omega, \qquad \int_\Omega u_*\, dx = \lambda$$

$$u^* = \frac{\lambda e^{v^*}}{\int_\Omega e^{v^*} dx} \quad \updownarrow \quad v_* = \log u_* - \frac{1}{|\Omega|}\int_\Omega \log u_*$$

$$-\Delta v_* = \lambda\left(\frac{e^{v_*}}{\int_\Omega e^{v_*}\, dx} - \frac{1}{|\Omega|}\right), \quad \int_\Omega v_*\, dx = 0 \quad \longleftrightarrow \quad v_* \in E_\lambda$$

⟷

$$J_\lambda(v) = \frac{1}{2}||\nabla v||_2^2 - \lambda \log \int_\Omega e^v\, dx$$

$$V_0 = \{v \in H^1(\Omega) \mid \int_\Omega v\, dx = 0\}$$

$0 < \lambda \leq 8\pi \longrightarrow$ $E_\lambda = \{0\}$

$$u(t) \to u_* \equiv \frac{\lambda}{|\Omega|} \quad \text{in } C^\infty \text{ topology}$$

elliptic theories

(a) $\lambda = 8\pi, \quad \Omega = S^2$ (Hamilton)

(b) $\lambda = 8\pi, \ \Omega = \mathcal{T} \equiv \mathbf{R}^2/a\mathbf{Z} \times b\mathbf{Z}, \ \frac{b}{a} \geq \frac{\pi}{4}$

Lin-Lucia 06

Remark E_λ may be a continuum

Trudinger-Moser-Fontana $\quad v \in V_0, \ \|\nabla v\|_2 \leq 1 \quad \Rightarrow \quad \int_\Omega e^{4\pi v^2}\, dx \leq C$

Theorem $T=+\infty,\ \|u(t)\|_\infty+\|u(t)^{-1}\|_\infty\le C$ ⟶ $\exists u_*\in F_\lambda$ $u(t)\to u_*$ in C^∞ topology

in algebraic order

u_* non-degenerate ⟶ in exponential order

⟷ $v_*=\log u_*-\frac{1}{|\Omega|}\int_\Omega \log u_*$ non-degenerate critical point of $J_\lambda=J_\lambda(v),\ v\in V_0$

⟷ $\psi\in H^2(\Omega),\ -\Delta\psi=u^*\psi \text{ in } \Omega,\ \int_\Omega \psi u^*\,dx=0 \ \Rightarrow\ \psi=0$

Proof $u=e^w$ $\frac{\partial e^w}{\partial t}=\Delta w+\lambda(\frac{e^w}{\int_\Omega e^w\,dx}-\frac{1}{|\Omega|}) \text{ in } \Omega\times(0,T)$ ⟷ $\frac{\partial e^w}{\partial t}=-\delta\mathcal{E}(w)$

$$\mathcal{E}(w)=\int_\Omega \frac{1}{2}|\nabla w|^2-e^w+\frac{\lambda}{|\Omega|}w\ dx,\quad w\in H^1(\Omega)=V$$

<u>Gradient inequality</u>

$w_*\in V,\ \delta\mathcal{E}(w_*)=0\ \Rightarrow\ 0<\exists\theta\le\frac{1}{2},\ \exists\varepsilon_0>0$

$\forall w\in V,\ \|w-w_*\|_V<\varepsilon_0\ \Rightarrow\ |\mathcal{E}(w)-\mathcal{E}(w_*)|^{1-\theta}\le C\|\delta\mathcal{E}(w)\|_{V^*}$

u_* non-degenerate ⟶ $\theta=\frac{1}{2}$

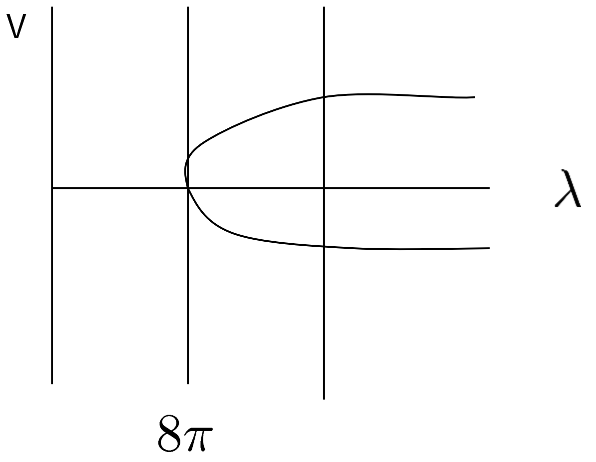
elliptic theory

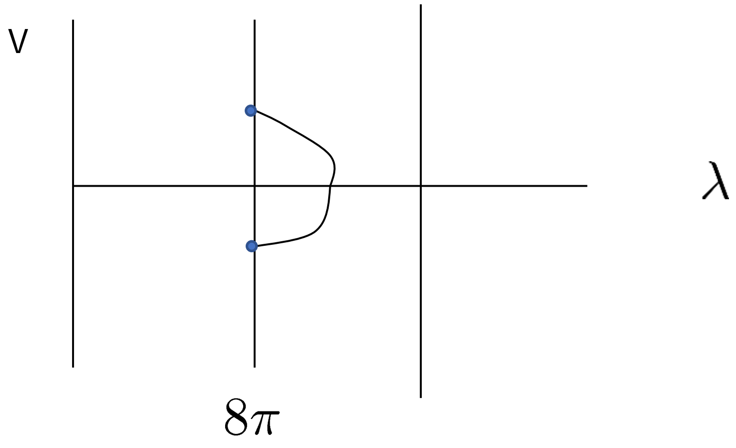
$$J_\lambda(v)=\frac{1}{2}\|\nabla v\|_2^2-\lambda\log\int_\Omega e^v\,dx,\ V_0=\{v\in H^1(\Omega)\mid \int_\Omega v\,dx=0\}$$

$$\delta J_\lambda(v_*)=0\ \Leftrightarrow\ -\Delta v_*=\lambda\Big(\frac{e^{v_*}}{\int_\Omega e^{v_*}\,dx}-\frac{1}{|\Omega|}\Big),\ \int_\Omega v_*\,dx=0$$

$\Omega=S^2$ (analytic proof)
Chanillo-Kiessling 95, Cheng-Lin 97, Lin 00

$\Omega=T^2=\mathbf{R}^2/a\mathbf{Z}\times b\mathbf{Z},\ \frac{b}{a}\geq\frac{\pi}{4}$ (Lin-Lucia 07)





2. Gradient Inequality (Lojasiewicz 63, Simon 83)

Lemma 1 $E:\mathbf{R}^n\to\mathbf{R}$ real analytic at $x=0$

$$E(0)=0,\ \delta E(0)=0 \Longrightarrow 0<\exists\theta\le\frac{1}{2}$$

$$|E(x)|^{1-\theta}\le C|\delta E(x)|,\ |x|\ll 1$$

Proof $$E(x)=\sum_\alpha C_\alpha x^\alpha$$

$$\alpha=(\alpha_1,\cdots,\alpha_n),\ x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$$

suffices to assume $E\not\equiv 0 \Longrightarrow \exists m\ge 2$

$$C_\alpha=0,\ |\forall\alpha|\le m-1$$
$$C_\alpha\neq 0,\ |\exists\alpha|=m$$

$$E(x)=E_0(x)+E_1(x),\ E_0(x)=\sum_{|\alpha|=m}C_\alpha x^\alpha$$

$$|E_1(x)|=o(|x|^m),\ |E_1'(x)|=o(|x|^{m-1})$$

$$|E_0(x)|=r^m|E_0(\omega)|\le Cr^m,\ x=r\omega,\ r=|x|$$

$$\begin{aligned}|E_0'(x)| &= \{\sum_{i=1}^n(\frac{\partial E_0}{\partial x_i})^2\}^{1/2}\\ &= r^{m-1}\{\sum_{i=1}^n(\frac{\partial E_0}{\partial x_i}(\omega))^2\}^{1/2}\\ &\ge c_1r^{m-1},\ \exists c_1>0\end{aligned}$$

$$\dot{x}=-\delta E(x) \qquad \frac{d}{dt}E(x)=-|\dot{x}|^2\le 0$$

assume; global in time solution with pre-compact orbit

theory of dynamical systems. LaSalle principle

$$\Longrightarrow \exists t_k\uparrow+\infty,\ x(t_k)\to\exists x_*,\ \delta E(x_*)=0$$

$$H=(E(x)-E(x_*))^\theta$$ well-defined

$$\lim_{t\uparrow+\infty}H(x(t))=0$$

$$\boxed{\dot{x} = -\delta E(x)}$$

Haraux-Jendoubi 01

$$0 < \exists \theta \leq \frac{1}{2} \qquad |E(x)|^{1-\theta} \leq C|\delta E(x)|, \ |x| \ll 1$$

$$\exists t_k \uparrow +\infty, \ x(t_k) \to \exists x_*, \ \delta E(x_*) = 0$$

$$H = (E(x) - E(x_*))^\theta \qquad \lim_{t \uparrow +\infty} H(x(t)) = 0$$

$$\begin{aligned} -\frac{dH}{dt} &= \theta(E(x) - E(x_*))^{\theta-1}(-\dot{x} \cdot \delta E(x)) \\ &= \theta(E(x) - E(x_*))^{\theta-1}|\delta E(x)|^2 \\ &= \theta(E(x) - E(x_*))^{\theta-1}|\delta E(x)||\dot{x}| \\ &\geq c_2|\dot{x}|, \ \exists c_2 > 0 \end{aligned}$$

$$\Longrightarrow \int_0^\infty |\dot{x}| \ dt < +\infty$$

$$\exists \lim_{t \uparrow +\infty} x(t) = x_*$$

$$\begin{aligned} -\frac{dH}{dt} &= \theta(E(x) - E(x_*))^{\theta-1}|\delta E(x)|^2 \\ &\geq c_3(E(x) - E(x_*))^{1-\theta} \\ &= c_3 H^{\frac{1}{\theta}-1}, \ \exists c_3 > 0 \end{aligned}$$

$$H \leq C\Phi, \ \Phi(t) = \begin{cases} t^{-\frac{\theta}{1-2\theta}}, & 0 < \theta < 1/2 \\ e^{-\delta_0 t}, & \theta = \frac{1}{2}, \ \delta_0 = c_3^{-1} \end{cases}$$

$$|x(s) - x(t)| \leq C\Phi(t), \ s > t \quad \longrightarrow \text{(left, to } c_2|\dot{x}| \text{ line)}$$

$$s \uparrow +\infty \Longrightarrow |x(t) - x_*| \leq C\Phi(t)$$

application to the NRF

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda(\frac{e^w}{\int_\Omega e^w\, dx} - \frac{1}{|\Omega|})$$

analytic

$$\frac{\partial e^w}{\partial t} = -\delta\mathcal{E}(w) \qquad \mathcal{E}(w) = \int_\Omega \frac{1}{2}|\nabla w|^2 - e^w + \frac{\lambda}{|\Omega|}w\ dx$$

$$w \in H^1(\Omega) = V$$

realized as a self-adjoint operator in

$$w_* \in V,\ \ \delta\mathcal{E}(w_*) = 0 \qquad X = L^2(\Omega)$$

$$\mathcal{L} \equiv \delta^2\mathcal{E}(w^*) = -\Delta - e^{w^*} : V \to V^*$$

linearized operator

$$X_1 \equiv \mathrm{Ker}\ \mathcal{L} = \{v \in D(\mathcal{L}) \big| \mathcal{L}v = 0\} \subset V = H^1(\Omega)$$

$\mathcal{P} : V^* \to X_1$ orthogonal projection $\dim\ X_1 = n$

Lemma 2 (Chill 03, 06) $w_* \in \exists U \subset V$ neighborhood

$$\mathcal{S} = \{w \in U \big| (I - \mathcal{P})\delta\mathcal{E}(w) = 0\}$$

critical manifold

local analytic manifold around w_* with dimension n

$$\exists g : U_1 = U \cap X_1 \to U_2 = (I - \mathcal{P})U$$

analytic

$$g(w_1^*) = w_2^*,\ w^* = w_1^* + w_2^* \in U_1 \oplus U_2$$

$$\mathcal{S} = \{w_1 + g(w_1) \mid w_1 \in U_1\}$$

⟶ $0 < \exists\theta \leq \frac{1}{2}$

$$w \in V,\ \|w - w^*\|_V < \exists\varepsilon_0 \ \Rightarrow$$

$$|\mathcal{E}(w) - \mathcal{E}(w^*)|^{1-\theta} \leq C\|\delta\mathcal{E}(w)\|_{V^*}$$

$$\sup_{t_0 \leq t < t_0+T} \|w(\cdot,t) - w^*\|_V$$

parabolic regularity

$$\leq C(\|w(\cdot,t_0) - w^*\|_V + \sup_{t_0 \leq t < t_0+T} \|w(\cdot,t) - w^*\|_2)$$

⟶ convergence in algebraic rate

role of non-degeneracy $w^* = \log u^*$

$$-\Delta \log u_* = u_* - \frac{1}{|\Omega|}\int_\Omega u_*\, dx, \ \int_\Omega u_* \ dx = \lambda$$

non-degenerate

⟺

$$v_* = \log u_* - \frac{1}{|\Omega|}\int_\Omega \log u_*$$

non-degenerate critical point of $J_\lambda = J_\lambda(v), \ v \in V_0$

⟺ non-degeneracy of the linearized operator

$$\mathcal{B}\phi = -\Delta\phi - u^*\phi + \frac{1}{\lambda}(\phi, u^*)u^*$$

$$\phi \in D(\mathcal{B}) = H^2(\Omega) \cap V_0$$

⟺ $\mathcal{M} = -\Delta - u^* : V = H^1(\Omega) \to V^*$

$$\phi \in V, \ \int_\Omega u^*\phi\, dx = 0 \quad \Rightarrow \quad ||\phi||_V \leq C||\mathcal{M}\phi||_{V^*}$$

⟺ implicit function theorem

$$\exists \varepsilon_0 > 0, \ w \in V, \ \int_\Omega e^w\, dx = \lambda, \ \|w - w^*\|_V < \varepsilon_0$$

$$\Rightarrow \ \|w - w^*\|_V \leq C\|\mathcal{M}(w - w^*)\|_{V^*}$$

Lemma 3 u_* non-degenerate ⟶ $\theta = \frac{1}{2}$

$$w \in V, \ \|w - w^*\|_V < \exists\varepsilon_0, \ \int_\Omega e^w = \lambda$$

$$\Rightarrow$$

$$|\mathcal{E}(w) - \mathcal{E}(w^*)|^{1-\theta} \leq C\|\delta\mathcal{E}(w)\|_{V^*}$$