

Smoluchowski-Poisson equations in statistical physics and cell biology

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Cell Movement ~ Top Down Approach

Keller-Segel 70

$$u_t = \nabla \cdot (d_1(u, v)\nabla u) - \nabla \cdot (d_2(u, v)\nabla v)$$

$$v_t = d_v \Delta v - k_1 v w + k_{-1} p + f(v)u$$

$$w_t = d_w \Delta w - k_1 v w + (k_{-1} + k_2)p + g(v, w)u$$

$$p_t = d_p \Delta p + k_1 v w - (k_{-1} + k_2)p$$

$u = u(x, t)$ cellular slime molds

$v = v(x, t)$ chemical substances

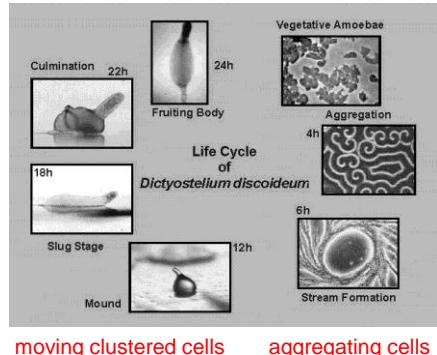
$w = w(x, t)$ enzymes

$p = p(x, t)$ comlices

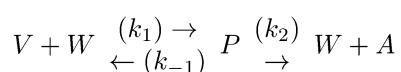
1. transport, gradient

- (a) diffusion u, v, w, p
- (b) chemotaxis $v \rightarrow u$

2. production $u \rightarrow (v, w)$



3. chemical reaction v, w, p



$$v_t = -k_1 v w + k_{-1} p$$

$$w_t = -k_1 v w + (k_{-1} + k_2)p$$

$$p_t = k_1 v w - (k_{-1} + k_2)p$$

Michaelis-Menten reduction (w, p)

1. quasi-static $k_1vw - (k_{-1} + k_2)p = 0$
2. mass conservation $w + p = c$

\Rightarrow

$$u_t = \nabla \cdot (d_1(u, v)\nabla u) - \nabla \cdot (d_2(u, v)\nabla v)$$

$$v_t = d_v\Delta v - k(v)v + f(v)u$$

$$k(v) = \frac{ck_1k_2}{(k_{-1} + k_2) + k_1v}$$



Nanjundiah 73

$$d_1(u, v), k(v), f(v) \text{ constant}$$

$$d_2(u, v) = u\chi'(v)$$

$$u_t = d_u\Delta u - \nabla \cdot (u\nabla\chi(v))$$

$$v_t = d_v\Delta v - b_1v + b_2u$$

Smoluchowski-Poisson system

Childress-Percus 81 Jäger-Luckhaus 92

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$-\Delta v = u - \frac{1}{|\Omega|} \int_{\Omega} u \text{ in } \Omega \times (0, T)$$

$$\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0, T)$$

$$\int_{\Omega} v = 0$$

zero-flux boundary condition

Smoluchowski-ODE system

$$q_t = \nabla \cdot (\nabla q - q\nabla\varphi(v))$$

$$v_t = q \text{ in } \Omega \times (0, T)$$

$$\left. \frac{\partial q}{\partial \nu} - q \frac{\partial \varphi(v)}{\partial \nu} \right|_{\partial\Omega} = 0$$

Cell Movement ~ Bottom Up Approach

continuous particle distribution

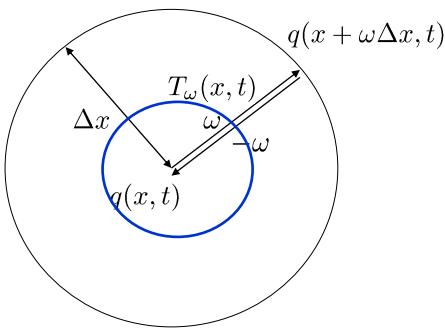
Ichikawa-Rouzi-S. 12

c.f. Othmer-Stevens 97

$q = q(x, t)$ particle density

$T = T_{\omega}(x, t)$ transient probability

τ mean waiting time



Master equation

$$q(x, t + \Delta t) - q(x, t) =$$

$$\int_{S^{N-1}} T_{-\omega}(x + \omega\Delta x, t)q(x + \omega\Delta x, t)d\omega$$

$$- \int_{S^{N-1}} T_{\omega}(x, t)d\omega \cdot q(x, t)$$

Renormalized barrier

$$\frac{1}{|\Delta t|} \int_{S^{N-1}} T_{\omega}(x, t)d\omega = \tau^{-1}$$

\Rightarrow

$$\frac{\tau}{\Delta t} T_{\omega}(x, t) = \frac{T(x + \omega \frac{\Delta x}{2}, t)}{\int_{S^{N-1}} T(x + \omega' \frac{\Delta x}{2}, t)d\omega'}$$

$$T = T(x, t)$$

\Rightarrow

$$\frac{\partial q}{\partial t} = D \nabla \cdot (\nabla q - q \nabla \log T)$$

$$\tau^{-1}(\Delta x)^2 = 2ND \quad \text{Einstein's formula}$$

Statistical Mechanics - Thermodynamics

system	consistency	kinetics	ensemble
isolated	energy	entropy	micro-canonical
closed	temperature	Helmholtz free energy	canonical
open	pressure	Gibbs free energy	grand-canonical

Top Down Approach particle density duality field

Smoluchowski equation \longleftrightarrow **Poisson equation**

$$u_t = \nabla \cdot (\nabla u - u \nabla v) \text{ in } \Omega \times (0, T)$$

$$\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial \Omega \times (0, T)$$

$$-\Delta v = u - \frac{1}{|\Omega|} \int_{\Omega} u, \left. \frac{\partial v}{\partial \nu} \right|_{\partial \Omega} = 0$$

$$\int_{\Omega} v = 0$$

$$\Leftrightarrow v = G * u = \int_{\Omega} G(\cdot, x') u(x') dx'$$

Helmholtz's free energy

$$\begin{aligned} \mathcal{F}(u) &= \int_{\Omega} u(\log u - 1) - \frac{1}{2} \langle G * u, u \rangle \\ \delta \mathcal{F}(u) &= \log u - G * u \end{aligned}$$

$$u_t = \nabla u \cdot \nabla \delta \mathcal{F}(u)$$

$$\left. \frac{\partial}{\partial \nu} \delta \mathcal{F}(u) \right|_{\partial \Omega} = 0 \quad \text{model (B) equation}$$

Statistical Physics ~ Bottom Up Approach

Kinetic Point Vortex Mean Field

Chavanis 08 Langevin equation

$$\begin{aligned} \frac{dx_i}{dt} &= \alpha \nabla_i^{\perp} \hat{H}_N - \mu \alpha^2 \nabla_i \hat{H}_N + \sqrt{2\nu} R_i(t) \\ i &= 1, 2, \dots, N, \mu > 0 \text{ mobility} \end{aligned}$$

$\nu > 0$ viscosity of the particles

\hat{H}_N point vortex Hamiltonian

BBGKY-like hierarchy to $\{P_i\}_{i=1,2,\dots,N}$
factorization (propagation of chaos)

$$P_N(x_1, x_2, \dots, x_N, t) = \prod_{i=1}^N P_1(x_i, t)$$



Kyoto 2011. 8. 28-31

$R_i(t)$ white noise, $\langle R_i(t) \rangle = 0$

$$\langle R_i^{\alpha}(t) R_j^{\beta}(t') \rangle = \delta_{ij} \delta_{\alpha\beta} \delta(t - t')$$

$P_N(x_1, \dots, x_N, t)$ N-pdf

\Rightarrow Fokker-Planck equation

$$\begin{aligned} \frac{\partial P_N}{\partial t} + \alpha \nabla^{\perp} \cdot \hat{H}_N \nabla P_N \\ = \nabla \cdot (\nu \nabla P_N + \mu \alpha^2 P_N \nabla \hat{H}_N) \end{aligned}$$

high-energy limit

$$\mu \hat{\beta} N \alpha^2 = \nu \beta, \alpha N = 1, \omega = P_1$$

\Rightarrow Euler-Smoluchowski-Poisson equation

$$\begin{aligned} \frac{\partial \omega}{\partial t} + \nabla^{\perp} \psi \cdot \nabla \omega &= \nu \nabla \cdot (\nabla \omega + \beta \alpha \omega \nabla \psi) \\ -\Delta \psi &= \omega, \psi|_{\partial \Omega} = 0 \end{aligned}$$

The model

$\Omega \subset \mathbf{R}^2$ bounded domain
 $\partial\Omega$ smooth

other Smoluchowski part

Chavanis 08
Brownian point vortices

$$u_t + \nabla \cdot u \nabla^\perp v = \nabla \cdot (\nabla u - u \nabla v)$$

1. Smoluchowski Part

$$\begin{aligned} u_t &= \nabla \cdot (\nabla u - u \nabla v) \\ \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \Big|_{\partial\Omega} &= 0 \\ u|_{t=0} &= u_0(x) > 0 \end{aligned}$$

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

other Poisson parts

a) Debye system (DD model)

$$\Delta v = u, \quad v|_{\partial\Omega} = 0$$

b) Childress-Percus-Jager-Luckhaus model (chemotaxis)

Sire-Chavanis 02
motion of the mean field of many self-gravitating Brownian particles

$$\begin{aligned} -\Delta v &= u - \frac{1}{|\Omega|} \int u \\ \frac{\partial v}{\partial \nu} \Big|_{\partial\Omega} &= 0, \quad \int_{\Omega} v = 0 \end{aligned}$$

DD model

$$v \leftrightarrow -v$$

$$\begin{aligned} u_t &= \nabla \cdot (\nabla u + u \nabla v), \quad -\Delta v = u \\ \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \Big|_{\partial\Omega} &= v|_{\partial\Omega} = 0 \end{aligned}$$

global-in-time existence with compact orbit
Biler-Hebisch-Nadzieja 94

1. total mass conservation

$$\frac{d}{dt} \|u(t)\|_1 = 0$$

2. free energy decreasing

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u(\log u - 1) + \frac{1}{2} uv \, dx \\ = - \int_{\Omega} u |\nabla(\log u + v)|^2 \leq 0 \end{aligned}$$

3. key estimate

$$\|u \nabla u \cdot \nabla v\|_2 \leq C \|u\|_2 \|\nabla u\|_2 \|\nabla v\|_6$$

chemotaxis system

$$\begin{aligned} u_t &= \nabla \cdot (\nabla u - u \nabla v) \\ -\Delta v &= u - \frac{1}{|\Omega|} \int_{\Omega} u, \quad \int_{\Omega} v = 0 \\ \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \Big|_{\partial\Omega} &= \frac{\partial v}{\partial \nu} \Big|_{\partial\Omega} = 0 \\ u|_{t=0} &= u_0(x) > 0 \end{aligned}$$

\Rightarrow

$$\frac{\partial u}{\partial \nu} \Big|_{\partial\Omega} = \frac{\partial v}{\partial \nu} \Big|_{\partial\Omega} = 0$$

compact Riemann surface without boundary

blowup threshold

a. Biler 98, Gajewski-Zacharias 98,
Nagai-Senba-Yoshida 97

b. Nagai 01, Senba-S. 01b

self-attractive Smoluchowski
– Neumann - Poisson
equation

quantized blowup
mechanism - kinetic level

$$\begin{aligned} u_t &= \nabla \cdot (\nabla u - u \nabla v) \\ -\Delta v &= u - \frac{1}{|\Omega|} \int_{\Omega} u \\ \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \\ \int_{\Omega} v &= 0 \end{aligned}$$

Theorem A1

[formation of collapse]

$$\begin{aligned} u(x, t) dx \rightarrow & \\ \sum_{x_0 \in S} m(x_0) \delta_{x_0}(dx) & \\ + f(x) dx & \end{aligned}$$

Senba-S. 01

Herrero-Velázquez 96

Nanjundiah 73

formation of sub-collapse
type II blowup rate

Theorem A2

[mass quantization]

$$\begin{aligned} m(x_0) &= m_*(x_0) \\ &\equiv \begin{cases} 8\pi, & x_0 \in \Omega \\ 4\pi, & x_0 \in \partial\Omega \end{cases} \\ &\text{c.f. threshold} \end{aligned}$$

Senba-S. 01b

Biler 98

Gajewski-Zacharias 98

Nagai-Senba-Yoshida 97

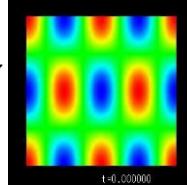
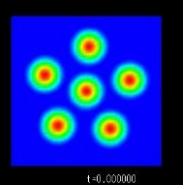
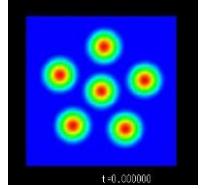
Nagai 95

Jäger-Luckhaus 92

Childress-Percus 81



S. Free Energy and Self-Interacting Particles 05



N. Saito 03

Simulations do not reach the critical stage.

quantized blowup mechanism – spectral level (Boltzmann-Poisson equation)

Theorem B1 [Nagasaki-S. 90a]

$\{(\lambda_k, v_k)\}$ solution sequence

$\lambda_k \rightarrow \lambda_0 \in (0, \infty), \|v_k\|_\infty \rightarrow \infty$

\Rightarrow

$\lambda_0 = 8\pi N, N \in \mathbb{N}$

\exists sub-sequence, $\exists \mathcal{S} \subset \Omega, |\mathcal{S}| = N$

$$\lambda = \|u\|_1 \quad u \xleftrightarrow{\text{duality}} v$$

$\Omega \subset \mathbf{R}^2$ bounded domain $\partial\Omega$ smooth

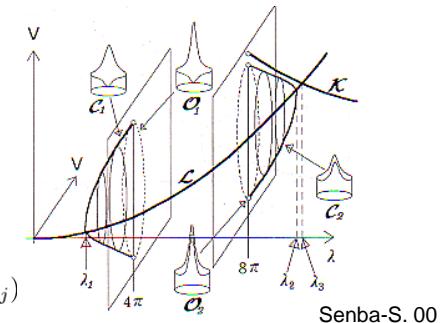
$\lambda > 0$ constant

$$-\Delta v = \frac{\lambda e^v}{\int_{\Omega} e^v} \text{ in } \Omega, v = 0 \text{ on } \partial\Omega$$

$G = G(x, x')$ Green's function

$$R(x) = \left[G(x, x') + \frac{1}{2\pi} \log |x - x'| \right]_{x'=x}$$

Robin function



$v_k \rightarrow v_0$ locally uniform in $\bar{\Omega} \setminus \mathcal{S}$

$$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0) \text{ singular limit}$$

$\mathcal{S} = \{x_1^*, \dots, x_N^*\}$ blowup set

$$\nabla_i H_N|_{(x_1, \dots, x_N) = (x_1^*, \dots, x_N^*)} = 0, 1 \leq i \leq N$$

$$H_N(x_1, \dots, x_N) = \frac{1}{2} \sum_i R(x_i) + \sum_{i < j} G(x_i, x_j)$$

$$\Omega \subset \mathbf{R}^2: \text{open set}, V \in C(\Omega)$$

$$-\Delta v = V(x)e^v, \quad 0 \leq V(x) \leq b \quad \text{in } \Omega$$

$$\int_{\Omega} e^v \leq C$$

Theorem B2 [Li-Shafrir 94]

$\{(V_k, v_k)\}$ solution sequence

$V_k \rightarrow V$ loc. unif. in Ω

\Rightarrow

\exists sub-sequence with the alternatives;

1. $\{v_k\}$: loc. unif. bdd in Ω

2. $\exists \mathcal{S} \subset \Omega, \#\mathcal{S} < +\infty$

$v_k \rightarrow -\infty$ loc. unif. in $\Omega \setminus \mathcal{S}$

$\mathcal{S} = \{x_0 \in \Omega \mid \exists x_k \rightarrow x_0 \text{ s.t.}$

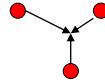
$v_k(x_k) \rightarrow +\infty\}$

$$V_k(x)e^{v_k} dx \rightarrow \sum_{x_0 \in \mathcal{S}} m(x_0) \delta_{x_0}(dx)$$

in $\mathcal{M}(\Omega)$, $m(x_0) \in 8\pi\mathbb{N}$

3. $v_k \rightarrow -\infty$ loc. unif. in Ω

Theorem B2 ~ local theory
(action through a medium)

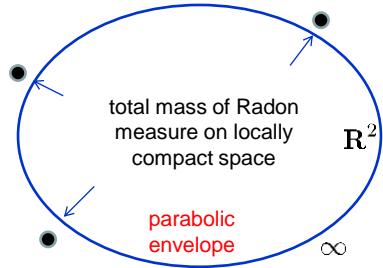


Theorem B1 ~ action at a distance
Boundary condition or Green's function of the Poisson part prohibits the collision of collapses



kinetic level

$$a(\mathbf{R}^2, s) = \lim_{R \uparrow +\infty} a(B_R, s) = m(x_0)$$



Green's function – potential of the action at a distance

$$x' \in \Omega$$

$$-\Delta G(\cdot, x') = \delta_{x'}, \quad G(\cdot, x')|_{\partial\Omega} = 0$$

fundamental solution

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

$$G = G(x, x') \in C^{2+\theta}(\overline{\Omega} \times \overline{\Omega} \setminus D)$$

$$D = \overline{\{(x, x) \mid x \in \Omega\}}, \quad 0 < \theta < 1$$

1. interior regularity

$$G(x, x') = \Gamma(x - x') + K(x, x')$$

$$K \in C^{2+\theta, 1}(\overline{\Omega} \times \Omega) \cap C^{1, 2+\theta}(\Omega \times \overline{\Omega})$$

2. boundary regularity

$$x_0 \in \partial\Omega$$

$$X : \overline{\Omega \cap B(x_0, 2R)} \rightarrow \overline{\mathbf{R}_+^2}, \quad X(x_0) = 0$$

conformal diffeo. into

$$\mathbf{R}_+^2 = \{(X_1, X_2) \in \mathbf{R}^2 \mid X_2 > 0\}$$

$$G(x, x') = E(x, x') + K(x, x')$$

$$E(x, x') = \Gamma(X - X') \square \Gamma(X - X'_*)$$

$$X_* = (X_1, -X_2), \quad X = (X_1, X_2)$$

$$\varphi \in C^2(\overline{\Omega}), \quad \left. \frac{\partial \varphi}{\partial \nu} \right|_{\partial\Omega} = 0 \Rightarrow \rho_\varphi \in L^\infty(\Omega \times \Omega)$$

$$\rho_\varphi(x, x') = \nabla \varphi(x) \cdot \nabla_x G(x, x') + \nabla \varphi(x') \cdot \nabla_{x'} G(x, x')$$

symmetrized localized Green's function
not-continuous

The result

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

$$u_t = \nabla \cdot (\nabla u - u \nabla v) \text{ in } \Omega \times (0, T)$$

$$\frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \Big|_{\partial\Omega} = 0$$

$$u|_{t=0} = u_0(x) > 0$$

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

$$T < +\infty \Rightarrow \lim_{t \uparrow T} \|u(\cdot, t)\|_\infty = +\infty$$

blowup set

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \rightarrow x_0, t_k \uparrow T \\ u(x_k, t_k) \rightarrow +\infty\} \neq \emptyset$$

Theorem 1

$$1. \ \#\mathcal{S} < +\infty, \boxed{\mathcal{S} \subset \Omega}$$

$$2. \ u(x, t) dx \rightharpoonup 8\pi \sum_{x_0 \in \mathcal{S}} \delta_{x_0}(dx)$$

$$+ f(x) dx \text{ in } \mathcal{M}(\overline{\Omega}), t \uparrow T$$

$$0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$$

$$3. \ \forall x_0 \in \mathcal{S}, \forall b > 0$$

$$\lim_{t \uparrow T} (T-t) \|u(\cdot, t)\|_{L^\infty(B(x_0, b(T-t)^{1/2}))} \\ = +\infty$$

formation of sub-collapse → type II blowup rate

Corollary

$$\lambda \equiv \|u_0\|_1 \leq 8\pi \Rightarrow T = +\infty$$

Remark

1. sub-critical mass → compactness of the orbit (Trudinger-Moser + free energy)
2. critical mass → unique continuation theorem

Theorem 2 (critical mass)

$$\|u_0\|_1 = 8\pi, \text{ } \exists \text{ stationary solution}$$

$$\Rightarrow T = +\infty, \limsup_{t \uparrow +\infty} \|u(\cdot, t)\|_\infty = +\infty$$

$$\forall t_k \uparrow +\infty, \lim_k \|u(\cdot, t_k)\|_\infty = +\infty$$

$$\exists \{t'_k\} \subset \{t_k\}$$

$$u(x, t + t'_k) dx \rightharpoonup 8\pi \delta_{x(t)}(dx)$$

$$\text{in } C_*(-\infty, \infty; \mathcal{M}(\overline{\Omega}))$$

$$t \in (-\infty, +\infty) \mapsto x(t) \in \Omega \text{ loc. a.c.}$$

$$\boxed{\frac{dx}{dt} = 4\pi \nabla R(x), \text{ a.e. } t \in \mathbf{R}}$$

Remark

domain close to a disc

→ no stationary solution with critical mass

→ blowup in infinite time

Corollary

$$u = u(|x|, t) \Rightarrow u(x, t+s) dx \rightharpoonup 8\pi \delta_0(dx)$$

$$s \uparrow +\infty \text{ in } C_*(-\infty, +\infty; \mathcal{M}(\overline{\Omega}))$$

Sire-Chavanis 02, Kavallaris-Souplet 09

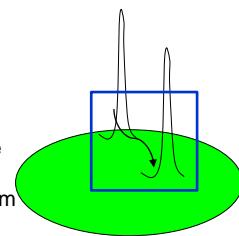
$$\|Q(\cdot, t) - 8\pi\|_1 \approx e^{-\sqrt{2t}}, t \uparrow +\infty$$

$$Q(x, t) = \int_{B(0, |x|)} u(\cdot, t)$$

Conjecture
(non-radial case)

collapse born on the boundary in infinite time

→ shifts to a local maximum of the **Robin function**

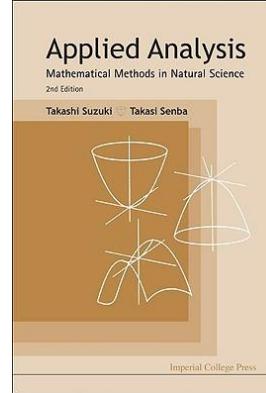


Abstract

We study Smoluchowski-Poisson equations in statistical physics and cell biology, particularly when Dirichlet boundary condition is provided to the Poisson part. The method of weak scaling limit reveals peculiar profiles of the solution, formation of collapses, mass quantization, type II blowup, simultaneous blowup, and mass separation. Exclusion of the boundary blowup is particularly emphasized.

Contents (10)

1. monotonicity formula (1)
2. formation of collapse (1)
3. blowup criterion (1)
- 4. weak solution (2)**
5. scaling limit (2)
6. parabolic envelope (1)
7. boundary blowup exclusion (2)



S. and T. Senba, Applied Analysis 11

1. monotonicity formula

1) total mass conservation

$$u_t = \nabla \cdot (\nabla u - u \nabla v) \\ \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \Big|_{\partial \Omega} = 0 \quad \text{Smoluchowski}$$

$$u \geq 0 \Rightarrow \|u(\cdot, t)\|_1 = \|u_0\|_1 \equiv \lambda$$

2) symmetrization

$$v = \int_{\Omega} G(\cdot, x') u(x') dx' \\ \Leftrightarrow \quad \text{Poisson} \\ -\Delta v = u, \quad v|_{\partial \Omega} = 0 \quad \text{- action at a distance}$$

$G(x, x') = G(x', x)$ action-reaction law

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \varphi(x) u(x, t) dx \\ &= - \int_{\Omega} \nabla \varphi \cdot (\nabla u - u \nabla v) dx \\ &= \int_{\Omega} \Delta \varphi(x) \cdot u(x, t) dx \\ &+ \frac{1}{2} \int_{\Omega \times \Omega} \rho_{\varphi}(x, x') u(x, t) u(x', t) dx dx' \end{aligned}$$

weak form

$$\begin{aligned} \rho_{\varphi}(x, x') &= \nabla \varphi(x) \cdot \nabla_x G(x, x') \\ &+ \nabla \varphi(x') \cdot \nabla_{x'} G(x, x') \end{aligned}$$

$\|\rho_{\varphi}\|_{\infty} \leq C \|\nabla \varphi\|_{C^1} \Rightarrow$ monotonicity formula

$$\left| \frac{d}{dt} \int_{\Omega} u \varphi \right| \leq C(\lambda + \lambda^2) \|\nabla \varphi\|_{C^1}$$

1/10

$\varphi \in C^2(\bar{\Omega}), \quad \frac{\partial \varphi}{\partial \nu} \Big|_{\partial \Omega} = 0 \Leftrightarrow (\text{def}) \quad \varphi \in Y$

2. formation of collapse

1) weak continuation

$\varphi \in Y \hookrightarrow C(\bar{\Omega})$ dense

$$\left| \frac{d}{dt} \int_{\Omega} u(\cdot, t) \varphi \right| \leq C_{\lambda} \|\nabla \varphi\|_{C^1}$$

$$\lambda = \|u(\cdot, t)\|_1 \Rightarrow$$

$$0 \leq \mu(dx, t) \in C_*([0, T], \mathcal{M}(\bar{\Omega}))$$

$$u(x, t) dx = \mu(dx, t), 0 \leq t < T$$

2) ε -regularity

$$\lim_{R \downarrow 0} \limsup_{t \uparrow T} \|u(\cdot, t)\|_{L^1(\Omega \cap B(x_0, R))} < \exists \varepsilon_0$$

$$\Rightarrow x_0 \notin \mathcal{S}$$

1)+2) \Rightarrow (parabolic-elliptic regularity)

$$\begin{aligned} \mu(\cdot, T) &= \sum_{x_0 \in \mathcal{S}} m(x_0) \delta_{x_0}(dx) + f(x) dx \\ m(x_0) &\geq \varepsilon_0, 0 \leq f \in L^1(\Omega), \#\mathcal{S} < +\infty \end{aligned}$$

proof of ε -regularity - Dirichlet case

localization of the global-in-time existence criterion of Jager-Luckhaus type

$$x_0 \in \bar{\Omega}, 0 < R \ll 1$$

Lemma 1

$$\limsup_{t \uparrow T} \|u(\cdot, t)\|_{L^1(\Omega \cap B(x_0, R))} < \varepsilon_0$$

$$\Rightarrow \limsup_{t \uparrow T} \int_{\Omega \cap B(x_0, R/2)} u(\log u - 1) < +\infty$$

Lemma 2

$$\limsup_{t \uparrow T} \int_{\Omega \cap B(x_0, R)} u(\log u - 1) < +\infty$$

$$\Rightarrow x_0 \notin \mathcal{S}$$

Gagliardo-Nirenberg inequality
Moser's iteration scheme

$$v \geq 0 \quad \|v\|_{W^{1,q}} \leq C(q), 1 \leq q < 2$$

2/10

3. blowup criterion from the weak form

Theorem (Senba-S. 01b)

$$x_0 \in \Omega, 0 < R \ll 1$$

$$\|u_0\|_{L^1(B(x_0, R))} > 8\pi$$

$$\||x - x_0|^2 u_0\|_{L^1(B(x_0, 4R))} \ll 1$$

$$\Rightarrow T < +\infty$$

cut-off function

$$x_0 \in \bar{\Omega}, 0 < R \ll 1, \varphi = \varphi_{x_0, R} \in Y$$

$$0 \leq \varphi \leq 1, \varphi = \begin{cases} 1, & x \in B(x_0, R/2) \\ 0, & x \in \mathbf{R}^2 \setminus B(x_0, R) \end{cases}$$

$$|\nabla \varphi| \leq CR^{-1}\varphi^{5/6}, |\nabla^2 \varphi| \leq CR^{-2}\varphi^{2/3}$$

interior regularity

$$G(x, x') = \frac{1}{2\pi} \log \frac{1}{|x - x'|} + K(x, x')$$

$$K \in C^{2+\theta, 1}(\bar{\Omega} \times \Omega) \cap C^{1, 2+\theta}(\Omega \times \bar{\Omega})$$

$$x_0 \in \Omega, 0 < R \ll 1, \varphi = |x - x_0|^2 \varphi_{x_0, R}$$

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u \varphi &= \int_{\Omega} \Delta \varphi \cdot u(\cdot, t) \\ &+ \frac{1}{2} \int_{\Omega \times \Omega} \rho_{\varphi}(x, x') u \otimes u \end{aligned}$$

$$\begin{aligned} \frac{dI_R}{dt} &\leq J_R(0) + a(R^{-1}t^{1/2}) \\ &+ CR^{-1}I_{2R}(t)^{1/2} \end{aligned}$$

$$a(s) = C(s^2 + s)$$

$$I_R = \int_{\Omega} |x - x_0|^2 u(\cdot, t) \varphi_{x_0, R}$$

$$J_R = 4M_R - \frac{M_R^2}{2\pi} + CR^{-1}I_{2R}$$

$$M_R = \int_{\Omega} u(\cdot, t) \varphi_{x_0, R}$$

$$\begin{aligned} &+ \text{monotonicity formula} \quad M_R(0) > 8\pi, I_{2R}(0) \ll 1 \\ &\Rightarrow I_R(\exists t) < 0 \end{aligned}$$

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4. weak solution (Senba-S. 02a)

$$0 \leq \mu(dx, t) \in C_*([0, T], \mathcal{M}(\bar{\Omega}))$$

$\mathcal{M}(\bar{\Omega}) = C(\bar{\Omega})'$ weak solution to

$$\begin{aligned} u_t &= \nabla \cdot (\nabla u - u \nabla v), \quad -\Delta v = u \\ \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \Big|_{\partial \Omega} &= v|_{\partial \Omega} = 0 \\ \Leftrightarrow & \text{(def)} \end{aligned}$$

$$(1) \forall \varphi \in Y$$

$$t \in [0, T] \mapsto \langle \varphi, \mu(dx, t) \rangle \text{ a.c.}$$

$$\begin{aligned} \frac{d}{dt} \langle \varphi, \mu(dx, t) \rangle &= \langle \Delta \varphi, \mu(dx, t) \rangle \\ + \frac{1}{2} \langle \rho_\varphi, \nu(t) \rangle_{\mathcal{X}, \mathcal{X}'} &\text{ a.e. } t \end{aligned}$$

$$Y = \left\{ \varphi \in C^2(\bar{\Omega}) \mid \frac{\partial \varphi}{\partial \nu} \Big|_{\partial \Omega} = 0 \right\}$$

$$\rho_\varphi(x, x') = \nabla \varphi(x) \cdot \nabla_x G(x, x')$$

$$+ \nabla \varphi(x') \cdot \nabla_{x'} G(x, x')$$

$$0 \leq \exists \nu(\cdot, t) \in L^\infty(0, T; \mathcal{X}')$$

multiplicate operator

$$\begin{aligned} \mathcal{X} &= [\{\rho_\varphi + \psi \mid \varphi \in Y, \psi \in C(\bar{\Omega} \times \bar{\Omega})\}] \\ &\hookrightarrow L^\infty(\Omega \times \Omega) \end{aligned}$$

$$(2) \text{ (linking)}$$

$$\nu(\cdot, t)|_{C(\bar{\Omega} \times \bar{\Omega})} = \mu(dx, t) \otimes \mu(dx', t)$$

a.e. t

Remark (positivity)

$$0 \leq \nu \in \mathcal{X}' \Leftrightarrow \text{(def)}$$

$$\forall f, g \in \mathcal{X}, |f| \leq g$$

$$|\langle f, \nu \rangle_{\mathcal{X}, \mathcal{X}'}| \leq \langle g, \nu \rangle_{\mathcal{X}, \mathcal{X}'}$$

c.f. Poupaud 02, Dolbeaut-Schmeiser 09
Luckhaus-Sugiyama-Velazquez 12 4/10

$$0 \leq \mu(dx, t) \in C_*([0, T], \mathcal{M}(\bar{\Omega}))$$

weak solution

1 (total mass conservation)

$$\|\mu(\cdot, t)\|_{\mathcal{M}} = \langle 1, \mu(dx, t) \rangle = \lambda$$

2 (monotonicity formula) $\forall \varphi \in Y$

$$\left| \frac{d}{dt} \langle \varphi, \mu(dx, t) \rangle \right| \leq C(\lambda + \|\nu\|_{\infty, \mathcal{X}'}) \|\nabla \varphi\|_{C^1}$$

positivity + linking of the multi-plicate operator

$$\varphi = |x - x_0|^2 \varphi_{x_0, R}$$

$$\frac{dI_R}{dt} \leq J_R(0) + a(R^{-1}t^{1/2})$$

$$+ CR^{-1} I_{2R}(t)^{1/2}, \quad R \downarrow 0$$

Proposition 1

$$\exists x_0 \in \Omega, \mu(\{x_0\}, 0) > 8\pi$$

$$\langle |x - x_0|^2 \varphi_{x_0, R}, \mu(dx, 0) \rangle = o(R^2), \quad R \downarrow 0$$

$$\Rightarrow T = 0$$

$$0 \leq \nu(\cdot, t) \in L^\infty(0, T; \mathcal{X}')$$

multiplicate operator

Proposition 2

$$0 \leq \mu_k(dx, t) \in C_*([0, T], \mathcal{M}(\bar{\Omega}))$$

weak solutions

$$\sup_k \left[\|\mu_k(\cdot, 0)\|_{\mathcal{M}} + \boxed{\sup_{t \in [0, T]} \|\nu_k(\cdot, t)\|_{\mathcal{X}'}} \right]$$

$< +\infty \Rightarrow \exists$ sub-sequence

weakly converging to a weak solution

Remark 2

1. $\mathcal{X} \subset L^\infty(\Omega \times \Omega)$ separable

2. $u = u(x, t) \geq 0$ classical solution

\Rightarrow

$$\mu(dx, t) = u(x, t) dx$$

$$\nu(\cdot, t) = u(\cdot, t) \otimes u(\cdot, t) \in L^1(\Omega \times \Omega)$$

$$\|\nu(\cdot, t)\|_{\mathcal{X}'} = \lambda^2, \quad \lambda = \|u_0\|_1$$

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5. scaling limit ~ scaling invariance

$$u_\mu(x, t) = \mu^2 u(\mu x, \mu^2 t), \mu > 0$$

1) backward self-similar transformation

$$\begin{aligned} y &= (x - x_0)/(T - t)^{1/2} & x_0 \in \mathcal{S} \\ s &= -\log(T - t) \\ z(y, s) &= (T - t)u(x, t) \end{aligned}$$

$$\begin{aligned} y &\in (T - t)^{-1/2}(\Omega - \{x_0\}) = \Omega_s \\ -\log T \leq s < +\infty, \|z(\cdot, s)\|_1 &= \lambda \end{aligned}$$

$$\begin{aligned} z_s &= \nabla \cdot (\nabla z - z \nabla(w + |y|^2/4)) \\ \frac{\partial z}{\partial \nu} - z \frac{\partial}{\partial \nu}(w + |y|^2/4) \Big|_{\partial \Omega_s} &= 0 \end{aligned}$$

$$\begin{aligned} w(\cdot, s) &= \int_{\Omega_s} G_s(\cdot, y') z(y', s) dy' \\ G_s(y, y') &= G(x, x') \end{aligned}$$

2) weak form $x_0 \in \Omega$

$$\varphi \in C_0^2(\mathbf{R}^2), s \gg 1$$

$$\begin{aligned} \frac{d}{ds} \int_{\mathcal{O}_s} z \varphi &= \int_{\mathcal{O}_s} (\partial_s \varphi + y \cdot \nabla \varphi + \Delta \varphi) z \\ + \frac{1}{2} \int_{\mathcal{O}_s \times \mathcal{O}_s} \rho_\varphi^s(y, y') z \otimes z \end{aligned}$$

$$\mathcal{O}_s = \Omega_s \times \{s\}$$

$$\begin{aligned} \rho_\varphi^s(y, y') &= [\nabla \varphi(y) \cdot \nabla_y G_s(y, y')] \\ &+ \nabla \varphi(y') \cdot \nabla_{y'} G_s(y, y') \end{aligned}$$

$$G(x, x') = \Gamma(x - x') + K(x, x')$$

$$(x, x') \in (\bar{\Omega} \times \Omega) \cup (\Omega \times \bar{\Omega})$$

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

$$\begin{aligned} G_s(y, y') &= [\Gamma(y - y')] - \frac{s}{4\pi} \\ &+ K(e^{-s}y + x_0, e^{-s}y' + x_0) \end{aligned}$$

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3) diagonal argument

$$\forall s_k \uparrow +\infty, \exists \{s'_k\} \subset \{s_k\}$$

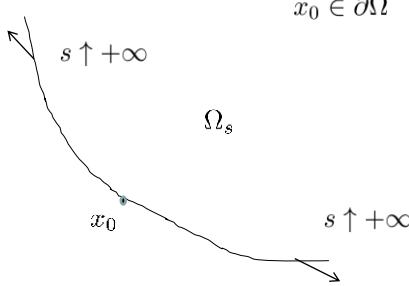
$$z(y, s + s'_k) dy \rightharpoonup \exists \zeta(dy, s)$$

in $C_*(-\infty, +\infty; \mathcal{M}_0(\mathbf{R}^2))$

$$\mathcal{M}_0(\mathbf{R}^2) = C_0(\mathbf{R}^2)'$$

$$C_0(\mathbf{R}^2)$$

$$= \{f \in C(\mathbf{R}^2 \cup \{\infty\}) \mid f(\infty) = 0\}$$



$$0 \leq \zeta(dy, s) \text{ finite measure}$$

$$\zeta(\mathbf{R}^2, s) \leq \lambda \equiv \|u_0\|_1, \forall s$$

$$\text{supp } \zeta(dy, s) \subset \overline{\mathbf{R}_+^2} \text{ (if } x_0 \in \partial\Omega\text{)}$$

weak solution to

$$\begin{aligned} \zeta_s &= \nabla \cdot (\nabla \zeta - \zeta \nabla(F * \zeta + |y|^2/4)) \\ \text{in } \mathbf{R}^2 \times (-\infty, +\infty) \end{aligned}$$

$$(F * \zeta)(\cdot, s) = \langle F(\cdot, y'), \zeta(dy', s) \rangle$$

$$F(y, y') = \begin{cases} \Gamma(y - y'), & x_0 \in \Omega \\ E(y, y'), & x_0 \in \partial\Omega \end{cases}$$

$$E(y, y') = \Gamma(y - y') - \Gamma(y - y'_*)$$

test functions

$$\varphi \in W \Leftrightarrow (\text{def}) \varphi \in C_0^2(\mathbf{R}^2)$$

$$\left. \frac{\partial \varphi}{\partial \nu} \right|_{\partial \mathbf{R}_+^2} = 0 \text{ (if } x_0 \in \partial\Omega\text{)}$$

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$$\begin{aligned}
0 &\leq \zeta(dy, s) \in C_*(-\infty, +\infty; \mathcal{M}_0(\mathbf{R}^2)) \\
\zeta(\mathbf{R}^2, s) &\leq \lambda \equiv \|u_0\|_1, \forall s \\
\text{supp } \zeta(dy, s) &\subset \overline{\mathbf{R}_+^2} \text{ (if } x_0 \in \partial\Omega)
\end{aligned}$$

$$\begin{aligned}
(1) \quad \forall \varphi \in W \\
s \in (-\infty, \infty) \mapsto \langle \varphi, \zeta(dy, s) \rangle \text{ loc. a.c.}
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{ds} \langle \varphi, \zeta(dy, s) \rangle \\
&= \langle \Delta \varphi + \frac{y}{2} \cdot \nabla \varphi, \zeta(dy, s) \rangle \\
&+ \frac{1}{2} \langle \rho_\varphi^0, \mathcal{K}(s) \rangle_{\mathcal{E}, \mathcal{E}'} \text{ a.e. } s
\end{aligned}$$

$$\begin{aligned}
0 \leq \exists \mathcal{K} = \mathcal{K}(\cdot, s) \in \mathcal{E}' \\
\|\mathcal{K}(\cdot, s)\|_{\mathcal{E}'} \leq \lambda^2 \text{ a.e. } s
\end{aligned}$$

multi-plicate operator

$$\begin{aligned}
\mathcal{E} &= [\{\rho_\varphi^0 + \psi \mid \varphi \in W, \psi \in Z\}] \\
&\hookrightarrow L^\infty(\mathbf{R}^2 \times \mathbf{R}^2)
\end{aligned}$$

$$\begin{aligned}
\rho_\varphi^0(y, y') &= \nabla \varphi(y) \cdot \nabla_y F(y, y') \\
&+ \nabla \varphi(y') \cdot \nabla_{y'} F(y, y') \\
F(y, y') &= \begin{cases} \Gamma(y - y'), & x_0 \in \Omega \\ \Gamma(y - y') - \Gamma(y - y_*), & x_0 \in \partial\Omega \end{cases} \\
Z &= C_0(\mathbf{R}^2 \times \mathbf{R}^2) \oplus [(C_0(\mathbf{R}^2) \oplus \mathbf{R}) \otimes \mathbf{R}] \\
&\oplus [\mathbf{R} \otimes (C_0(\mathbf{R}^2) \oplus \mathbf{R})] \\
C_0(\mathbf{R}^2 \times \mathbf{R}^2) &\dots \text{continuous functions on} \\
&(\mathbf{R}^2 \cup \{\infty\}) \times (\mathbf{R}^2 \cup \{\infty\}) \\
&\text{vanishing at } [(\mathbf{R}^2 \cup \{\infty\}) \times \{\infty\}] \\
&\cup [\{\infty\} \times (\mathbf{R}^2 \cup \{\infty\})]
\end{aligned}$$

$$\begin{aligned}
(2) \text{ (linking)} \\
\mathcal{K}(\cdot, s)|_Z &= \zeta(dy, s) \otimes \zeta(dy', s) \text{ a.e. } s
\end{aligned}$$

6. parabolic envelope

$$x_0 \in \mathcal{S}$$

$$\begin{aligned}
1) \text{ total mass, } 0 < R \leq 1 \\
\left| \frac{d}{dt} \int_{\Omega} u(\cdot, t) \varphi_{x_0, R} \right| &\leq C_\lambda R^{-2}, \quad \int_t^T dt \\
|\langle \varphi_{x_0, R}, u(\cdot, t) dx \rangle - \langle \varphi_{x_0, R}, \mu(dx, T) \rangle| &\leq C_\lambda (T-t)/R^2
\end{aligned}$$

$$\begin{aligned}
s \in \mathbf{R}, b > 0 \text{ fixed, } k \gg 1 \\
s'_k + s = -\log(T-t), \quad R = b(T-t)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&|\langle \varphi_{0,b}, z(\cdot, s+s'_k) dy \rangle \\
&- \langle \varphi_{x_0, b e^{-(s+s'_k)/2}}, \mu(dx, T) \rangle| \leq C_\lambda/b^2
\end{aligned}$$

$$k \rightarrow \infty$$

$$\mu(\cdot, T) = \sum_{x_0 \in \mathcal{S}} m(x_0) \delta_{x_0}(dx) + f(x) dx$$

$$\begin{aligned}
|\langle \varphi_{0,b}, \zeta(dy, s) \rangle - m(x_0)| &\leq C_\lambda/b^2, \quad b \uparrow +\infty \\
m(x_0) &= \zeta(\mathbf{R}^2, s), \quad -\infty < s < +\infty
\end{aligned}$$

2) second moment:

$$\begin{aligned}
\left| \frac{d}{dt} \int_{\Omega} |X|^2 \varphi_{x_0, R} u \right| &\leq C \\
X &= \begin{cases} x - x_0, & x_0 \in \Omega \\ X_{x_0}(x), & x_0 \in \partial\Omega \end{cases}
\end{aligned}$$

\Rightarrow

$$\langle |y|^2, \zeta(dy, s) \rangle \leq C, \quad -\infty < s < +\infty$$

Parabolic envelope .. Infinitely wide parabolic region

pre-scaled collapse mass = total mass of the weak scaling limit

uniformly bounded total second moment of the limit measure

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7. boundary blowup exclusion

$$x_0 \in \partial\Omega, \text{ supp } \zeta(dy, s) \subset \overline{\mathbf{R}_+^2}$$

$$0 \leq \zeta = \zeta(dy, s)$$

$$\in C_*(-\infty, +\infty; \mathcal{M}_0(\mathbf{R}^2))$$

$$\zeta(\mathbf{R}^2, s) = m(x_0) \text{ parabolic envelope I}$$

$$(1) \varphi \in C_0^2(\mathbf{R}^2), \frac{\partial \varphi}{\partial \nu} \Big|_{\partial \mathbf{R}_+^2} = 0$$

$$[\langle \varphi, \zeta(dy, s) \rangle]_{s=s_1}^{s=s_2}$$

$$= \int_{s_1}^{s_2} \left[\langle \Delta \varphi + \frac{y}{2} \cdot \nabla \varphi, \zeta(dy, s) \rangle \right. \\ \left. + \frac{1}{2} \langle \rho_\varphi, \mathcal{K}(s) \rangle_{\mathcal{E}, \mathcal{E}'} \right] ds$$

$$0 \leq \mathcal{K}(\cdot, s), \|\mathcal{K}(\cdot, s)\|_{\mathcal{E}'} \leq \lambda^2$$

$$\rho_\varphi^0(y, y') = \nabla \varphi(y) \cdot \nabla_y F(y, y') \\ + \nabla \varphi(y') \cdot \nabla_{y'} F(y, y')$$

$$F(y, y') = \Gamma(y - y') - \Gamma(y - y'_*)$$

$$\Gamma(y) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

(2) (linking)

$$\mathcal{K}(\cdot, s)|_Z = \zeta(dy, s) \otimes \zeta(dy', s) \text{ a.e.s}$$

1) measure part

$$\varphi = \varphi_R = |y|^2 \psi_R, \psi_R(y) = \psi(y/R)$$

$$\psi = \varphi_{0,2}$$

$$\Delta \varphi_R = 4\psi_R + 4 \frac{y}{R} \cdot \nabla \psi(\frac{y}{R}) + \frac{|y|^2}{R^2} \Delta \psi(\frac{y}{R})$$

$$y \cdot \nabla \varphi_R = 2|y|^2 \psi_R + |y|^2 \frac{y}{R} \cdot \nabla \psi(\frac{y}{R})$$

$$I(s) = \langle |y|^2, \zeta(dy, s) \rangle \leq C$$

parabolic envelope II

\Rightarrow (dominated convergence theorem)

$$\lim_{R \uparrow +\infty} \int_{s_1}^{s_2} \langle \Delta \varphi_R + \frac{y}{2} \cdot \nabla \varphi_R, \zeta(dy, s) \rangle$$

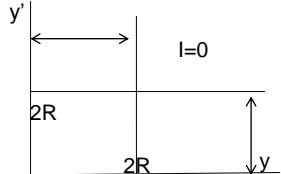
$$= 4(s_2 - s_1)m(x_0) + \int_{s_1}^{s_2} I(s) ds$$

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$$2) \text{ multi-plicate part} \quad \rho_\varphi^0(y, y') = I + II + III$$

$$I = \psi_R(y) \nabla |y|^2 \cdot \nabla_y F(y, y') + \psi_R(y') \nabla |y'|^2 \cdot \nabla_{y'} F(y, y')$$

$$|I| \leq C \left(\varphi_{0,4R}(y) \frac{|y|}{R} + \varphi_{0,4R}(y') \frac{|y'|}{R} \right), y, y' \in L$$



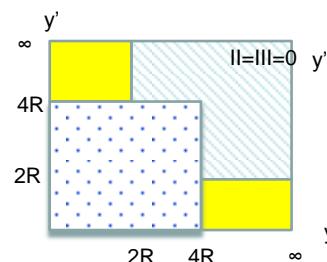
$$II = (|y|^2 - |y'|^2) \nabla \psi_R(y) \cdot \nabla_y F(y, y')$$

$$III = |y'|^2 (\nabla \psi_R(y) - \psi_R(y')) \cdot \nabla_{y'} F(y, y')$$

$$|II| + |III| \leq$$

$$C (\varphi_{0,8R}(y)(1 + |y|) + \varphi_{0,8R}(y')(1 + |y'|))$$

$$\cdot \left\{ \frac{|y|}{R} + \frac{|y|^2}{R^2} + \frac{|y'|}{R} + \frac{|y'|^2}{R^2} \right\}, \quad y, y' \in L$$



$$\langle 1 + |y|^2, \zeta(dy, s) \rangle \leq C$$

positivity \rightarrow linking \rightarrow dominated convergence theorem

$$\lim_{R \uparrow +\infty} \int_{s_1}^{s_2} \langle \rho_\varphi^0, \mathcal{K}(\cdot, s) \rangle_{\mathcal{E}, \mathcal{E}'} ds = 0$$

$$\frac{dI}{ds} = 4m(x_0) + I(s) \text{ a.e. } s$$

\Rightarrow

$$\lim_{R \uparrow +\infty} I(s) = +\infty, \text{ a contradiction}$$

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1. Senba-S. 01	weak formulation monotonicity formula	formation of collapse
2. Senba-S. 02a	weak solution	weak solution generation instant blowup for over mass concentrated initial data
3. Kurokiba-Ogawa 03	scaling invariance	non-existence of over mass entire solution without concentration
4. S. 05	backward self-similar transformation scaling limit parabolic envelope (1) scaling invariance of the scaling limit a local second moment	collapse mass quantization
5. Senba 07 Naito-S. 08	parabolic envelope (2)	type II blowup rate formation of sub-collapse
6. S. 08	scaling back	limit equation simplification
7. Senba-S. 11	translation limit	concentration-cancelation simplification
8. Espejo-Stevens-S. 12	quantization without blowup threshold	simultaneous blowup mass separation for systems

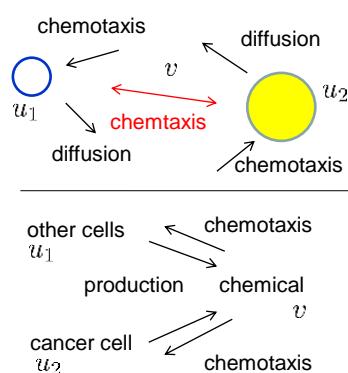
classical analysis

1. Nagai-Senba-Yoshida 97, Biler 98, Gajewski-Zacharias 98 global-in-time existence
2. Biler-Hilhorst-Nadieja 94, Nagai 95, Nagai 01, Senba-S. 02b blowup in finite time

Multi-Component Systems

DD model (hetero-separative, homo-aggregative type)
Kurokiba-Ogawa 03
Espejo-Stevens-Velazquez 10

$$\begin{aligned} u_{1t} &= d\Delta u_1 - \chi \nabla \cdot u_1 \nabla v \\ u_{2t} &= d\Delta u_2 + \chi \nabla \cdot u_2 \nabla v \\ -\Delta v &= u_1 - u_2 \text{ in } \mathbf{R}^2 \times (0, T) \\ u_i|_{t=0} &= u_{i0}(x) \geq 0, i = 1, 2 \end{aligned}$$

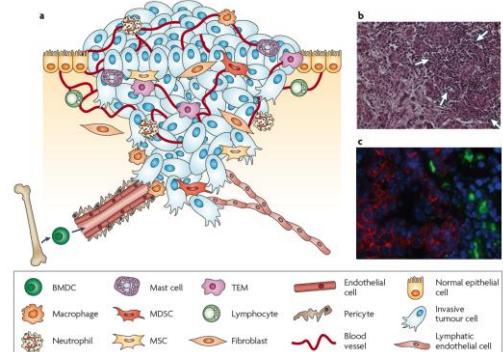


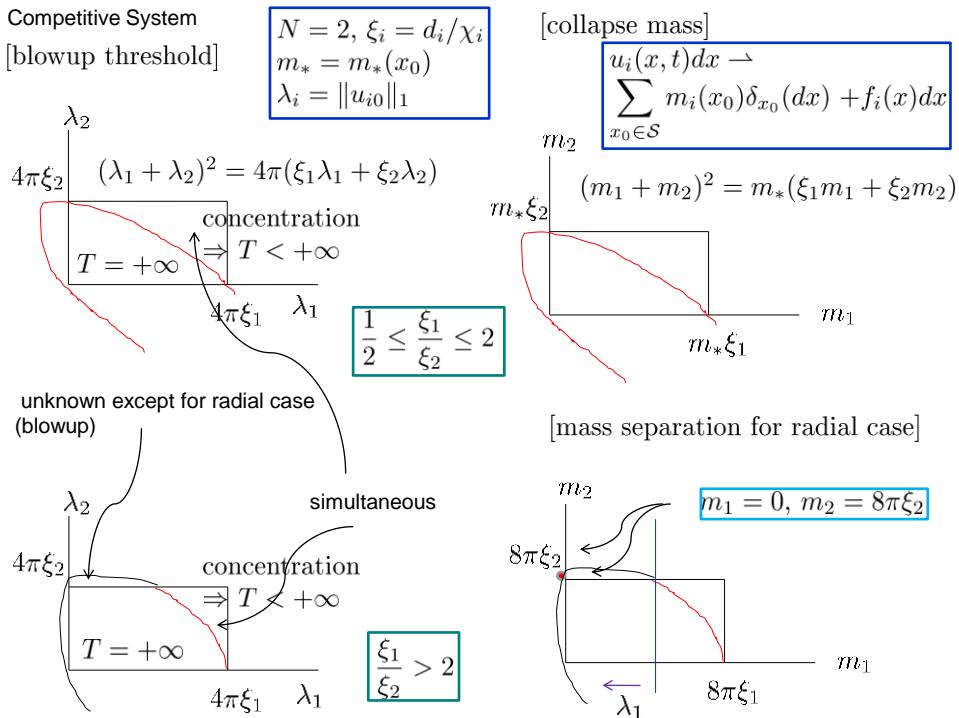
competitive system of chemotaxis (hetero-homo-aggregative)

Espejo-Stevens-Velazquez 09 simultaneous blowup
Espejo-Stevens-S. 12 collapse mass separation

$$\begin{aligned} u_{it} &= d_i \Delta u_i - \chi_i \nabla \cdot u_i \nabla v & -\Delta v &= u - \frac{1}{|\Omega|} \int_{\Omega} u \\ d_i \frac{\partial u_i}{\partial \nu} - \chi_i u_i \frac{\partial v}{\partial \nu} \Big|_{\partial \Omega} &= 0 & \frac{\partial v}{\partial \nu} \Big|_{\partial \Omega} &= 0 \\ u_i|_{t=0} &= u_{i0}(x) \geq 0 & i &= 1, 2, \dots, N \\ \int_{\Omega} v = 0, u &= \sum_{i=1}^N u_i \end{aligned}$$

tumor-associated micro-environment





cross chemotaxis system

$$\partial_t u_1 = d_1 \Delta u_1 - \chi_1 \nabla \cdot u_1 \nabla v_2$$

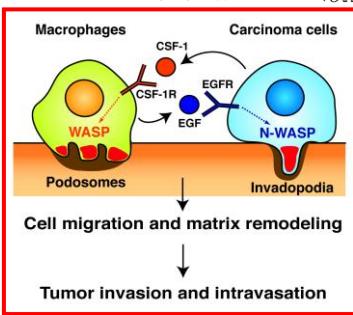
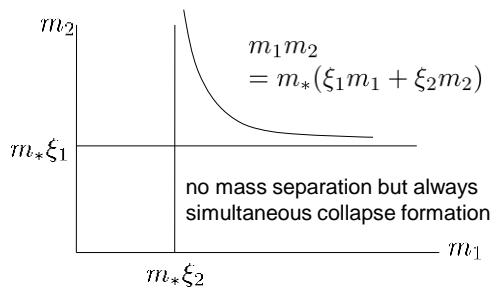
$$\partial_t u_2 = d_2 \Delta u_2 - \chi_2 \nabla \cdot u_2 \nabla v_1$$

$$d_1 \frac{\partial u_1}{\partial \nu} - \chi_1 u_1 \frac{\partial v_2}{\partial \nu} \Big|_{\partial \Omega} = 0$$

$$d_2 \frac{\partial u_2}{\partial \nu} - \chi_2 u_2 \frac{\partial v_1}{\partial \nu} \Big|_{\partial \Omega} = 0$$

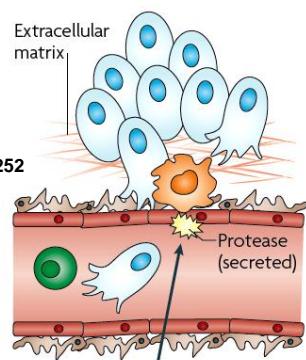
$$u_i|_{t=0} = u_{i0}(x) \geq 0, \int_{\Omega} v_i \, dx = 0$$

$$-\Delta v_i = u_i - \frac{1}{|\Omega|} \int_{\Omega} v_i, \frac{\partial v_i}{\partial \nu} \Big|_{\partial \Omega} = 0, i = 1, 2$$



J. Joyce, and J. Pollard.
Nat Rev Cancer 9: 239-252
(2009)

H. Yamaguchi et al. *Eur J Cell Biol* 85: 213-218 (2006)



Protease degradation and tumour cell intravasation

Lotka-Volterra system

$$\begin{aligned}\frac{du_1}{dt} &= (u_2 - u_3)u_1 \\ \frac{du_2}{dt} &= (u_3 - u_2)u_2 \quad \text{Painleve IV} \\ \frac{du_3}{dt} &= (u_1 - u_2)u_3\end{aligned}$$

$$\xi_j = \log u_j, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

\Rightarrow

$$\frac{d\xi}{dt} = H(\xi) \times a$$

$$H(\xi) = \begin{pmatrix} e^{\xi_1} \\ e^{\xi_2} \\ e^{\xi_3} \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\frac{d}{dt} a \cdot \xi = \frac{d}{dt} a \cdot H(\xi) = 0 \quad \text{order 2, elliptic}$$

$$u_{1t} = d_1 \Delta u_1 + (u_2 - u_3)u_1$$

$$u_{2t} = d_2 \Delta u_2 + (u_3 - u_1)u_2$$

$$u_{3t} = d_3 \Delta u_3 + (u_1 - u_2)u_3$$

$$\frac{\partial u_j}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad j = 1, 2, 3$$

monotonicity formula

$$\varphi \in C^2(\bar{\Omega}), \quad \frac{\partial \varphi}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt} \int_{\Omega} a \cdot u \varphi = \int_{\Omega} d \cdot u \Delta \varphi$$

$$\text{scaling} \quad u_j^\mu(x, t) = \mu^2 u_j(\mu x, \mu^2 t), \quad \mu > 0$$

Theorem

$$n \leq 2 \Rightarrow T = +\infty, \quad \sup_{t \geq 0} \|u(\cdot, t)\|_\infty < +\infty$$

$\exists \mathcal{O} \subset \mathbf{R}^3$ Jordan curve

$$\lim_{t \uparrow +\infty} \text{dist}_C(u(\cdot, t), \mathcal{O}) = 0$$

exclusion of blowup in both finite and infinite time

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