Point vortex mean field theories multi-intensities and kinetic model

Takashi Suzuki

A. Static Theory ~ what should it be?

Hamiltonian → statistical mechanics (Gibbs)

micro-canonical statistics

H total energy

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \ 1 \leq i \leq N$$

$\mathbf{R}^{6N}/\{H\}$ micro-canonical ensemble

$x = (q_1, \ldots, q_N, p_1, \ldots, p_N)$

$dx = dH \cdot \frac{d\Sigma(H)}{|\nabla H|}$

$d\Sigma(H) \leftrightarrow \{x \in \mathbf{R}^{6N} \mid H(x) = H\}$

$d\mu^{H,N} = \frac{1}{\Omega(H)} \cdot \frac{d\Sigma(H)}{|\nabla H|}$ micro-canonical measure

$\Omega(H) = \int_{\{H(x)=H\}} \frac{d\Sigma(H)}{|\nabla H|}$

thermo-dynamical relation

$\beta = \frac{\partial}{\partial H} \log \Omega(H)$

equivalent in heat equilibrium

canonical statistics

$\mathbf{R}^{6N}/\{T\}$ canonical ensemble

$\beta = 1/(kT)$ inverse temperature

$d\mu^{\beta,N} = \frac{e^{-\beta H}dx}{Z(\beta, N)}$ canonical (Gibbs) measure

$Z(\beta, N) = \int_{\mathbf{R}^{6N}} e^{-\beta H} dx$

weight-factor

equal a priori probabilities N→∞ … mean field limit

Onsager 49

ordered structure in negative inverse temperature

2D Euler equation of motion on simply-connected domain

$$v_t + (v\cdot\nabla)v = -\nabla p$$
$$\nabla\cdot v = 0 \text{ in } \Omega\times(0,T)$$
$$\nu\cdot v = 0 \text{ on } \partial\Omega\times(0,T)$$

$$\omega = \nabla\times v \text{ vorticity}$$

$$\omega(dx,t) = \sum_{i=1}^{N}\alpha_i\delta_{x_i(t)}(dx)$$

$$\frac{dx_i}{dt} = \nabla_i^{\perp}\hat{H}_N,\ 1\le i\le N$$

$$\nabla_i = \begin{pmatrix}\partial/\partial x_{i1}\\ \partial/\partial x_{i2}\end{pmatrix},\ \nabla_i^{\perp} = \begin{pmatrix}\partial/\partial x_{i2}\\ -\partial/\partial x_{i1}\end{pmatrix}$$
$$x_i = (x_{i1}, x_{i2})$$

$$\hat{H}_N(x_1,\dots,x_N) =$$
$$\sum_i\frac{\alpha_i^2}{2}R(x_j) + \sum_{i<j}\alpha_i\alpha_j G(x_i,x_j)$$

$G = G(x,x')$ Green's function

$$R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x-x'|\right]_{x'=x}$$

Robin function

point vortex Hamiltonian $\to$ **Gibbs measure**

Joyce-Montgomery 73 high energy limit

$$\alpha_i = \alpha,\ N\uparrow+\infty,\ \alpha N = 1,\ \hat{H}_N = H$$

$\Rightarrow$ point vortex mean field equation

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}},\ \alpha^2 N\hat{\beta} = \beta$$
$$\psi = \int_\Omega G(\cdot,x')\rho(x')dx' \text{ in } \Omega$$

Rigorous derivation

Caglioti-Lions-Marchioro-Pulvirenti 92, 95

Kiessling 93

a) bounded Boltzmann weight factors $\{Z\}$

b) uniqueness of the solution to the mean field equation

$\Rightarrow$ canonical statistics

1) convergence to the limit

2) canonical - micro canonical equivalence in the mean field level

3) propagation of chaos

a), b) OK if $\lambda < 8\pi$, $\lambda\ge 8\pi$?

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}},\ \lambda = -\beta$$
$$\psi = \int_\Omega G(\cdot,x')\rho(x')dx' \text{ in } \Omega$$

A formal derivation

$$\rho_k^N(x_1,\cdots,x_k)dx_1\cdots dx_k \quad \text{k-point pdf}$$
$$= \int_{\Omega^{N-k}}\mu^n(dx_{k+1}\cdots dx_N)$$

$$\lim_{N\to\infty}\langle\omega_N(x)\rangle = \rho(x) \quad \text{mean filed limit}$$
$$= \lim_{N\to\infty}\rho_1^N(x)$$

$$\rho_k^N \rightharpoonup \rho^{\otimes k} = \Pi_{i=1}^k\rho(x_i)$$ propagation of chaos

Theorem A1 [S. 92]

$0<\lambda<8\pi \Rightarrow \exists 1$ solution

Mean Field Equation

$\Omega\subset\mathbf{R}^2$ bounded domain $\partial\Omega$ smooth

$\lambda>0$ constant

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega,\ v=0 \text{ on } \partial\Omega$$

point vortex mean field equation (stream function formulation)

Theorem A2 [Nagasaki-S. 90]

$\{(\lambda_k, v_k)\}$ solution sequence

$\lambda_k \to \lambda_0 \in (0,\infty)$, $\|v_k\|_\infty \to \infty$

$\Rightarrow$

$\lambda_0 = 8\pi N$, $N \in \mathbf{N}$

$\exists$sub-sequence $\exists \mathcal{S} \subset \Omega$, $\sharp \mathcal{S} = N$

$v_k \to v_0$ locally uniform in $\overline{\Omega} \setminus \mathcal{S}$

$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0)$ singular limit

$\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$ blowup set

$\nabla_i H_N|_{(x_1,\cdots,x_N)=(x_1^*,\cdots,x_N^*)} = 0$, $1 \le i \le N$

$H_N(x_1,\ldots,x_N) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$

$\Omega \subset \mathbf{R}^2$ bounded domain $\partial\Omega$ smooth
$\lambda > 0$ constant

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega,\ v = 0 \text{ on } \partial\Omega$$

quantized blowup mechanism

recursive hierarchy

B. Impact of the EllipticTheory

Theorem A2 [Nagasaki-S. 90]

$\{(\lambda_k, v_k)\}$ solution sequence s.t.

$\lambda_k \to \lambda_0 \in [0,\infty)$, $\|v_k\|_\infty \to \infty$

$\Rightarrow \lambda_0 = 8\pi\ell$, $\ell \in \mathbf{N}$

$\exists$sub-sequence, $\exists \mathcal{S} \subset \Omega$, $\sharp \mathcal{S} = \ell$, s.t.

$v_k \to v_0$ loc. unif. in $\overline{\Omega} \setminus \mathcal{S}$

$v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0)$

$\nabla_{x_i} H_\ell(x_1^*, \cdots, x_\ell^*) = 0$, $1 \le i \le \ell$,

where

$G = G(x, x')$ the Green's function

$\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$

$$H_\ell(x_1,\ldots,x_\ell) = \frac{1}{2}\sum_{i=1}^{\ell} R(x_i) + \sum_{i<j} G(x_i, x_j)$$

$$R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x - x'|\right]_{x=x'}$$

Theorem B1 [Baraket-Pacard 98]

$(x_1^*, \ldots, x_\ell^*) \in \Omega \times \ldots \times \Omega$: non-degenerate critical point of $H(x_1, \ldots, x_\ell)$

$\Rightarrow$

$\exists$solution sequence $\{(\lambda_k, v_k)\}_k$ satisfying the conclusion of Theorem A2

Theorem B2 [C.-C. Chen-C.-S. Lin 03]

$d(\lambda)$: total degree, g: geneus $\Rightarrow \forall m \in \mathbf{N}$

$$d(\lambda) = \begin{pmatrix} m+1-g \\ m \end{pmatrix},\ 8\pi m < \lambda < 8\pi(m+1)$$

Hence $g \ge 1$, $\lambda \notin 8\pi\mathbf{N} \Rightarrow \exists$ solution

Theorem A1 [S. 92]

$0 < \lambda < 8\pi \Rightarrow \exists 1$ solution

1. non-radial bifurcation on annulus
(S.S. Lin 89, Nagasaki-S. 90)

2. effective bound of blowup points for simply-connected domain
(S.-Nagasaki 89)

3. classification of singular limits
(Nagasaki-S. 90)

4. spherical mean value theorem
(S. 90)

5. localization
(Brezis-Merle 91)

6. entire solution
(Chen-Li 91)

7. sup + inf inequality
(Shafrir 92)

8. uniqueness
(S. 92)

9. field-particle duality
(S. 92, Wolansky 92)

10. singular perturbation
(Weston 78, Moseley 83, S. 93, Baraket-Pacard 98)

11. blowup analysis
(Li-Shafrir 94)

12. Chern-Simons theory
(Tarantello 96)

13. global bifurcation
(S.-Nagasaki 89, Mizoguchi-S. 97 Chang-Chen-Lin 03)

14. min-max solution
(Ding-Jost-Li-Wang 99)

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega\text{: 2D, } v = 0 \text{ on } \partial\Omega$$

15. local uniform estimate
(Y.Y. Li 99)

16. variable coefficient
(Ma-Wei 01)

17. refined aysmptotics
(C.C. Chen- C.S. Lin 02)

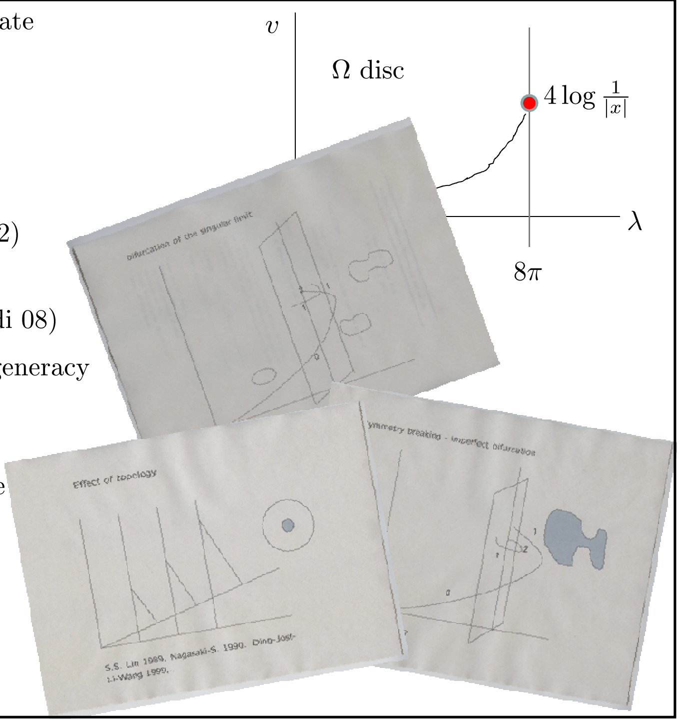
18. topological degree
(Chen-Lin 03, Malchiodi 08)

19. asymptotic non-degeneracy
(Gladiali-Grossi 04, Grossi-Ohtsuka-S. 11)

20. isoperimetric profile
(Lin-Lucia 06)

21. deformation lemma
(Lucia 07)

22. Morse index
(Gladiali-Grossi 09)

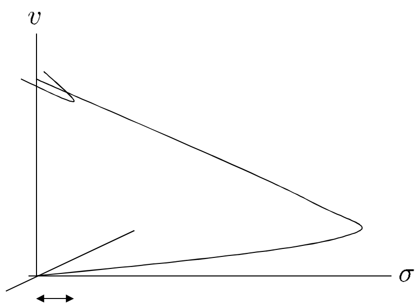


Theorem B1 [Baraket-Pacard 98]

$(x_1^*, \ldots, x_\ell^*) \in \Omega \times \ldots \times \Omega$: non-degenerate critical point of $H(x_1, \ldots, x_\ell)$

$\Rightarrow$

$\exists \ell$-point blowup solution sequence $\{(\sigma_k, v_k)\}$



Theorem B3 [Grossi-Ohtsuka-S. 11]

The linearized operator $-\Delta_D - \sigma_k e^{v_k}$ non-degenerate for $0 < \sigma_k \ll 1$ in the above theorem

Remark

1. only one point blowup and $\exists 1$ blowup spot for convex domain (Gross-F. Takahashi 10)
2. effective bound of the number of blowup points for simply connected domain
3. asymptotic uniqueness under the presense of symmetry
4. domain homology and Hamiltonian (Cao 10)
5. critical points of effective Hamiltonians H_ℓ, $\ell = 1, 2, \cdots \ell_*$ generically equivalent to the solution to the Gel'fand problem, $0 < \sigma \ll 1$
6. correspondence of the Morse indices (Gladiali-Grossi 09)
7. inhomogeneous coefficients, mean field equations, equations on manifold, etc., OK (c.f. Ohtsuka-Sato-S.)
8. one-point blowup case (Gladiali-Grossi 04)
9. refined asymptotics (Gladiali-Grossi 09)
10. asymptotic non-degeneracy in multi-blowup (Grossi-Ohtsuka-S. 11)

$-\Delta v = \sigma e^v$ in Ω
$v = 0$ on $\partial\Omega$

Liouville-Gel'fand eqation

C. Other Models

1. multi-intensity

Ω (closed) Riemannian surface

$E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$

$P(d\alpha)$ probability measure on $[-1, 1]$

c.f. $P(d\alpha) = \frac{1}{2}(\delta_{-1} + \delta_1)$

$$-\Delta v = \lambda \int_I \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}} - \frac{1}{|\Omega|} \right) P(d\alpha)$$

distribution of vortices with renormalized intensity α

mean field of vortices with deterministic multi-intensities (Sawada-S. 08, Onsager's note)

$$-\Delta v = \frac{\lambda}{2} \left(\frac{e^v}{\int_\Omega e^v dx} - \frac{e^{-v}}{\int_\Omega e^{-v} dx} \right)$$

$$-\Delta v = \lambda \left(\frac{\int_I \alpha e^{\alpha v} P(d\alpha)}{\int_I \int_\Omega e^{\alpha v} P(d\alpha)} - \frac{1}{|\Omega|} \right)$$

probability for vorticities to take renormalized intensity α

(Neri 04)

$$-\Delta v = \frac{\lambda}{2} \frac{e^v - e^{-v}}{\int_\Omega e^v + e^{-v} \, dx}$$

blowup analysis (Ohtsuka-Ricciardi-S.10)

$\{v_k\}_{k=1}^{\infty} \subset E,\ v = v_k$: non-compact

$$-\Delta v = \lambda \int_I \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}} - \frac{1}{|\Omega|} \right) P(d\alpha)$$

$\Rightarrow$

$$\mu_k(dxd\alpha) = \frac{\lambda_k e^{\alpha v_k}}{\int_\Omega e^{\alpha v_k}} P(d\alpha)dx$$

$\rightharpoonup$

$$\mu(dxd\alpha) = \Big[\sum_{p\in\mathcal{S}} m(\alpha,p)\delta_p(dx)$$
$$+ r(\alpha,x)dx\Big] P(d\alpha)$$

$$\forall p \in \mathcal{S},\ 8\pi \int_I m(\alpha,p)P(d\alpha)$$
$$= \left\{ \int_I \alpha m(\alpha,p)P(d\alpha) \right\}^2$$

residual vanishing (Ricciardi-Takahashi-S.)

$\mathrm{supp} P \subset I_\pm \Rightarrow r = 0$

Trudinger-Moser inequality (Ricciardi-S.)

$$J_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2$$
$$-\lambda \int_I \left(\log \int_\Omega e^{\alpha v} \right) P(d\alpha) \qquad v \in E$$

bounded if

$$\lambda < \overline{\lambda} =$$
$$\inf \left\{ \frac{8\pi P(K_\pm)}{\left[\int_{K_\pm} \alpha P(d\alpha)\right]^2} \,\middle|\, K_\pm \subset I_\pm \cap \mathrm{supp}\, P \right\}$$

$I_- = [-1,0],\ I_+ = [0,1]$

1. optimal

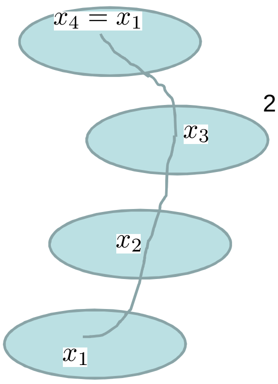
2. extremal case in progress

broken path model

$$-\Delta_i v_i = \lambda \int_{[-1,1]} \frac{\alpha K_i e^{\alpha v_i}}{\int_\Omega K_i e^{\alpha v_i}} P(d\alpha)$$

$v_i|_{\partial\Omega} = 0,\ i = 1,\cdots,p,\ x_{p+1} = x_1$

$$K_i(x_i) = \int_{\Omega^{p-1}} e^{\gamma \sum_{i=1}^{p} |x_{i+1}-x_i|^2}$$
$$\cdot e^{\alpha \sum_{j\neq i} v_j(x_j)} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_p$$



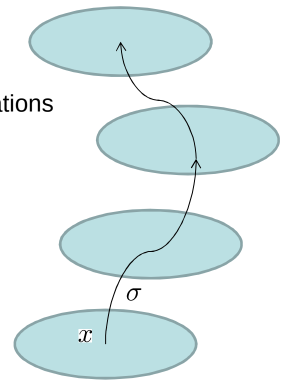
continuous path model

$$\frac{\partial \rho}{\partial \sigma} + \beta\psi\rho = \frac{1}{2}\Delta\rho$$

$$\psi(\cdot,\sigma) = \int_\Omega G(\cdot,x')\rho(x',\sigma)dx'$$

2. vortex filaments mean field equations

Sawada-S.
/ formal derivation
/ variational structure



D. Kinetic Theory (Boltzmann) ∼ where is it going?

system	consistency	kinetics	ensemble
isolated	energy	entropy	micro-canonical
closed	temperature	Helmholtz free energy	canonical
open	pressure	Gibbs free energy	grand-canonical

thermodynamics

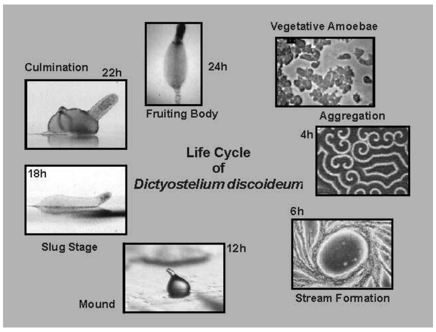
bottom up ↑ ↓ top down

statistical mechanics

Hamilton system → particle collision → time irreversible kinetics

All ensembles equivalent in the range of short interaction (the state which the system takes ultimately in the sense of Gibbs)

variational structure of a chemotaxis model in the context of thermodynamics



moving clustered cells　　aggregating cells

1. transport, gradient

(a) diffusion (b) chemotaxis

2. production　3. chemical reaction

2D Smolchowski-Poisson equation (Childress-Percus, Jager-Luckhaus model)

$$u_t = \nabla\cdot(\nabla u - u\nabla v)$$
$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$$
$$\int_\Omega v = 0$$

$$\frac{du}{dt} = u^2,\ u(0) = T^{-1} > 0$$
$$\Rightarrow$$
$$u(t) = (T-t)^{-1}$$
$$\lim_{t\uparrow T} u(t) = +\infty$$

Blowup of the solution

Theorem D1

[formation of collapse]

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

Senba-S. 01

Herrero-Velázquez 96

Nanjundiah 73

quanitzed blowup mechanism in the kinetic level

Theorem D2

[mass quantization]

$$m(x_0) = m_*(x_0) \equiv \begin{cases} 8\pi, & x_0\in\Omega \\ 4\pi, & x_0\in\partial\Omega \end{cases}$$

c.f. threshold

Senba-S. 01b

Biler 98

Gajewski-Zacharias 98

Nagai-Senba-Yoshida 97

Nagai 95

Jäger-Luckhaus 92

Childress-Percus 81

S. Free Energy and Self-Interacting Particles, Birkhäuser Boston, 05

Smoluchowski equation

$$u_t = \nabla \cdot (\nabla u - u\nabla v) \text{ in } \Omega \times (0,T)$$

$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,T)$$

particle density ↑

duality — mean field description of self-interaction

field ↓

Poisson equation

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\int_\Omega v = 0$$

$\Leftrightarrow$

$$v = G * u = \int_\Omega G(\cdot, x')u(x')dx'$$

variational structure

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle G * u, u\rangle$$

Helmholtz's free energy

$$\delta\mathcal{F}(u) = \log u - G * u$$

$$u_t = \nabla u \cdot \nabla \delta\mathcal{F}(u)$$

$$\left.\frac{\partial}{\partial \nu}\delta\mathcal{F}(u)\right|_{\partial\Omega} = 0$$

model (B) equation

↓

$$\frac{d}{dt}\|u(t)\|_1 = 0$$

total mass conservation

$$\frac{d}{dt}\mathcal{F}(u(t)) = -\int_\Omega u|\nabla\delta\mathcal{F}(u)|^2$$

free energy decrease

control of stationary states

$$u \geq 0, \quad \frac{d}{dt}\|u(\cdot,t)\|_1 = 0$$

$$\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 dx$$

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

$$\int_\Omega v = 0$$

stationary

$\Rightarrow$

$\log u - v = \text{constant}$

$\|u\|_1 = \lambda$ total mass

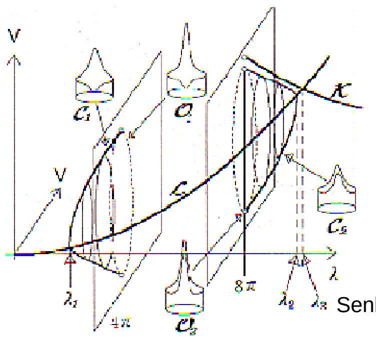
$\Rightarrow$

$$u = \frac{\lambda e^v}{\int_\Omega e^v}$$

$$-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right), \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

$\int_\Omega v = 0$, point vortex mean field equation

$\Rightarrow$ total mass quantization



Senba-S. 00

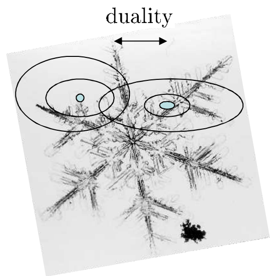
$\lambda = 8\pi \ (4\pi)$

... interior (boundary) threshold

stationary quantization → kinetic quantization

1. recursive hierarchy

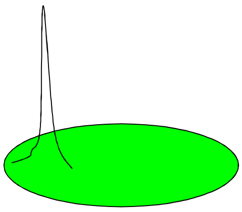
point vortex mean field equation
Smoluchowski-Poisson equation

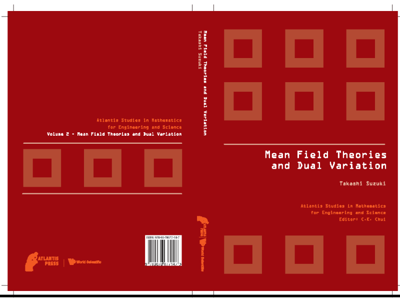


4. nonlinear spectral dynamics

3. field-particle duality

2. quantized blowup mechanics





static theory
$\downarrow$
kinetic theory

TS
Mean Field Theories and Dual Variation
Atlantis Press
Amsterdam 2008

Kinetic Mean Field Theories

Chavanis 08 micro-canonical theory

long-range interactions

$\Rightarrow$

non-equivalence of the ensembles

Langevin equation

$$\frac{dx_i}{dt} = \alpha\nabla_i^{\perp}\hat{H}_N - \mu\alpha^2\nabla_i\hat{H}_N + \sqrt{2\nu}R_i(t)$$

$i = 1, 2, \cdots, N$

$\mu > 0$ mobility

$\nu \geq 0$ diffusion coefficient
(viscosity of the particles)

$R_i(t)$ white noise

$\langle R_i(t)\rangle = 0$

$\langle R_i^{\alpha}(t)R_j^{\beta}(t')\rangle = \delta_{ij}\delta_{\alpha\beta}\delta(t-t')$

$P_N(x_1, \cdots, x_N, t)$

N-body distribution function

$\Rightarrow$

Fokker-Planck equation

$$\frac{\partial P_N}{\partial t} + \alpha\nabla^{\perp}\cdot\hat{H}_N\nabla P_N = \nabla\cdot(\nu\nabla P_N + \mu\alpha^2 P_N\nabla\hat{H}_N)$$

$\Rightarrow$

BBGKY-like hierarchy to $\{P_i\}_{i=1,2,\cdots,N}$

inverse temperature $\hat{\beta}$

$$\frac{1}{g(\hat{E})}\frac{\partial}{\partial\hat{E}}[g(\hat{E})P_j] = \hat{\beta}P_j + \frac{\partial P_j}{\partial\hat{E}}$$

$$g(\hat{E}) = \int \delta[\hat{E} - \hat{H}(x_1, \cdots, x_N)]\prod_{i=1}^{N} dx_i$$

factorization (propagation of chaos)

$$P_N(x_1, x_2, \cdots, x_N, t) = \prod_{i=1}^{N} P_1(x_i, t)$$

high-energy limit

$\hat{\beta} N \alpha^2 = \beta,\ \alpha N = 1,\ \omega = P_1$

$\Rightarrow$

$$\frac{\partial \omega}{\partial t} + \nabla^{\perp}\psi \cdot \nabla\omega = \nu \nabla \cdot (\nabla\omega + \beta\alpha\omega\nabla\psi)$$

$$-\Delta\psi = \omega,\ \psi|_{\partial\Omega} = 0$$

$\beta = -\lambda$

renormalized inverse temperature

1. vortex term
   $\nabla^{\perp}\psi \cdot \nabla\omega = \nabla \cdot \omega\nabla^{\perp}\psi$
2. Dirichlet boundary condition for the Poisson part

Smoluchowski-Poisson equation

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$

$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

$$\int_\Omega v = 0$$

two-species model

$$\frac{\partial \omega_\pm}{\partial t} + \nabla \cdot \omega_\pm \nabla^{\perp}\psi = \nu\nabla \cdot (\nabla\omega_\pm \pm \beta\alpha\omega_\pm\nabla\psi)$$

$$-\Delta\psi = \omega_+ + \omega_-$$

$$\frac{\partial \omega_\pm}{\partial \nu} \pm \beta\alpha\omega_\pm \frac{\partial \psi}{\partial \nu}\Big|_{\partial\Omega} = \psi|_{\partial\Omega} = 0$$

$$\omega_\pm|_{t=0} = \omega_{\pm 0}$$

$$\nu > 0,\ \alpha > 0,\ \beta = -\lambda < 0,\ \pm\omega_{\pm 0} \geq 0$$

$\longleftrightarrow$

Euler-Smoluchowski part

$$\frac{\partial u_1}{\partial t} + \nabla \cdot u_1\nabla^{\perp}v = d\Delta u_1 - \chi\nabla \cdot u_1\nabla v$$

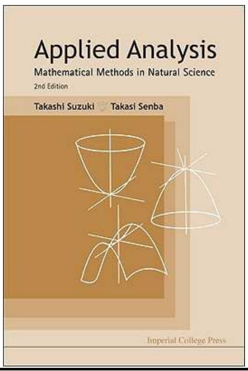
$$\frac{\partial u_2}{\partial t} + \nabla \cdot u_2\nabla^{\perp}v = d\Delta u_2 + \chi\nabla \cdot u_2\nabla v$$

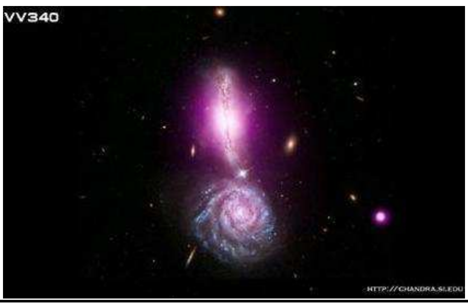
$$d\frac{\partial u_1}{\partial \nu} - \chi u_1\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = d\frac{\partial u_2}{\partial \nu} + \chi u_2\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

$$u_i|_{t=0} = u_{i0} \geq 0,\ i = 1, 2$$

Poisson part

$$-\Delta v = u,\ v|_{\partial\Omega} = 0,\ u = u_1 - u_2$$





1. vortex terms
2. hetero-separative homo-aggregative

1. Senba-S. 01	weak formulation monotonicity formula	formation of collapse
2. Senba-S. 02	weak solution	instant blowup for over mass collapse initial data
3. Kurokiba-Ogawa 03	scaling invariance	non-existence of over mass entire solution without concentration
4. S. 05	backward self-similar transformation scaling limit parabolic envelope (1) scaling invariance of the scaling limit	collapse mass quantization
5. Senba 07 Naito-S. 08	parabolic envelope (2)	type II blowup rate formation of sub-collapse
6. S. 08	scaling back	
7. Senba-S. 11	translation limit	limiting process simplification
8. Espejo-Stevens-S. 12	quantization without blowup threshold	simultaneous blowup mass separation

classical analysis Mathematical Structure of the Smoluchowski-Poisson equation

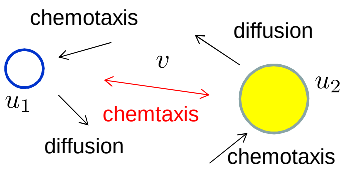
1. Nagai-Senba-Yoshida 97, Biler 98, Gajewski-Zacharias 98 global-in-time existence
2. Biler-Hilhorst-Nadieja 94, Nagai 95, Nagai 01, Senba-S. 02 blowup in finite time

DD model
(hetero-separative,
homo-aggregative type)

Kurokiba-Ogawa 03
Espejo-Stevens-Velazquez 10

$$u_{1t} = d\Delta u_1 - \chi\nabla\cdot u_1\nabla v$$
$$u_{2t} = d\Delta u_2 + \chi\nabla\cdot u_2\nabla v$$

$$-\Delta v = u_1 - u_2 \text{ in } \mathbf{R}^2\times(0,T)$$
$$u_i|_{t=0} = u_{i0}(x)\geq 0,\ i=1,2$$



collapse mass separation

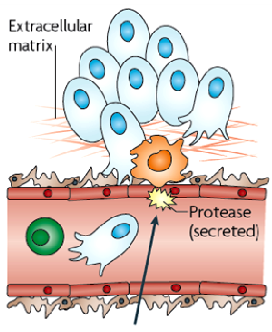
competitive system of chemotaxis
(hetero-homo-aggregative type)

Espejo-Stevens-Velazquez 09
Espejo-Stevens-S. 12

$$u_{it} = d_i\Delta u_i - \chi_i\nabla\cdot u_i\nabla v$$
$$d_i\frac{\partial u_i}{\partial\nu} - \chi_i u_i\frac{\partial v}{\partial\nu}\Big|_{\partial\Omega} = 0$$
$$u_i|_{t=0} = u_{i0}(x)\geq 0,\ i=1,2,\cdots,N$$
$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$
$$\frac{\partial v}{\partial\nu}\Big|_{\partial\Omega} = 0$$
$$\int_\Omega v = 0,\ u = \sum_{i=1}^N u_i$$

macrophage u_1 — chemotaxis — production — chemical v — cancer cell u_2 — chemotaxis

J. Joyce, and J. Pollard. *Nat Rev Cancer* 9: 239-252 (2009)



simultaneous blowup
hetero-homo-aggregation type

Typical Results - DD model

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

$\partial_t u_1 = d_1 \Delta u_1 - \chi_1 \nabla \cdot u_1 \nabla v$

$\partial_t u_2 = d_2 \Delta u_2 + \chi_2 \nabla \cdot u_2 \nabla v$

$$d_1 \frac{\partial u_1}{\partial \nu} - \chi_1 u_1 \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = d_2 \frac{\partial u_2}{\partial \nu} + \chi_2 u_2 \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

$u_i|_{t=0} = u_{i0}(x) \geq 0, \ i = 1,2$

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \ \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

$\int_\Omega v = 0, \ u = u_1 - u_2$

$\lambda_i = \|u_{i0}\|_1$ **mass**

$\xi_i = d_i/\chi_i$ **inverse motility**

Theorem 1 [classical analysis]

1. $\lambda_i < 4\pi\xi_i, \ i = 1,2 \Rightarrow T = +\infty$

2. $(\lambda_1(x_0) - \lambda_2(x_0))^2 > m_*(x_0) \sum_{i=1}^{2} \xi_i \lambda_i(x_0)$

$\||x - x_0|^2 u_{i0}\|_{L^1(B(x_0,2R))} \ll 1, \ x_0 \in \overline{\Omega}$

$\Rightarrow T < +\infty$

$\lambda_i(x_0) = \|u_{i0}\|_{L^1(B(x_0,R))}, \ 0 < R \ll 1$

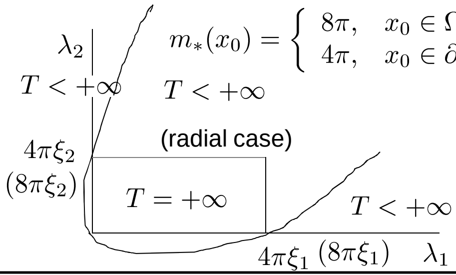
3. $u_i = u_i(|x|, t), \ d_1 = d_2, \ \chi_1 = \chi_2$

$\lambda_i > 8\pi\xi_i, \ \||x|^2 u_{i0}\|_{L^1(B_r)} \ll 1$

$0 < r \ll 1 \Rightarrow \exists T_i < +\infty$

$\lim_{t \uparrow T_i} \|u_i(\cdot, T)\|_\infty = +\infty, \ i = 1,2$

$$m_*(x_0) = \begin{cases} 8\pi, & x_0 \in \Omega \\ 4\pi, & x_0 \in \partial\Omega \end{cases}$$



Theorem 2 [blowup analysis]

1. $T < +\infty \Rightarrow \sharp\, \mathcal{S} < +\infty$

$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists t_k \uparrow T, \exists x_k \to x_0$

$\sum_{i=1}^{2} u_i(x_k, t_k) \to +\infty\}$

2. $i = 1,2, \ u_i(x,t)dx \rightharpoonup$

$\sum_{x_0 \in \mathcal{S}} m_i(x_0)\delta_{x_0}(dx) + f_i(x)dx$ in $\mathcal{M}(\overline{\Omega})$

$0 \leq f_i = f_i(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$

$\forall x_0 \in \mathcal{S}, \ m_i(x_0) \geq 0$

$\sum_{i=1}^{2} m_i(x_0) > 0$

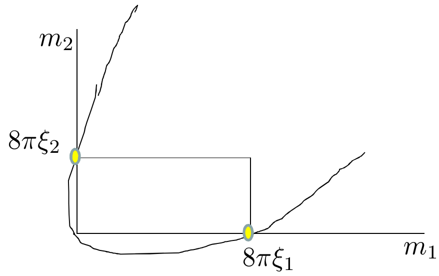
$(m_1(x_0) - m_2(x_0))^2$

$= m_*(x_0) \sum_{i=1}^{2} \xi_i m_i(x_0)$

3. (mass separation)

$d_1 = d_2, \ \chi_1 = \chi_2, \ u_{i0} = u_{i0}(|x|), \ \forall i \Rightarrow$

$(m_1(x_0), m_2(x_0)) = (8\pi\xi_1, 0)$ or $(0, 8\pi\xi_2)$



1. Neumann boundary condition of the Poisson part $\Rightarrow \frac{\partial u_i}{\partial \nu}\Big|_{\partial\Omega} = 0, \ i = 1,2$

2. parameter variation method for logarithmic HLS inequality of Shafrir-Wolansky 05

$\Rightarrow$ (sharp) global-in-time region

Euler-Smoluchowski-Poisson equation

$$\frac{\partial u_1}{\partial t} + \nabla \cdot u_1 \nabla^{\perp} v = d\Delta u_1 - \chi \nabla \cdot u_1 \nabla v$$

$$\frac{\partial u_2}{\partial t} + \nabla \cdot u_2 \nabla^{\perp} v = d\Delta u_2 + \chi \nabla \cdot u_2 \nabla v$$

$$d\frac{\partial u_1}{\partial \nu} - \chi u_1 \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega}$$

$$= d\frac{\partial u_2}{\partial \nu} + \chi u_2 \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

$$u_i|_{t=0} = u_{i0} \geq 0, \ i = 1, 2$$

$$-\Delta v = u, \ v|_{\partial\Omega} = 0, \ u = u_1 - u_2$$

Technical Difficulties

1. vortex terms
2. Dirichlet bounday condtion for the Poisson part ← main obstruction
3. two-species

1. global-in-time existence criterion valid (Biler-Herbisch-Nadzieja 94):

$$\left| \int_\Omega u \nabla u \cdot \nabla v \ dx \right| \leq \|u\|_3 \|\nabla u\|_2 \|\nabla v\|_6$$

2. blowup criterion valid in either DD model or single species ESP

3. blowup analysis valid to the interior boundary point in the above systems

4. boundary blowup excluded in single species DD model

Collapse mass quantization

$$u_t = \nabla \cdot (\nabla u - u \nabla v)$$

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$

$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

$$\int_\Omega v = 0$$

1. total mass conservation: $\|u(t)\|_1 = \lambda$

2. free energy decreasing: $\frac{d}{dt}\mathcal{F}(u) \leq 0$

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle$$

3. weak fom: $\forall \varphi \in C^2(\overline{\Omega})$, $\frac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} = 0$

$$\frac{d}{dt}\int_\Omega u\varphi - \int_\Omega u\Delta\varphi = \frac{1}{2}\int_\Omega\int_\Omega \rho_\varphi u \otimes u$$

$$\rho_\varphi(x, x') = \nabla\varphi(x) \cdot \nabla_x G(x, x')$$
$$+\nabla\varphi(x') \cdot \nabla_{x'} G(x, x')$$

Formation of collapse

1. free energy + Trudinger-Moser $\Rightarrow$ ε-regularity;

$$\lim_{R\downarrow 0} \limsup_{t\uparrow T} \|u(\cdot, t)\|_{L^1(\Omega \cap B(x_0, R))} < \exists \varepsilon_0$$

$\Rightarrow x_0 \notin \mathcal{S}$

2. weak fomulation $\Rightarrow$ monotonicity formula;

$$\varphi \in C^2(\overline{\Omega}), \ \frac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} = 0$$

$$\left|\frac{d}{dt}\int_\Omega u(\cdot, t)\varphi\right| \leq C_\varphi(\lambda + \lambda^2)$$

3. formation of collapse

$$u(x, t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx)$$
$$+f(x)dx \text{ in } \mathcal{M}(\overline{\Omega}), \ t \uparrow T$$

A1

method of the weak scaling limit

1. backward self-similar transformation

$y = (x - x_0)/(T - t)$

$s = -\log(T - t),\ t < T$

$z(y, s) = (T - t)u(x, t)$

$\forall s_k \uparrow +\infty,\ \exists \{s_k'\} \subset \{s_k\}$ s.t.

$z(y, s + s_k')dy \rightharpoonup \exists \zeta(dy, s)$

in $C_*(-\infty, +\infty; \mathcal{M}_0(\mathbf{R}^2))$

with 0-extension and
reflection (for boundary blowup point)

weak solution to

$\zeta_s = \nabla \cdot (\nabla \zeta - \zeta \nabla(\Gamma * \zeta + |y|^2/4))$

in $\mathbf{R}^2 \times (-\infty, +\infty),\ \mathcal{M}_0 = C_0(\overline{\mathbf{R}}^2)'$

$f \in C_0(\overline{\mathbf{R}}_+^2) \Leftrightarrow$
$f \in C(\mathbf{R}^2 \cup \{\infty\}),\ f(\infty) = 0$

$$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$$

2. parabolic envelope (1)

$\hat{m}(x_0) = \zeta(\mathbf{R}^2, s),\ -\infty < s < +\infty$

$$\hat{m}(x_0) = \begin{cases} m(x_0), & x_0 \in \Omega \\ 2m(x_0), & x_0 \in \partial\Omega \end{cases}$$

3. parabolic envelope (2)

$0 \leq \langle |y|^2, \zeta(dy, s) \rangle = I(s) \leq C$

4. scaled second moment

$\frac{dI}{ds} = I - \sigma(x_0),\ -\infty < s < +\infty$

$\Rightarrow$

$$I = \sigma(x_0) \equiv \frac{1}{2\pi}\hat{m}(x_0)^2 - 4\hat{m}(x_0) \geq 0$$

A2

5. scaling back

$\zeta(dy, s) = e^{-s}A(dy', s')$

$y' = e^{-s/2}y,\ s' = -e^{-s}$

$\Rightarrow (y, s) \in \mathbf{R}^2 \times (-\infty, 0)$

$A_s = \nabla \cdot (\nabla A - A \nabla \Gamma * A)$

$A = A(dy, s) \geq 0,\ A(\mathbf{R}^2, s) = \hat{m}(x_0)$

6. weak translation limit

$\forall s_k \uparrow +\infty,\ \exists \{s_k'\} \subset \{s_k\}$

$A(dy, s - s_k') \rightharpoonup a(dy, s)$

in $C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2))$

$\mathcal{M}(\mathbf{R}^2) = [C_0(\mathbf{R}^2) \oplus \mathbf{R}]'$ envelopes
the total scaling mass

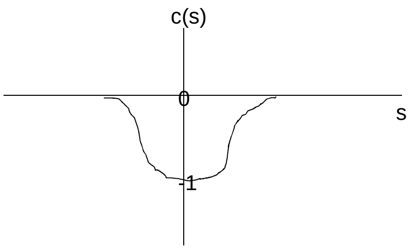
$a_s = \nabla \cdot (\nabla a - a \nabla \Gamma * a)$

$a(dy, s) \geq 0,\ (y, s) \in \mathbf{R}^2 \times (-\infty, +\infty)$

$a(\mathbf{R}^2, s) = \hat{m}(x_0)$

7. scaling argument applied to the weak solution

1) local second moment



2) scale invariance

$a(y, s) \mapsto a_\mu(y, s) = \mu^2 a(\mu y, \mu^2 s)$

$\Rightarrow$

$$\sigma(x_0) \equiv \frac{1}{2\pi}\hat{m}(x_0)^2 - 4\hat{m}(x_0) \leq 0$$

A3

Summary

1. In Onsager's static theory of point vortices we have recursive hierarchy, quantized blowup mechanism, field-particle duality, and nonlinear spectral mechanics
2. Kinetic theory of Chavanis induces Euler-Smoluchowski-Poisson equations as a mean field limit
3. Its two-species model without vortex terms is a drift-diffusion system with hetero-separative-homo-aggregative gradients
4. If the Poisson part is provided with the Neumann boundary condition, we have a complete classical analysis; existence and non-existence of the solution global-in-time. Then the blowup analysis guarantees the standard results; formation of collapse, mass quantization, mass separation, formation of sub-collapse, and type II blowup rate
5. Three factors - vortex term, Dirichlet boundary condition, and two species - are main technical difficulties in the Euler-Smoluchowski-Poisson equation. Except for the global-in-time criterion so far the results are restricted
6. Yet all the blowup analysis is done for the DD model concerning aggregative single species, excluding boundary blowup points

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