Mean Field Theories
and
Dual Variation

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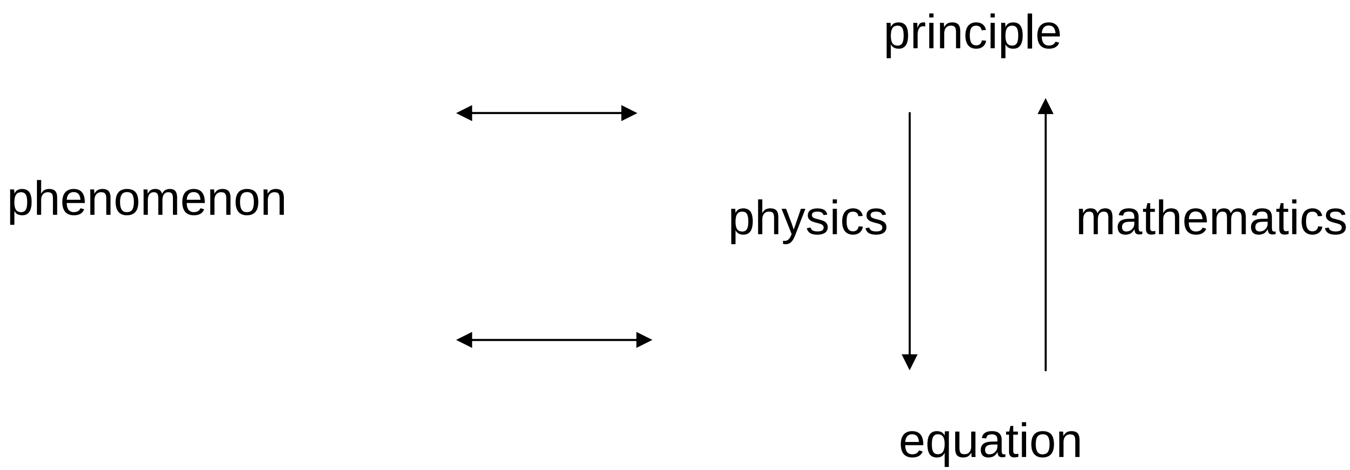
Mathematical Science and Mathematical Analysis
for Self-Interacting Particles


principle
phenomenon
physics
mathematics
equation

Mathematical analysis for differential equations
- Paradigm of Poincare-Hadamard

1. Fundamental theorem…well-posedness;
existence, uniqueness, continuous dependence of the solution

2. Qualitative study

If A is a square matrix, then the well-posedness of $Ax = b$ is summarized as $\det A \neq 0$,

so that uniqueness of x (for each b), existence of x (for each b), and continuous dependence of x (w.r.t. b) are equivalent.

Infinite dimensional version is valid if $A = I + K$ with compact K, which covers most standard linear pde's.

...Fredholm, Banach-Steinhaus, Riesz-Schauder,...

List of standard linear pde's

Schrodinger
(dispersive)

$$iv_t - \Delta v = f$$

Hyperbolic
(conservative)

$$v_{tt} - \Delta v = f$$

Elliptic
(static)

$$-\Delta v = f$$

Parabolic
(dissipative)

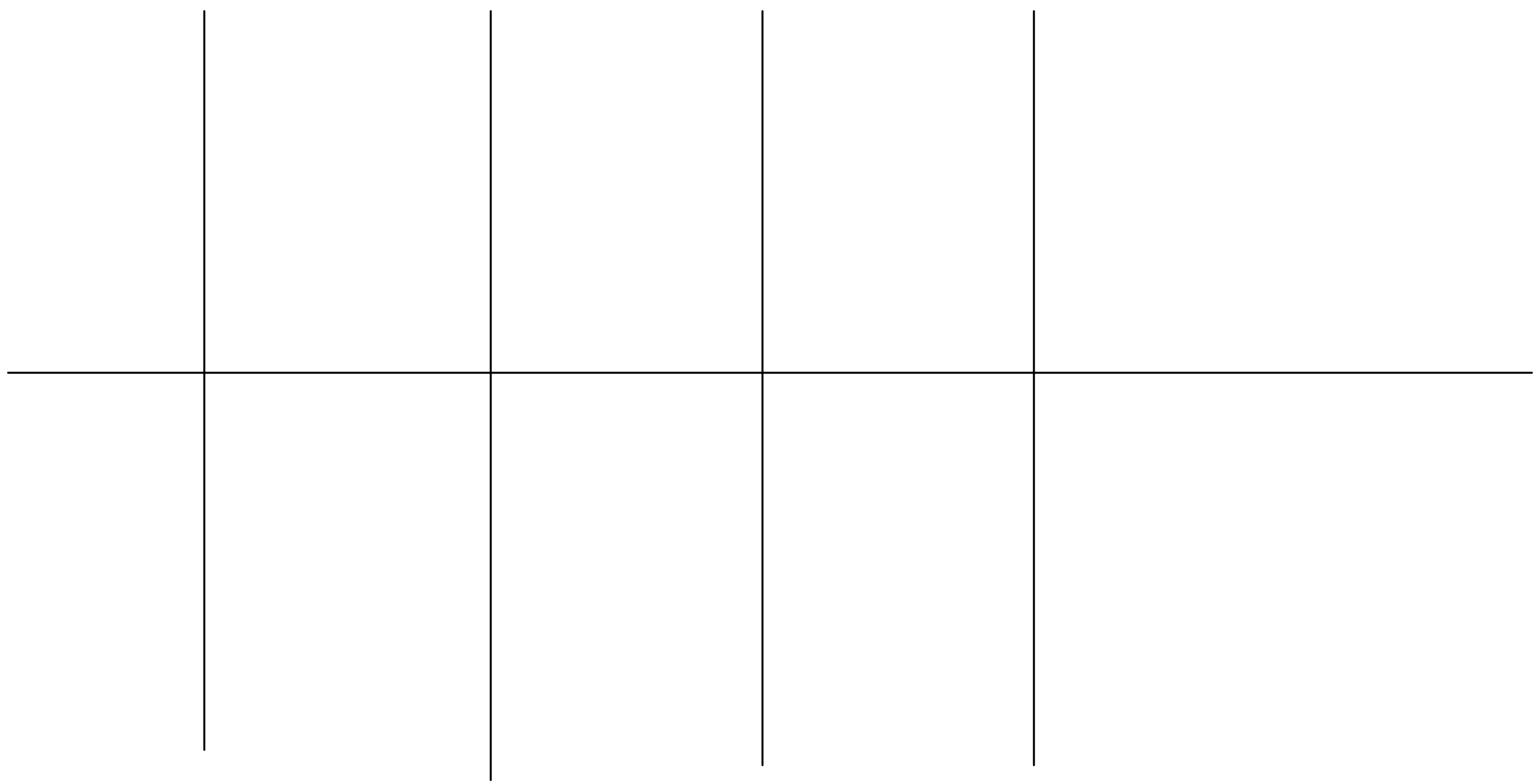
$$v_t - \Delta v = f$$

Weak solution is already necessary to guarantee the well-posedness of the equation, for example,

$$v_{tt} = v_{xx} \quad \Leftrightarrow \quad v(x,t) = f(x+t) + g(x-t)$$

Importance of the notion of well-posedness is the looking of the total set of the solution.

Spectrum of the harmonic oscillator $-\frac{d^2}{dx^2}+x^2$ is quantized as $\{2n+1 \mid n=0,1,\cdots\}$.



In nonlinear problems, those three notions - existence, uniqueness, continuous dependence, are not equivalent.

Uniqueness is the most difficult!

Linear – Semi-linear – Quasi-linear – Fully Nonlinear

Single – System

Standard methods in the theory of nonlinear pde:

1. Method of Perturbation

2. Method of Energy / Variation

3. Method of Dynamical System

4. Method of Exact Solution

5. Method of Comparison

List of weak solutions valid to nonlinear problems

1. Schwartz

2. Lax-Kato

3. Minty-Browder

4. Nonlinear Semigroup

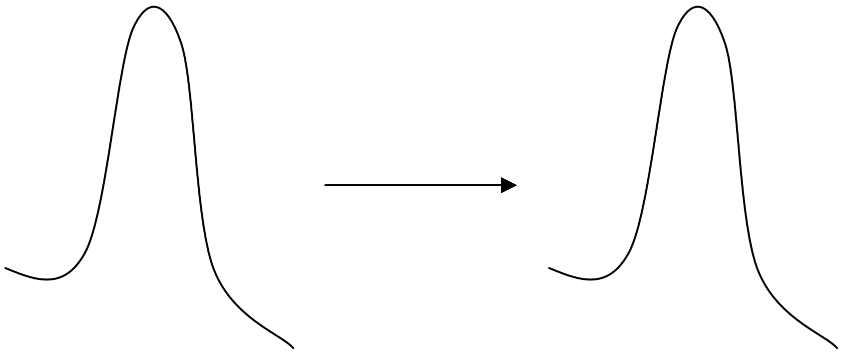
5. Viscosity

6. Renormalization

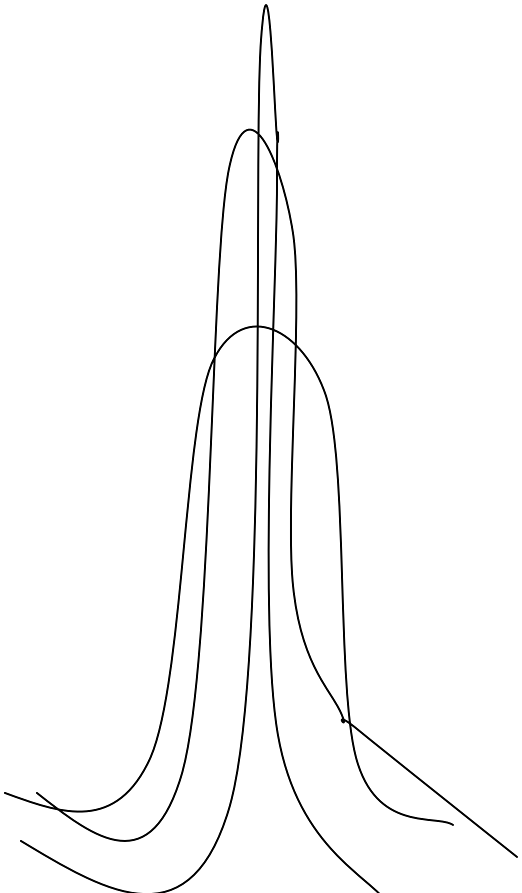
7. Varifold

Breaking down of the generation of weak solutions

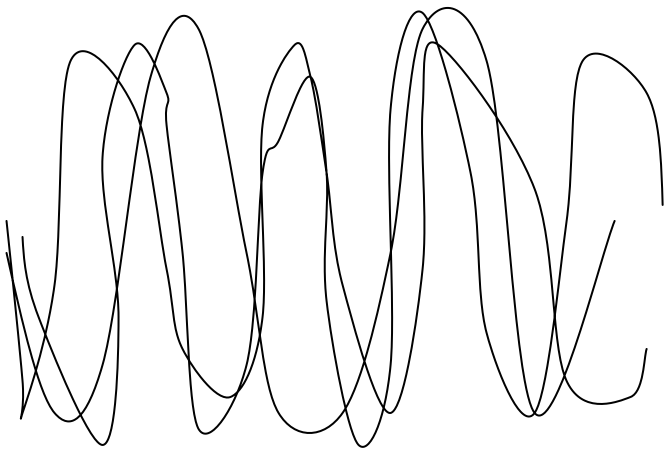
1. flight



2. Concentration



3. Oscillation



Method of weak convergence in the theory of nonlinear partial differential equations controls those phenomena and gets the solution by excluding weak converging approximate solutions which are strongly non-compact.

Chandrasekhar's critical mass in steller astronomy (Yamabe problem in geomerty):

$$-\Delta v = v^5, \quad v > 0 \quad \text{in} \quad \Omega$$
$$v = 0 \quad \text{on} \quad \partial\Omega,$$

where $\Omega \subset \mathbf{R}^3$ is a bounded domain.

1. If Ω is star-shaped, then there is no solution (Pokhozaev 1965).

2. If Ω has a hole, then there is a solution (Bahri 1988).

Scaling invariance:

$$-\Delta v = v^p \Rightarrow -\Delta v_\lambda = v_\lambda^p,$$

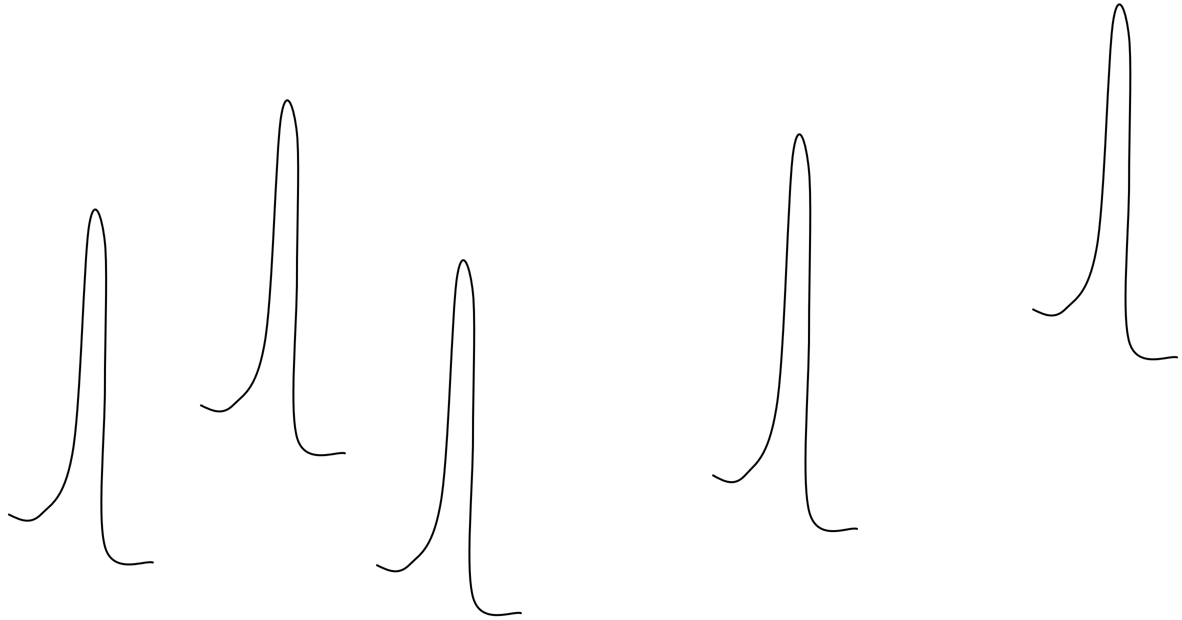
where $v_\lambda(x) = \lambda^{2/(p-1)} v(\lambda x)$.

General blowup mechanism for $u_t = L(D^2u, Du, u, x, t)$

1. Blowup… $T_{\max} < \infty$

2. Rigidness-Threshold… $\| u_0 \| < C_* \Rightarrow T_{\max} = \infty, \| \exists u_0 \| > C^* \Rightarrow T_{\max} < \infty$

3. Bubble – Quantization



Mean field equation of self-gravitating particles
…movements of nebula, cellular slime molds,…

$$u_t = \nabla \cdot (\nabla u - u \nabla v)$$

$$\Omega \times (0, T)$$

$$0 = \Delta v - av + u$$

$$\partial u / \partial \nu = \partial v / \partial \nu = 0 \qquad \partial\Omega \times (0, T)$$

$\Omega \subset R^2$ bounded $\partial \Omega$ smooth

$a > 0$ constant

$u = u(x,t)$ denotes the mass distribution of many self-gravitating particles, and $v = v(x,t)$ is the field created by u.

$$u_0(x) \geq 0 \quad \Rightarrow \quad u(x,t) \geq 0$$

$$\frac{d}{dt}\int_\Omega u(x,t) = \int_\Omega u_t = \int_\Omega \nabla \cdot (\nabla u - u\nabla v)$$
$$= \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right) = 0$$

$$\|u(t)\|_1 = \|u_0\|_1 \equiv \lambda$$

total mass conservation

$$\frac{d}{dt}W = -\int_\Omega u\,|\nabla(\log u - v)|^2 \leq 0$$

decrease of the free energy

$$W = \int_\Omega u(\log u - 1) - \frac{1}{2}\left(\|\nabla v\|_2^2 + a\,\|v\|_2^2\right)$$

$$= \int_\Omega u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,y)u \otimes u$$

Quantized blowup mechanism

1. Formation of collapses

$$T_{\max} < +\infty \Rightarrow$$

$$u(x,t)dx \rightharpoonup_{t \to T_{\max}} \sum_{x_0 \in S} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$$0 \leq f \in L^1(\Omega) \cap C\left(\overline{\Omega} \setminus S\right)$$

2. mass quantization $\quad m(x_0) = m_*(x_0)$

$$m_*(x_0) = \begin{array}{ll} 8\pi & x_0 \in \Omega \\ 4\pi & x_0 \in \partial\Omega \end{array}$$

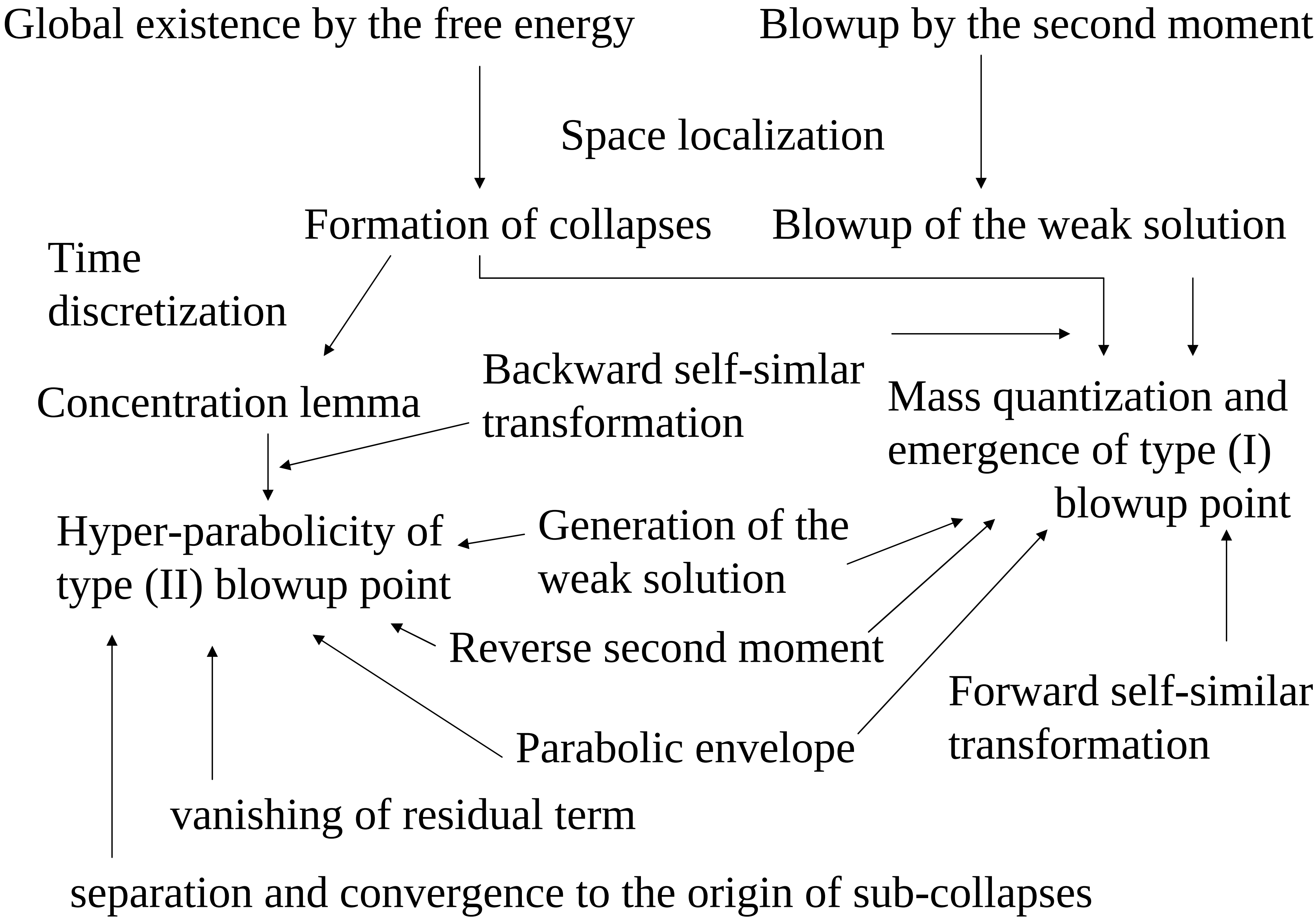
$$S=\{x_0\in\overline{\Omega}\mid \exists x_k\to x_0, \exists t_k\uparrow T_{\max}, u(x_k,t_k)\to+\infty\}$$

…the blowup set.

$$\| u(t)\|_1=\| u_0\|_1=\lambda$$
$$\Rightarrow 2\#(\Omega\cap S)+\#(\partial\Omega\cap S)\le \| u_0\|_1/(4\pi)$$

The number of collapses is estimated from above by the total mass.

Patlak (1953),
Keller-Segel (1970),
Nanjundiah (1973),
Childress-Percus (1981),
Percus (1984),
Jäger-Luckhaus (1991),
Nagai (1995),
Herrero-Velazquez (1996),
Nagai-Senba-Yoshida (1997),
Biler (1998),
Gajewski-Zacharias (1998), ...


Global existence by the free energy
Blowup by the second moment
Space localization
Formation of collapses
Blowup of the weak solution
Time discretization
Concentration lemma
Backward self-simlar transformation
Mass quantization and emergence of type (I) blowup point
Hyper-parabolicity of type (II) blowup point
Generation of the weak solution
Reverse second moment
Forward self-similar transformation
Parabolic envelope
vanishing of residual term
separation and convergence to the origin of sub-collapses

S.

Free Energy and Self-Interacting Particles

/A Mathematical Approach

Birkhäuser (2004)

ODE	Newton	Langevin
Kinetic	Jeans-Vlasov	Fokker-Planck
Conservation of Mass	Euler	Keller-Segel
Equilibrium	Semilinear elliptic eigenvalue problem (non-linearity unknown)	Semilinear elliptic eigenvalue problem (exponential non-linearity)
Clustered Particles	Hamiltonian flow	Gradient flow
Physics	Conservation laws	Free energy
Mathematics	Chaotic motion	Quantized blow-up mechanism

Theory of nonlinear quantum mechanics

1. Each equilibrium state is formulated as a nonlinear eigenvalue problem with non-local term, where eigenvalue is associated with conservative quantities.

2. Total set of equilibrium states obeys the profile of the quantized blowup mechanism.

3. This fact induces the same phenomenon to the non-equilibrium state.

Equilibrium state of the system of chemotaxis:

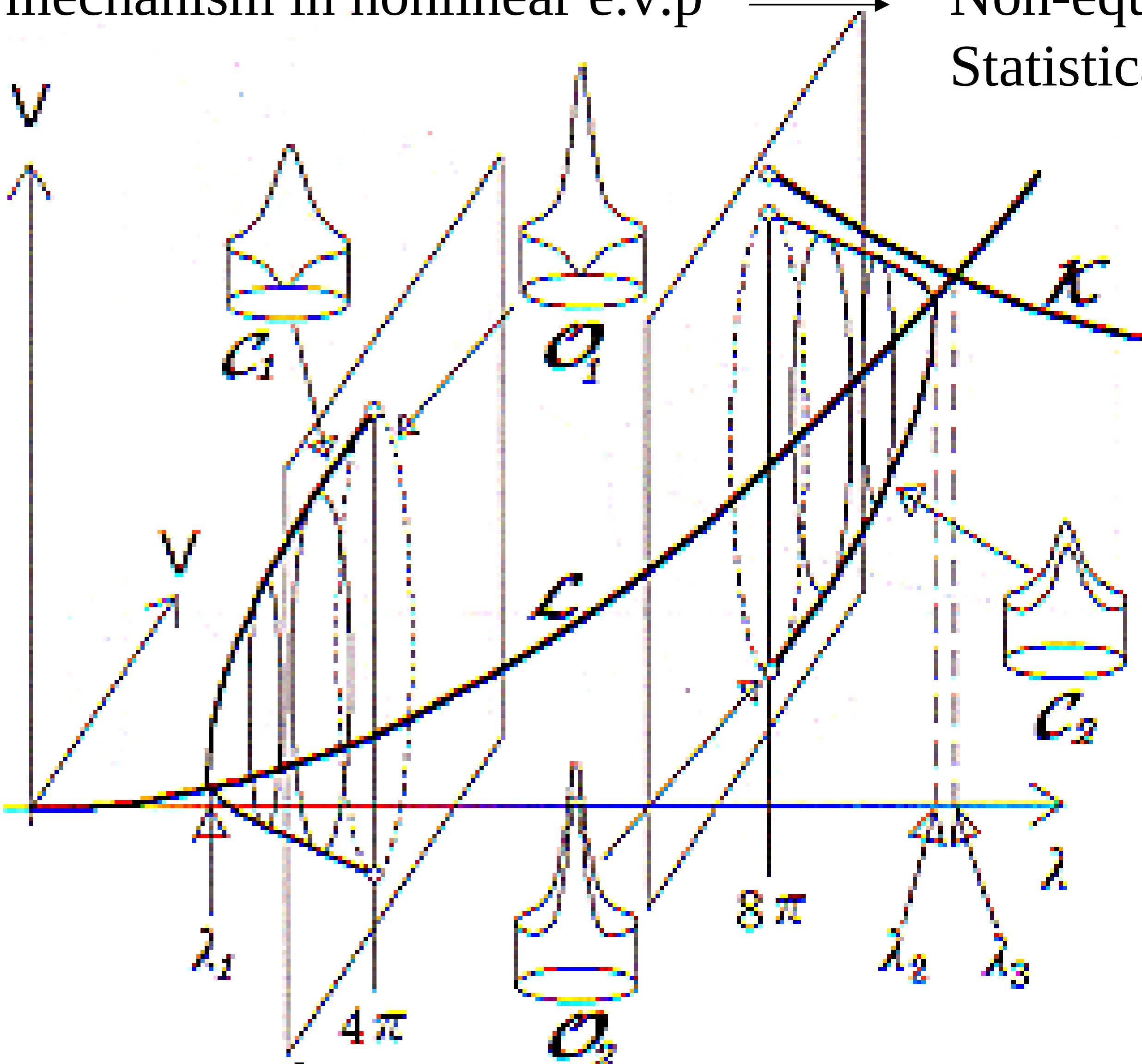
$$-\Delta v + av = \lambda \frac{e^v}{\int_\Omega e^v} \quad \text{in} \quad \Omega$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,$$

where $\lambda = \|u_0\|_1$.

Blowup mechanism in nonlinear e.v.p

Non-equilibrium
Statistical mechanics



Condensate.super-conductivity.
Gauge theory

Nonlinear quantum
mechanics

Theory of dual variation

1. The equilibrium state is formulated in terms of particles and field independently. Each of them has its own variational structure.

2. They accept Legendre transformation, take unfolding by Lagrange function, and are dynamically equivalent.

Full-system of chemotaxis:

$$\left.\begin{array}{l} u_t = \nabla \cdot (\nabla u - u \nabla v) \\ \tau v_t = \Delta v + u \end{array}\right\} \quad \text{in} \quad \Omega \times (0,T)$$

$$\left.\begin{array}{l} \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = 0 \\ v = 0 \end{array}\right\} \quad \text{on} \quad \partial\Omega \times (0,t).$$

has the Lyapunov function

$$\mathcal{W}(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}\|\nabla v\|_2 - \langle v, u \rangle :$$

$$\frac{d}{dt}\mathcal{W}(u,v) + \tau \|v_t\|_2^2 + \int_\Omega u \left|\nabla(\log u - v)\right|^2 = 0.$$

In the equilibrium state, we have

$$v = (-\Delta_D)^{-1}u \quad \text{and} \quad u = \frac{\lambda e^v}{\int_\Omega e^v}$$

and it holds that

$$\log u - (-\Delta_D)^{-1}u = \text{constant} \quad \text{and} \quad \|u\|_1 = \lambda$$

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{with} \quad v|_{\partial\Omega} = 0.$$

Those problems are formulated as the variational problems

$$\delta\mathcal{F}(u) = 0 \quad \text{on} \quad \|u\|_1 = \lambda$$
$$\delta\mathcal{J}_\lambda(v) = 0,$$

where

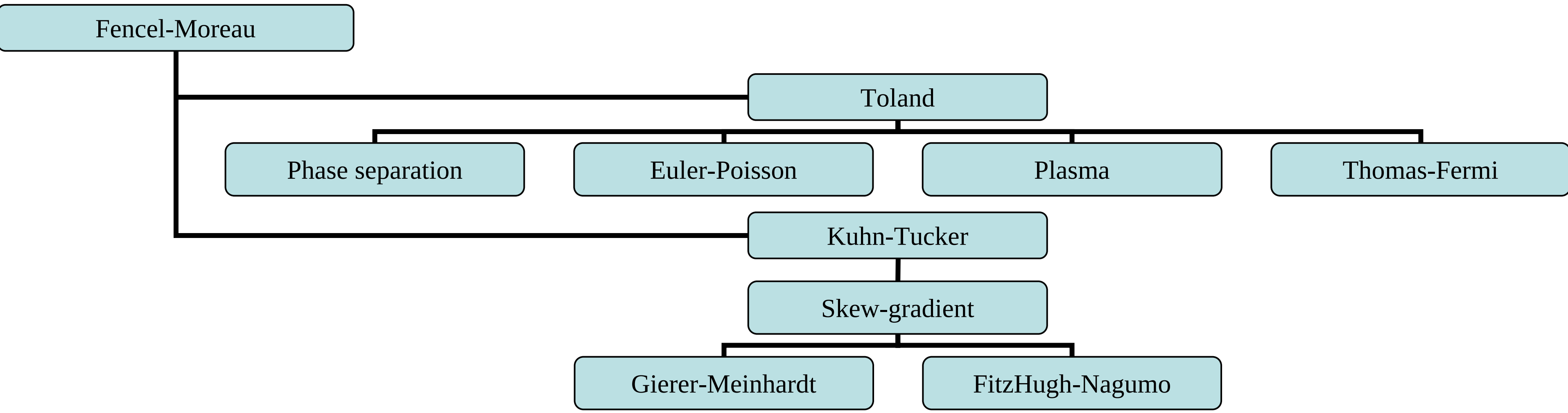
$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\int\int_{\Omega\times\Omega} G(x,x')u\otimes u$$
$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda\log\left(\int_\Omega e^v\right) + \lambda\log\lambda - \lambda.$$

Those variational structures are transformed to each other by the Legendre transformation, and the Lagrange function

$$\mathcal{W} = \mathcal{W}(u, v)$$

takes the role of unfolding; more precisely,

$$\mathcal{W}\big|_{v=(-\Delta_D)^{-1}u} = \mathcal{F}$$
$$\mathcal{W}\big|_{u=\frac{\lambda e^v}{\int_\Omega e^v}} = \mathcal{J}_\lambda.$$



Mean Field Theories and Dual Variation

Self-dual Gauge theory
…super-conductivity,
condensate

Statistical mechanics
…turbulence,
transportation,
self-gravitating particles

Mean field theories
…phase separation,
phase transition,
pattern formation,
self-organization

Mathematical Biology
…chemotaxis,
morphogenesis
angiogenesis,
pulse interaction

Physical principle
…mass conservation
free energy

Mathematical principle
…nonlinear quantum mechanics
quantized blowup mechanism
dual variation

System biology
…backward causality
non-equilibrium
emergence
hyper-circle