Exclusion of boundary blowup for 2D chemotaxis system with Dirichlet boundary condition for the Poisson part

2012. 09

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The model

$\Omega \subset \mathbf{R}^2$ bounded domain
$\partial\Omega$ smooth

1. Smoluchowski Part

$$u_t = \nabla \cdot (\nabla u - u\nabla v)$$
$$\left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$
$$u|_{t=0} = u_0(x) > 0$$

2. Poisson Part

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

Sire-Chavanis 02
motion of the mean field of many self-gravitating Brownian particles

other Poisson parts

a) Debye system (DD model)

$$\Delta v = u, \quad v|_{\partial\Omega} = 0$$

b) Childress-Percus-Jager-Luckhaus model (chemotaxis)

$$-\Delta v = u - \frac{1}{|\Omega|}\int u$$
$$\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \ \int_\Omega v = 0$$

DD model $v \leftrightarrow -v$

$$u_t = \nabla\cdot(\nabla u + u\nabla v), \ -\Delta v = u$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = v|_{\partial\Omega} = 0$$

global-in-time existence with compact orbit
Biler-Hebisch-Nadzieja 94

1. total mass conservation
$$\frac{d}{dt}\|u(t)\|_1 = 0$$

2. free energy decreasing
$$\frac{d}{dt}\int_\Omega u(\log u - 1) + \frac{1}{2}uv\ dx$$
$$= -\int_\Omega u|\nabla(\log u + v)|^2 \le 0$$

3. key estimate
$$\|u\nabla u\cdot\nabla v\|_2 \le C\|u\|_2\|\nabla u\|_2\|\nabla v\|_6$$

chemotaxis system

$$u_t = \nabla\cdot(\nabla u - u\nabla v)$$
$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \ \int_\Omega v = 0$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$
$$u|_{t=0} = u_0(x) > 0$$

$\Rightarrow$
$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

compact Riemann surface without boundary

blowup threshold

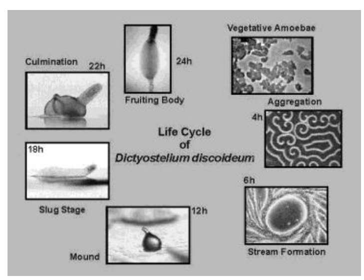
a. Biler 98, Gajewski-Zacharias 98, Nagai-Senba-Yoshida 97

b. Nagai 01, Senba-S. 01b

Nemann case … self-attractive Smoluchowski - Poisson equation

quantized blowup mechanism - kinetic level

$$u_t = \nabla\cdot(\nabla u - u\nabla v)$$
$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$$
$$\int_\Omega v = 0$$



moving clustered cells

aggregating cells

Theorem A1

$\boxed{\text{formation of collapse}}$

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

Senba-S. 01

Herrero-Velázquez 96

Nanjundiah 73

formation of sub-collapse
type II blowup rate

Theorem A2

$\boxed{\text{mass quantization}}$

$$m(x_0) = m_*(x_0) \equiv \begin{cases} 8\pi, & x_0\in\Omega \\ 4\pi, & x_0\in\partial\Omega \end{cases}$$

c.f. threshold

Senba-S. 01b

Biler 98

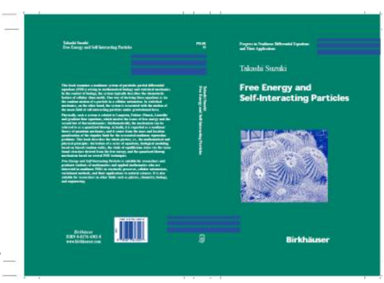
Gajewski-Zacharias 98

Nagai-Senba-Yoshida 97

Nagai 95

Jäger-Luckhaus 92

Childress-Percus 81



S. Free Energy and Self-Interacting Particles, Birkhäuser Boston, 05 quantized blowup mechanism – spectral level (Boltzmann-Poisson equation)

Theorem B1 [Nagasaki-S. 90a]

$\{(\lambda_k, v_k)\}$ solution sequence

$\lambda_k \to \lambda_0 \in (0,\infty)$, $\|v_k\|_\infty \to \infty$

$\Rightarrow$

$\lambda_0 = 8\pi N$, $N \in \mathbf{N}$

$\exists$sub-sequence, $\exists \mathcal{S} \subset \Omega$, $\sharp \mathcal{S} = N$

$\lambda = \|u\|_1$ $\quad$ u $\xleftrightarrow{\text{duality}}$ v

$\Omega \subset \mathbf{R}^2$ bounded domain $\partial\Omega$ smooth

$\lambda > 0$ constant

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega,\ v = 0 \text{ on } \partial\Omega$$

$G = G(x,x')$ Green's function

$$R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x-x'| \right]_{x'=x}$$

Robin function

$v_k \to v_0$ locally uniform in $\overline{\Omega} \setminus \mathcal{S}$

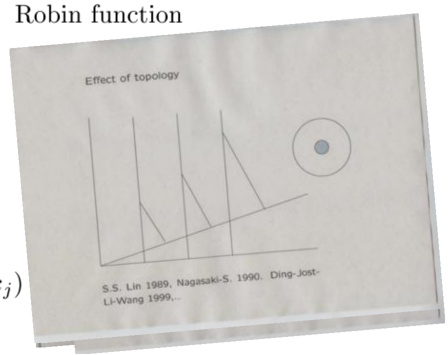
$v_0(x) = \boxed{8\pi} \sum_{x_0 \in \mathcal{S}} G(x, x_0)$ singular limit

$\mathcal{S} = \{x_1^*, \ldots, x_N^*\}$ blowup set

$\nabla_i H_N|_{(x_1,\cdots,x_N)=(x_1^*,\cdots,x_N^*)} = 0,\ 1 \le i \le N$

$$H_N(x_1,\ldots,x_N) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i,x_j)$$

Hamiltonian



$\Omega \subset \mathbf{R}^2$: open set, $V \in C(\Omega)$

$-\Delta v = V(x)e^v$, $\ 0 \le V(x) \le b \quad$ in Ω

$\int_\Omega e^v \le C$

Theorem B2 [Li-Shafrir 94]

$\{(V_k, v_k)\}$ solution sequence

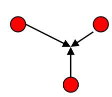
$V_k \to V$ loc. unif. in Ω

$\Rightarrow$

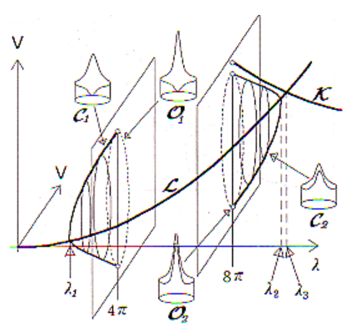
$\exists$ sub-sequence with the alternatives;

1. $\{v_k\}$: loc. unif. bdd in Ω
2. $\exists \mathcal{S} \subset \Omega$, $\sharp\mathcal{S} < +\infty$
$v_k \to -\infty$ loc. unif. in $\Omega \setminus \mathcal{S}$
$\mathcal{S} = \{x_0 \in \Omega \mid \exists x_k \to x_0$ s.t. $v_k(x_k) \to +\infty\}$
$V_k(x)e^{v_k}dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx)$
in $\mathcal{M}(\Omega)$, $m(x_0) \boxed{\in 8\pi\mathbf{N}}$
3. $v_k \to -\infty$ loc. unif. in Ω

local theory (short range interaction)



Long-range interaction (boundary condition or Green's function of the Poisson part) prohibits the collision of collapses



nonlinear spectral mechanics -
Senba-S. 00 (Neumann case)

Green's function – potential of long range interaction due to the action at a distance

$x' \in \Omega$

$-\Delta G(\cdot, x') = \delta_{x'}, \boxed{G(\cdot, x')|_{\partial\Omega} = 0}$

fundamental solution

$\Gamma(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$

$G = G(x, x') \in C^{2+\theta}(\overline{\Omega} \times \overline{\Omega} \setminus D)$

$D = \overline{\{(x, x) \mid x \in \Omega\}}, \ 0 < \theta < 1$

1. interior regularity

$G(x, x') = \Gamma(x - x') + K(x, x')$

$K \in C^{2+\theta,1}(\overline{\Omega} \times \Omega) \cap C^{1,2+\theta}(\Omega \times \overline{\Omega})$

2. boundary regularity

$x_0 \in \partial\Omega$

$X : \overline{\Omega \cap B(x_0, 2R)} \to \overline{\mathbf{R}^2_+}, \ X(x_0) = 0$

conformal diffeo. into

$\mathbf{R}^2_+ = \{(X_1, X_2) \in \mathbf{R}^2 \mid X_2 > 0\}$

$G(x, x') = E(x, x') + K(x, x')$

$K \in C^{2+\theta,1} \cap C^{1,2+\theta}$

$(\overline{\Omega \cap B(x_0, R)} \times \overline{\Omega \cap B(x_0, R)})$

$E(x, x') = \Gamma(X - X') \boxed{-} \Gamma(X - X'_*)$

$X_* = (X_1, -X_2), \ X = (X_1, X_2)$

$\boxed{\varphi \in C^2(\overline{\Omega}), \ \left.\frac{\partial \varphi}{\partial \nu}\right|_{\partial\Omega} = 0} \Rightarrow \boxed{\rho_\varphi \in L^\infty(\Omega \times \Omega)}$

slg (symmetrized localized Green's function)

$\rho_\varphi(x, x') = \nabla\varphi(x) \cdot \nabla_x G(x, x') + \nabla\varphi(x') \cdot \nabla_{x'} G(x, x')$

The result

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

$u_t = \nabla \cdot (\nabla u - u\nabla v) \ \text{ in } \Omega \times (0, T)$

$\left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$

$u|_{t=0} = u_0(x) > 0$

$-\Delta v = u, \quad v|_{\partial\Omega} = 0$

$T < +\infty \Rightarrow \boxed{\lim_{t \uparrow T} \|u(\cdot, t)\|_\infty = +\infty}$

blowup set

$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0, \ t_k \uparrow T$

$u(x_k, t_k) \to +\infty\} \neq \emptyset$

Theorem 1

1. $\sharp\mathcal{S} < +\infty, \boxed{\mathcal{S} \subset \Omega}$

2. $u(x, t)dx \rightharpoonup 8\pi \sum_{x_0 \in \mathcal{S}} \delta_{x_0}(dx)$

$+f(x)dx$ in $\mathcal{M}(\overline{\Omega}), \ t \uparrow T$

$0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$

3. $\forall x_0 \in \mathcal{S}, \forall b > 0$

$\lim_{t \uparrow T} (T - t)\|u(\cdot, t)\|_{L^\infty(B(x_0, b(T-t)^{1/2})}$

$= +\infty$

formation of sub-collapse → type II blowup rate

Corollary

$\lambda \equiv \|u_0\|_1 \leq 8\pi \Rightarrow T = +\infty$

Remark

1. sub-critical mass → compact orbit
2. critical mass → ∃ blowup in infinite time

Theorem 2 (critical mass)

$\|u_0\|_1 = 8\pi$, $\not\exists$ stationary solution

$\Rightarrow (T = +\infty)$

$\forall t_k \uparrow +\infty \ \exists \{t_k'\} \subset \{t_k\}$

$u(x, t + t_k')dx \rightharpoonup 8\pi\delta_{x(t)}(dx)$

in $C_*(-\infty, \infty; \mathcal{M}(\overline{\Omega}))$

$t \in (-\infty, +\infty) \mapsto x(t) \in \Omega$ loc. a.c.

$$\boxed{\frac{dx}{dt} = 4\pi\nabla R(x), \text{ a.e. } t \in \mathbf{R}}$$

Remark

domain close to a disc

→ no stationary solution with critical mass

→ blowup in infinite time

$$\lim_{t\uparrow+\infty} \|u(\cdot, t)\|_\infty = +\infty$$

Corollary

$u = u(|x|, t) \Rightarrow u(x, t + s)dx \rightharpoonup 8\pi\delta_0(dx)$

$s \uparrow +\infty$ in $C_*(-\infty, +\infty; \mathcal{M}(\overline{\Omega}))$

<u>Sire-Chavanis 02</u>, <u>Kavallaris-Souplet 09</u>

$\|Q(\cdot, t) - 8\pi\|_1 \approx e^{-\sqrt{2t}}$, $t \uparrow +\infty$

$$Q(x, t) = \int_{B(0,|x|)} u(\cdot, t)$$

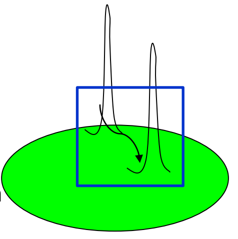
Conjecture

(non-radial case)

collapse born on the boundary in infinite time

→

shifts to a local maximum of the Robin function



Contents (17)

1. monotonicity formula

1) total mass conservation

$$u_t = \nabla\cdot(\nabla u - u\nabla v)$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$ Smoluchowski
$$u \geq 0 \Rightarrow \|u(\cdot,t)\|_1 = \|u_0\|_1 \equiv \lambda$$

2) symmetrization

$$v = \int_\Omega G(\cdot,x')u(x')dx'$$
$$\Leftrightarrow$$
$$-\Delta v = u,\ v|_{\partial\Omega} = 0$$ Poisson - action at a distance

$$G(x,x') = G(x',x)$$ action –reaction law

$$\boxed{\varphi \in C^2(\overline{\Omega}),\ \frac{\partial\varphi}{\partial\nu}\Big|_{\partial\Omega} = 0 \Leftrightarrow (\mathrm{def})\ \varphi \in Y}$$

$$\frac{d}{dt}\int_\Omega \varphi(x)u(x,t)dx$$
$$= -\int_\Omega \nabla\varphi\cdot(\nabla u - u\nabla v)dx$$
$$= \int_\Omega \Delta\varphi(x)\cdot u(x,t)dx$$
$$+\frac{1}{2}\int_{\Omega\times\Omega}\rho_\varphi(x,x')u(x,t)u(x',t)dxdx'$$

weak form

$$\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x')$$
$$+\nabla\varphi(x')\cdot\nabla_{x'}G(x,x')$$ slg

$$\|\rho_\varphi\|_\infty \leq C\|\nabla\varphi\|_{C^1} \Rightarrow$$ monotonicity formula
$$\left|\frac{d}{dt}\int_\Omega u\varphi\right| \leq C(\lambda+\lambda^2)\|\nabla\varphi\|_{C^1}$$

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2. formation of collapse

1) weak continuation

$$\varphi \in Y \hookrightarrow C(\overline{\Omega}) \text{ dense}$$
$$\left|\frac{d}{dt}\int_\Omega u(\cdot,t)\varphi\right| \leq C_\lambda\|\nabla\varphi\|_{C^1}$$
$$\lambda = \|u(\cdot,t)\|_1 \Rightarrow$$

$$0 \leq \mu(dx,t) \in C_*([0,T],\mathcal{M}(\overline{\Omega}))$$
$$u(x,t)dx = \mu(dx,t),\ 0 \leq t < T$$

2) ε-regularity

$$\lim_{R\downarrow 0}\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \exists\varepsilon_0$$
$$\Rightarrow x_0 \notin \mathcal{S}$$

1)+2) $\Rightarrow$ parabolic-elliptic regularity

$$\mu(\cdot,T) = \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$
$$m(x_0) \geq \varepsilon_0,\ 0 \leq f \in L^1(\Omega),\ \sharp\mathcal{S} < +\infty$$

proof of ε-regularity - $\boxed{\text{Dirichlet case}}$

localization of the global-in-time existence criterion of Jager-Luckhaus type

$$x_0 \in \overline{\Omega},\ 0 < R \ll 1$$

Lemma 1

$$\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R)} < \varepsilon_0$$
$$\Rightarrow \limsup_{t\uparrow T}\int_{\Omega\cap B(x_0,R/2)} u(\log u - 1) < +\infty$$

Lemma 2

$$\limsup_{t\uparrow T}\int_{\Omega\cap B(x_0,R)} u(\log u - 1) < +\infty$$
$$\Rightarrow x_0 \notin \mathcal{S}$$

Gagliardo-Nirenberg inequality

Moser's iteration scheme

$$\boxed{v \geq 0}\ \|v\|_{W^{1,q}} \leq C(q),\ 1 \leq q < 2$$

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3. blowup criterion from the weak form

Theorem (Senba-S. 01b)
$x_0 \in \Omega,\ 0 < R \ll 1$
$\|u_0\|_{L^1(B(x_0,R))} > 8\pi$
$\||x-x_0|^2 u_0\|_{L^1(B(x_0,4R))} \ll 1$
$\Rightarrow T < +\infty$

cut-off function

$$x_0 \in \overline{\Omega},\ 0 < R \ll 1, \varphi = \varphi_{x_0,R} \in Y$$
$$0 \le \varphi \le 1,\ \varphi = \begin{cases} 1, & x \in B(x_0, R/2) \\ 0, & x \in \mathbf{R}^2 \setminus B(x_0,R) \end{cases}$$
$$|\nabla \varphi| \le CR^{-1}\varphi^{5/6},\ |\nabla^2 \varphi| \le CR^{-2}\varphi^{2/3}$$

interior regularity

$$G(x,x') = \frac{1}{2\pi}\log\frac{1}{|x-x'|} + K(x,x')$$
$$K \in C^{2+\theta,1}(\overline{\Omega}\times\Omega)\cap C^{1,2+\theta}(\Omega\times\overline{\Omega})$$

$$x_0 \in \Omega,\ 0 < R \ll 1,\ \varphi = |x-x_0|^2\varphi_{x_0,R}$$
$$\frac{d}{dt}\int_\Omega u\varphi = \int_\Omega \Delta\varphi\cdot u(\cdot,t) + \frac{1}{2}\int_{\Omega\times\Omega}\rho_\varphi(x,x')u\otimes u$$

$$\boxed{\frac{dI_R}{dt} \le J_R(0) + a(R^{-1}t^{1/2}) + CR^{-1}I_{2R}(t)^{1/2}}$$

$$a(s) = C(s^2+s)$$
$$I_R = \int_\Omega |x-x_0|^2 u(\cdot,t)\varphi_{x_0,R}$$
$$J_R = 4M_R - \frac{M_R^2}{2\pi} + CR^{-1}I_{2R}$$
$$M_R = \int_\Omega u(\cdot,t)\varphi_{x_0,R}$$

+ monotonicity formula

$$M_R(0) > 8\pi,\ I_{2R}(0) \ll 1 \Rightarrow I_R(\exists t) < 0$$

4. weak solution (Senba-S. 02a)

$0 \le \mu(dx,t) \in C_*([0,T],\mathcal{M}(\overline{\Omega}))$
$\mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})'$ weak solution to

$$u_t = \nabla\cdot(\nabla u - u\nabla v),\ -\Delta v = u$$
$$\left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = v|_{\partial\Omega} = 0$$
$\Leftrightarrow$ (def)

(1) $\forall \varphi \in Y$
$t \in [0,T] \mapsto \langle \varphi, \mu(dx,t)\rangle$ a.c.

$$\frac{d}{dt}\langle\varphi,\mu(dx,t)\rangle = \langle\Delta\varphi,\mu(dx,t)\rangle + \frac{1}{2}\langle\rho_\varphi,\nu(t)\rangle_{\mathcal{X},\mathcal{X}'} \text{ a.e. } t$$

$$Y = \left\{\varphi \in C^2(\overline{\Omega}) \mid \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0\right\}$$

$$\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x')$$

slg

$0 \le \exists\nu(\cdot,t) \in L^\infty(0,T;\mathcal{X}')$
multiplicate operator

$$\mathcal{X} = \left[\{\rho_\varphi + \psi \mid \varphi \in Y, \psi \in C(\overline{\Omega}\times\overline{\Omega})\}\right] \hookrightarrow L^\infty(\Omega\times\Omega)$$

(2) (linking)
$\nu(\cdot,t)|_{C(\overline{\Omega}\times\overline{\Omega})} = \mu(dx,t)\otimes\mu(dx',t)$
a.e. t

Remark 1 (positivity)
$0 \le \nu \in \mathcal{X}' \Leftrightarrow$ (def)
$\forall f,g \in \mathcal{X},\ |f| \le g$
$|\langle f,\nu\rangle_{\mathcal{X},\mathcal{X}'}| \le \langle g,\nu\rangle_{\mathcal{X},\mathcal{X}'}$

c.f. Poupaud 02, Dolbeaut-Schmeiser 09
Luckhaus-Sugiyama-Velazquez 12

$0 \le \mu(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$
weak solution

1 (total mass conservation)
$\|\mu(\cdot,t)\|_{\mathcal{M}} = \langle 1, \mu(dx,t)\rangle = \lambda$
2 (monotonicity formula) $\forall \varphi \in Y$
$$\left|\frac{d}{dt}\langle \varphi, \mu(dx,t)\rangle\right| \le C(\lambda + \|\nu\|_{\infty,\mathcal{X}'})\|\nabla\varphi\|_{C^1}$$

positivity + linking of the multi-plicate operator

$\varphi = |x-x_0|^2\varphi_{x_0,R}$
$$\frac{dI_R}{dt} \le J_R(0) + a(R^{-1}t^{1/2})$$
$+CR^{-1}I_{2R}(t)^{1/2}$, $R \downarrow 0$

Proposition 1

$\exists x_0 \in \Omega$, $\mu(\{x_0\},0) > 8\pi$
$\langle |x-x_0|^2\varphi_{x_0,R}, \mu(dx,0)\rangle = o(R^2)$, $R \downarrow 0$
$\Rightarrow T = 0$

$0 \le \nu(\cdot,t) \in L^\infty(0,T;\mathcal{X}')$
multiplicate operator

Proposition 2

$0 \le \mu_k(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$
weak solutions
$$\sup_k \left[\|\mu_k(\cdot,0)\|_{\mathcal{M}} + \boxed{\sup_{t\in[0,T]} \|\nu_k(\cdot,t)\|_{\mathcal{X}'}}\right]$$
$< +\infty \Rightarrow \exists$ sub-sequence
weakly converging to a weak solution

Remark 2

1. $\mathcal{X} \subset L^\infty(\Omega\times\Omega)$ separable
2. $u = u(x,t) \ge 0$ classical solution

$\Rightarrow$
$\mu(dx,t) = u(x,t)dx$
$\nu(\cdot,t) = u(\cdot,t)\otimes u(\cdot,t) \in L^1(\Omega\times\Omega)$
$\|\nu(\cdot,t)\|_{\mathcal{X}'} = \lambda^2$, $\lambda = \|u_0\|_1$

4. scaling limit

$x_0 \in \mathcal{S}$

1) backward self-similar transformation
$y = (x-x_0)/(T-t)^{1/2}$
$s = -\log(T-t)$
$z(y,s) = (T-t)u(x,t)$

$y \in (T-t)^{-1/2}(\Omega - \{x_0\}) = \Omega_s$
$-\log T \le s < +\infty$, $\|z(\cdot,s)\|_1 = \lambda$

$z_s = \nabla\cdot(\nabla z - z\nabla(w + |y|^2/4))$
$$\left.\frac{\partial z}{\partial\nu} - z\frac{\partial}{\partial\nu}(w + |y|^2/4)\right|_{\partial\Omega_s} = 0$$

$$w(\cdot,s) = \int_{\Omega_s} G_s(\cdot,y')z(y',s)dy'$$
$\boxed{G_s(y,y') = G(x,x')}$

2) weak form, $x_0 \in \Omega$
$\varphi \in C_0^2(\mathbf{R}^2)$, $s \gg 1$
$$\frac{d}{ds}\int_{\mathcal{O}_s} z\varphi = \int_{\mathcal{O}_s}(\partial_s\varphi + y\cdot\nabla\varphi + \Delta\varphi)z$$
$$+\frac{1}{2}\int_{\mathcal{O}_s\times\mathcal{O}_s}\rho_\varphi^s(y,y')z\otimes z$$

$\mathcal{O}_s = \Omega_s \times \{s\}$
$\rho_\varphi^s(y,y') = \boxed{\nabla\varphi(y)\cdot\nabla_y G_s(y,y')}$
$+\nabla\varphi(y')\cdot\nabla_{y'}G_s(y,y')$

$G(x,x') = \Gamma(x-x') + K(x,x')$
$(x,x') \in (\overline{\Omega}\times\Omega) \cup \boxed{(\Omega\times\overline{\Omega})}$
$$\Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$G_s(y,y') = \boxed{\Gamma(y-y')} - \frac{s}{4\pi}$$
$+K(e^{-s}y + x_0, e^{-s}y' + x_0)$

3) weak form, $x_0 \in \partial\Omega$

$X = X_{x_0} : \overline{\Omega \cap B(x_0, 2R)} \to \overline{\mathbf{R}^2_+}$
$X(x_0) = 0$, conformal diffeo. into

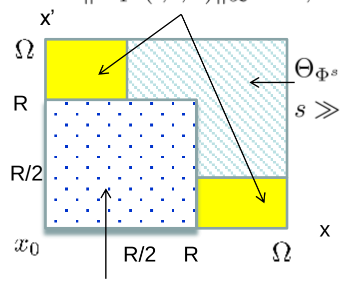
$$\boxed{\begin{array}{l} \varphi \in C_0^2(\overline{\mathbf{R}^2_+}), \ \left. \dfrac{\partial \varphi}{\partial \nu} \right|_{\partial \mathbf{R}^2_+} = 0 \\ \Phi^s = \varphi \circ Y_s, \ s \gg 1 \\ Y_s(y) = e^{s/2} X(e^{-s/2} y + x_0) \end{array}}$$

$$\frac{d}{ds} \int_{\mathcal{O}_s} z \Phi^s =$$
$$\int_{\mathcal{O}_s} (\partial_s \Phi^s + y \cdot \nabla \Phi^s / 2 + \Delta \Phi^s) z$$
$$+ \frac{1}{2} \int_{\mathcal{O}_s \times \mathcal{O}_s} \Theta_{\Phi^s}(y, y', s) z \otimes z$$

Liouville formula
pre-scaled variables

$$\Theta_{\Phi^s}(y, y', s) = \nabla \Phi^s(y) \cdot \nabla_y G_s(y, y')$$
$$+ \nabla \Phi^s(y') \cdot \nabla_{y'} G_s(y, y')$$

$G \in C^{2+\theta}(\overline{\Omega} \times \overline{\Omega} \setminus D)$
$\|\Theta_{\Phi^s}(\cdot, \cdot, s)\|_\infty \to 0, \ s \uparrow +\infty$



$\Theta_{\Phi^s}(\cdot, \cdot, s) = 0$
$s \gg 1$

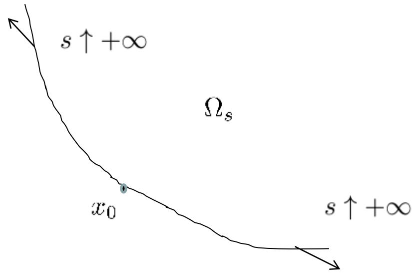
$$G_s(y, y') = E_s(y, y')$$
$$+ K(e^{-s} y + x_0, e^{-s} y' + x_0)$$

$$E_s(y, y')$$
$$= \boxed{\Gamma(Y_s - Y_s') - \Gamma(Y_s - Y_{s*}')}$$

4) zero-extension, diagonal argument

$\forall s_k \uparrow +\infty, \ \exists \{s_k'\} \subset \{s_k\}$
$z(y, s + s_k') dy \rightharpoonup \exists \zeta(dy, s)$
in $C_*(-\infty, +\infty; \mathcal{M}_0(\mathbf{R}^2))$

$\mathcal{M}_0(\mathbf{R}^2) = C_0(\mathbf{R}^2)'$
$C_0(\mathbf{R}^2)$
$= \{ f \in C(\mathbf{R}^2 \cup \{\infty\}) \mid f(\infty) = 0 \}$



$$\boxed{\begin{array}{l} 0 \le \zeta(dy, s) \text{ finite measure} \\ \zeta(\mathbf{R}^2, s) \le \lambda \equiv \|u_0\|_1, \ \forall s \\ \text{supp } \zeta(dy, s) \subset \overline{\mathbf{R}^2_+} \ (\text{if } x_0 \in \partial\Omega) \end{array}}$$

weak solution to
$\zeta_s = \nabla \cdot (\nabla \zeta - \zeta \nabla (F * \zeta + |y|^2/4))$
in $\mathbf{R}^2 \times (-\infty, +\infty)$

$$(F * \zeta)(\cdot, s) = \langle F(\cdot, y'), \zeta(dy', s) \rangle$$

$$F(y, y') = \begin{cases} \Gamma(y - y'), & x_0 \in \Omega \\ E(y, y'), & x_0 \in \partial\Omega \end{cases}$$

$$E(y, y') = \Gamma(y - y') - \Gamma(y - y_*')$$

test functions
$\varphi \in W \Leftrightarrow (\text{def}) \ \varphi \in C_0^2(\mathbf{R}^2)$
$\left. \dfrac{\partial \varphi}{\partial \nu} \right|_{\partial \mathbf{R}^2_+} = 0 \ (\text{if } x_0 \in \partial\Omega)$

$$\boxed{\begin{aligned}&0 \le \zeta(dy,s) \in C_*(-\infty,+\infty;\mathcal{M}_0(\mathbf{R}^2))\\ &\zeta(\mathbf{R}^2,s) \le \lambda \equiv \|u_0\|_1,\ \forall s\\ &\text{supp } \zeta(dy,s) \subset \overline{\mathbf{R}^2_+}\ (\text{if } x_0 \in \partial\Omega)\end{aligned}}$$

(1) $\forall \varphi \in W$

$s \in (-\infty,\infty) \mapsto \langle \varphi, \zeta(dy,s)\rangle$ loc. a.c.

$$\frac{d}{ds}\langle \varphi, \zeta(dy,s)\rangle$$
$$= \langle \Delta\varphi + \frac{y}{2}\cdot\nabla\varphi, \zeta(dy,s)\rangle$$
$$+\frac{1}{2}\langle \rho^0_\varphi, \mathcal{K}(s)\rangle_{\mathcal{E},\mathcal{E}'} \text{ a.e. } s$$

$0 \le \exists \mathcal{K} = \mathcal{K}(\cdot,s) \in \mathcal{E}'$

$\|\mathcal{K}(\cdot,s)\|_{\mathcal{E}'} \le \lambda^2$ a.e. s

multi-plicate operator

$$\mathcal{E} = [\{\rho^0_\varphi + \psi \mid \varphi \in W, \psi \in Z\}]$$
$$\hookrightarrow L^\infty(\mathbf{R}^2\times\mathbf{R}^2)$$

$$\rho^0_\varphi(y,y') = \nabla\varphi(y)\cdot\nabla_y F(y,y')$$
$$+\nabla\varphi(y')\cdot\nabla_{y'}F(y,y')$$

slg

$$F(y,y') = \begin{cases} \Gamma(y-y'), & x_0 \in \Omega \\ \Gamma(y-y') - \Gamma(y-y'_*), & x_0 \in \partial\Omega \end{cases}$$

$$Z = C_0(\mathbf{R}^2\times\mathbf{R}^2) \oplus [(C_0(\mathbf{R}^2)\oplus\mathbf{R})\otimes\mathbf{R}]$$
$$\oplus[\mathbf{R}\otimes(C_0(\mathbf{R}^2)\oplus\mathbf{R})]$$

$C_0(\mathbf{R}^2\times\mathbf{R}^2)$ .. continuous functions on $(\mathbf{R}^2\cup\{\infty\})\times(\mathbf{R}^2\cup\{\infty\})$ vanishing at $[(\mathbf{R}^2\cup\{\infty\})\times\{\infty\}] \bigcup [\{\infty\}\times(\mathbf{R}^2\cup\{\infty\})]$

(2) (linking)

$\mathcal{K}(\cdot,s)|_Z = \zeta(dy,s)\otimes\zeta(dy',s)$ a.e.s 9/17

5. parabolic envelope $x_0 \in \mathcal{S}$

1) total mass, $\boxed{0 < R \le 1}$

$$\left|\frac{d}{dt}\int_\Omega u(\cdot,t)\varphi_{x_0,R}\right| \le C_\lambda R^{-2}, \quad \int_t^T dt$$

$$\boxed{\begin{aligned}&|\langle \varphi_{x_0,R}, u(\cdot,t)dx\rangle - \langle \varphi_{x_0,R}, \mu(dx,T)\rangle|\\ &\le C_\lambda(T-t)/R^2\end{aligned}}$$

$s \in \mathbf{R}$, $b > 0$ fixed, $k \gg 1$

$s'_k + s = -\log(T-t),\ R = b(T-t)^{1/2}$

$$|\langle \varphi_{0,b}, z(\cdot, s+s'_k)dy\rangle$$
$$-\langle \varphi_{x_0,be^{-(s+s'_k)/2}}, \mu(dx,T)\rangle| \le C_\lambda/b^2$$

$k \to \infty$

$$\mu(\cdot,T) = \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

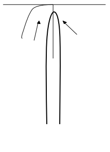
$$|\langle \varphi_{0,b}, \zeta(dy,s)\rangle - m(x_0)| \le C_\lambda/b^2, \quad b\uparrow+\infty$$
$$m(x_0) = \zeta(\mathbf{R}^2,s),\ -\infty < s < +\infty$$

2) second moment:

$$\left|\frac{d}{dt}\int_\Omega |X|^2\varphi_{x_0,R}u\right| \le C$$
$$X = \begin{cases} x - x_0, & x_0 \in \Omega \\ X_{x_0}(x), & x_0 \in \partial\Omega \end{cases}$$

$\Rightarrow$

$$\langle |y|^2, \zeta(dy,s)\rangle \le C,\ -\infty < s < +\infty$$



$t = T$

Parabolic envelope .. Infinitely wide parabolic region

pre-scaled collapse mass = total mass of the weak scaling limit

uniformly bounded total second moment of the limit measure 10/17

6. boundary blowup exclusion

$x_0 \in \partial\Omega$, supp $\zeta(dy,s) \subset \overline{\mathbf{R}^2_+}$

$0 \leq \zeta = \zeta(dy,s) \in C_*(-\infty,+\infty;\mathcal{M}_0(\mathbf{R}^2))$

$\boxed{\zeta(\mathbf{R}^2,s) = m(x_0)}$

(1) $\varphi \in W \Leftrightarrow \varphi \in C^2_0(\mathbf{R}^2),\ \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\mathbf{R}^2_+} = 0$

$$[\langle \varphi, \zeta(dy,s)\rangle]_{s=s_1}^{s=s_2} = \int_{s_1}^{s_2} \boxed{\langle \Delta\varphi + \frac{y}{2}\cdot\nabla\varphi, \zeta(dy,s)\rangle} + \frac{1}{2}\boxed{\langle \rho^0_\varphi, \mathcal{K}(s)\rangle_{\mathcal{E},\mathcal{E}'}}\ ds$$

$0 \leq \mathcal{K}(\cdot,s), \quad \|\mathcal{K}(\cdot,s)\|_{\mathcal{E}'} \leq \lambda^2$ mo

$\rho^0_\varphi(y,y') = \nabla\varphi(y)\cdot\nabla_y F(y,y') + \nabla\varphi(y')\cdot\nabla_{y'}F(y,y')$ slg

$F(y,y') = \Gamma(y-y') - \Gamma(y-y'_*)$

(2) (linking)

$\mathcal{K}(\cdot,s)|_Z = \zeta(dy,s)\otimes\zeta(dy',s)$ a.e.s

1) measure part

$\varphi = |y|^2\psi_R,\ \psi_R(y) = \psi(y/R)$

$\psi = \varphi_{0,2}(|y|)$

$\Delta\varphi = 4\psi_R + 4\frac{y}{R}\cdot\nabla\psi(\frac{y}{R}) + \frac{|y|^2}{R^2}\Delta\psi(\frac{y}{R})$

$y\cdot\nabla\varphi = 2|y|^2\psi_R + |y|^2\frac{y}{R}\cdot\nabla\psi(\frac{y}{R})$

$\langle 1+|y|^2, \zeta(dy,s)\rangle \leq C$

$\Rightarrow$ (dominated convergence theorem)

$$\lim_{R\uparrow+\infty}\int_{s_1}^{s_2}\langle \Delta\varphi + \frac{y}{2}\cdot\nabla\varphi, \zeta(dy,s)\rangle = 4(s_2-s_1)m(x_0) + \int_{s_1}^{s_2} I(s)ds$$

$\boxed{I(s) = \langle |y|^2, \zeta(dy,s)\rangle}$

2) multi-plicate part

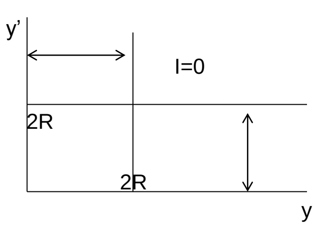
$\rho^0_\varphi(y,y') = \nabla\varphi(y)\cdot\nabla_y F(y,y') + \nabla\varphi(y')\cdot\nabla_{y'}F(y,y')$

$= I + II + III,\ \varphi = |y|^2\psi_R$

$I = \psi_R(y)\nabla|y|^2\cdot\nabla_y F(y,y') + \psi_R(y')\nabla|y'|^2\cdot\nabla_{y'}F(y,y')$

$II = (|y|^2 - |y'|^2)\nabla\psi_R(y)\cdot\nabla_y F(y,y')$

$III = |y'|^2(\nabla\psi_R(y) - \nabla\psi_R(y'))\cdot\nabla_{y'}F(y,y')$



$\boxed{\rho^0_{|y|^2}(y,y') = 0} \Rightarrow$

$I = (\psi_R(y) - \psi_R(y'))\nabla|y|^2\cdot\nabla_y F(y,y')$

$|y| < 2R \Rightarrow$

$|I| \leq \|\nabla\psi\|_\infty \cdot \frac{|y-y'|}{R}\cdot 2|y|\cdot\frac{1}{\pi|y-y'|}$

$\leq \frac{2}{\pi}\|\nabla\psi\|_\infty\varphi_{0,4R}(y)\cdot\frac{|y|}{R}$

$|I| \leq C\left(\varphi_{0,4R}(y)\frac{|y|}{R} + \varphi_{0,4R}(y')\frac{|y'|}{R}\right)$

$y, y' \in \mathbf{R}^2$

$0 \leq \varphi_{0,4R}(y)\frac{|y|}{R} + \varphi_{0,4R}(y')\frac{|y'|}{R} \leq C$

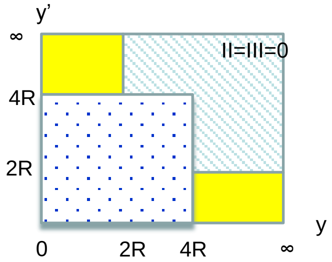
$\to 0$

$\forall y, y'$ as $R\uparrow+\infty$

$$|II| + |III| \leq CH_R(y, y'), \quad y, y' \in \mathbf{R}^2$$

$$0 \leq H_R(y,y') =$$
$$(\varphi_{0,8R}(y)(1+|y|) + \varphi_{0,8R}(y')(1+|y'|))$$
$$\cdot \left[\frac{|y|}{R} + \frac{|y|^2}{R^2} + \frac{|y'|}{R} + \frac{|y'|^2}{R^2} \right]$$
$$\leq C(1+|y|^2)(1+|y'|^2)$$

$$\lim_{R\uparrow+\infty} H_R(y,y') = 0, \quad \forall y, y' \in \mathbf{R}^2$$



1. positivity → linking

2. dominated convergence theorem

$$\langle 1+|y|^2, \zeta(dy,s) \rangle \leq C$$

$$\lim_{R\uparrow+\infty} \int_{s_1}^{s_2} \langle \rho^0_\varphi(y,y'), \mathcal{K}(\cdot,s) \rangle_{\mathcal{E},\mathcal{E}'} \, ds = 0$$

3)

$$I(s_2) - I(s_1) = \int_{s_1}^{s_2} 4m(x_0) + I(s) \ ds$$
$$I(s) = \langle |y|^2, \zeta(dy,s) \rangle \leq C$$
$$\frac{dI}{ds} = 4m(x_0) + I(s) \ \text{a.e.} \ s$$
$$\Rightarrow$$
$$\lim_{R\uparrow+\infty} I(s) = +\infty, \ \text{a contradiction}$$

7. interior blowup control

1. estimate from below

$$x_0 \in \Omega \Rightarrow F = \Gamma(y - y')$$
$$\rho^0_{|y|^2}(y,y') = -\frac{1}{2\pi}$$
$$\frac{dI}{ds} = \boxed{4m(x_0) - \frac{m(x_0)^2}{2\pi}} + I(s)$$
$$4m(x_0) \leq \frac{m(x_0)^2}{2\pi} \Rightarrow m(x_0) \geq 8\pi$$

2. scaling back

$$\zeta(dy,s) = e^{-s} A(dy', s')$$
$$y' = e^{-s/2} y, \ s' = -e^{-s}$$

$$A_s = \nabla \cdot (\nabla A - A \nabla \Gamma * A)$$
$$A = A(dy,s) \geq 0, \ (y,s) \in \mathbf{R}^2 \times (-\infty, 0)$$
$$A(\mathbf{R}^2, s) = m(x_0)$$

weak solution with uniformly bounded (λ^2) multi-plicate operator

3. translation limit

$$\forall s_k \uparrow +\infty, \ \exists \{s'_k\} \subset \{s_k\}$$
$$A(dy, s - s'_k) \rightharpoonup a(dy,s)$$
$$\text{in } C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2))$$

$$a(\mathbf{R}^2, s) = \langle 1, a(dy,s) \rangle$$
$$= \lim_{k\to\infty} \langle 1, A(dy, s - s'_k) \rangle = m(x_0)$$

$\mathcal{M}(\mathbf{R}^2) = [C_0(\mathbf{R}^2) \oplus \mathbf{R}]'$ envelopes the total scaling mass

$$a_s = \nabla \cdot (\nabla a - a \nabla \Gamma * a)$$
$$a(dy,s) \geq 0, \ (y,s) \in \mathbf{R}^2 \times (-\infty, +\infty)$$
$$a(\mathbf{R}^2, s) = m(x_0)$$

scaling argument valid to the weak solution without the total second moment convergence

$0 \le a = a(dy, s)$
$\in C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2))$
$\boxed{a(\mathbf{R}^2, s) = m(x_0)}$

$0 \le \exists \mathcal{K} = \mathcal{K}(\cdot, s),\ \|\mathcal{K}(\cdot, s)\|_{\mathcal{E}'} \le \lambda^2$
$\mathcal{K}(\cdot, s)|_Z = a(dy, s) \otimes a(dy', s)$ a.e. s

$\forall \varphi \in W \oplus \mathbf{R}$
$s \in (-\infty, +\infty) \mapsto \langle \varphi, a(dy, s) \rangle$ loc. a.c.

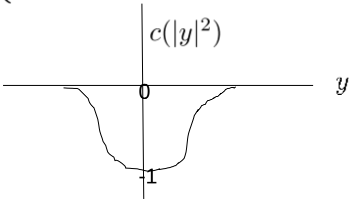
$$\frac{d}{ds}\langle \varphi, a(dy, s)\rangle = \langle \Delta\varphi, a(dy, s)\rangle$$
$$+\frac{1}{2}\langle \rho_\varphi^0, \mathcal{K}(s)\rangle_{\mathcal{E},\mathcal{E}'} \text{ a.e. } s$$

$$\rho_\varphi^0(y, y') = \frac{\nabla\varphi(y) - \nabla\varphi(y')}{2\pi} \cdot (y - y')$$

4. local second moment

$0 \le c'(t) \le 1,\ t \ge 0$
$-1 \le c(t) \le 0,\ t \ge 0$

$$c(t) = \begin{cases} t - 1, & 0 \le t \le 1/4 \\ 0, & t \ge 4 \end{cases}$$



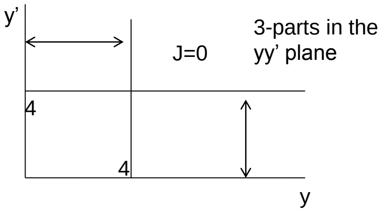
$$\frac{d}{ds}\langle c(|y|^2) + 1, a(dy, s)\rangle$$
$$= \langle 4c''(|y|^2)|y|^2 + 4c'(|y|^2)$$
$$-\boxed{\frac{m(x_0)}{2\pi}}c'(|y|^2), a(dy, s)\rangle - \langle J, \mathcal{K}(s)\rangle_{\mathcal{E},\mathcal{E}'}$$

$$\boxed{\begin{aligned} &J = J(y, y') \\ &= \frac{(c'(|y|^2) - c'(|y'|^2))(|y|^2 - |y'|^2)}{4\pi|y - y'|^2} \end{aligned}}$$

$$\frac{d}{ds}\langle c(|y|^2) + 1, a(dy, s)\rangle$$
$$= \langle 4c''(|y|^2)|y|^2 + 4c'(|y|^2)$$
$$-\frac{m(x_0)}{2\pi}c'(|y|^2), a(dy, s)\rangle - \langle J, \mathcal{K}(s)\rangle_{\mathcal{E},\mathcal{E}'}$$

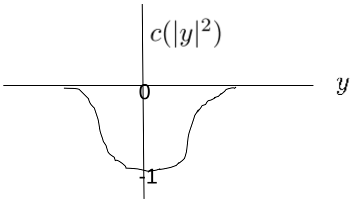
$$J = J(y, y')$$
$$= \frac{(c'(|y|^2) - c'(|y'|^2))(|y|^2 - |y'|^2)}{4\pi|y - y'|^2}$$

$$|J| \le C_1(\varphi_{0,8}(y) + \varphi_{0,8}(y'))$$
$$\cdot \left\{(c(|y|^2) + 1) + (c(|y'|^2) + 1)\right\}$$



$0 \le c'(t) \le 1,\ t \ge 0$
$-1 \le c(t) \le 0,\ t \ge 0$

$$c(t) = \begin{cases} t - 1, & 0 \le t \le 1/4 \\ 0, & t \ge 4 \end{cases}$$



$4c''(t)t \le C_2(c(t) + 1)$

$$\boxed{\begin{aligned} &\frac{d}{ds}\langle c(|y|^2) + 1, a(dy, s)\rangle \\ &\le C_3\langle c(|y|^2) + 1, a(dy, s)\rangle \\ &+(4 - \frac{m(x_0)}{2\pi})\langle c'(|y|^2), a(dy, s)\rangle \end{aligned}}$$

$c'(0) = 1 \Rightarrow$
$c(t) + 1 + c'(t) \ge \exists\delta > 0$

$m(x_0) > 8\pi$

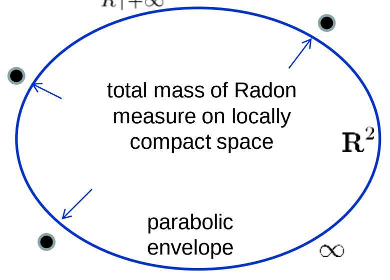
$$\Rightarrow \quad \frac{d}{ds}\langle c(|y|^2)+1, a(dy,s)\rangle \leq C_4\langle c(|y|^2)+1, a(dy,s)\rangle + \delta m(x_0)\left\{4 - \frac{m(x_0)}{2\pi}\right\}$$

$$\Rightarrow$$

$$\boxed{\langle c(|y|^2)+1, a(dy,0)\rangle \geq \eta > 0}$$

$$\eta = \delta m(x_0)\left\{\frac{m(x_0)}{2\pi} - 4\right\}/C_4$$

$$a(\mathbf{R}^2, s) = \lim_{R\uparrow+\infty} a(B_R, s) = m(x_0)$$



5. scaling invariance

$$a(y,s) \mapsto a_\mu(y,s) = \mu^2 a(\mu y, \mu^2 s)$$

$$\langle c(|y|^2)+1, a^\mu(dy,0)\rangle \geq \eta,\ \mu > 0$$

$$\Rightarrow$$

$$\langle c(\mu^{-2}|y|^2)+1, a(dy,0)\rangle \geq \eta$$

$$0 \leq c(\mu^{-2}|y|^2)+1 \leq 1$$

$$\forall y \in \mathbf{R}^2,\ c(\mu^{-2}|y|^2)+1 \to 0,\ \mu\uparrow+\infty$$

$$\Rightarrow \text{(dominated convergence theorem)}$$

$$0 \geq \eta, \text{ contradiction, } m(x_0) \leq 8\pi$$

(concentration – cancellation)

6. formation of sub-collapse

$$m(x_0) = 8\pi \Rightarrow \frac{dI}{ds} = I(s),\ I(s) \leq C$$

$$I(s) \equiv \langle |y|^2, \zeta(dy,s)\rangle = 0$$

$$\zeta(dy,s) = 8\pi\delta_0(dy)$$

17/17

1. Senba-S. 01	weak formulation monotonicity formula	formation of collapse
2. Senba-S. 02a	weak solution	weak solution generation instant blowup for over mass concentrated initial data
3. Kurokiba-Ogawa 03	scaling invariance	non-existence of over mass entire solution without concentration
4. S. 05	backward self-similar transformation scaling limit parabolic envelope (1) scaling invariance of the scaling limit a local second moment	collapse mass quantization
5. Senba 07 Naito-S. 08	parabolic envelope (2)	type II blowup rate formation of sub-collapse
6. S. 08	scaling back	limit equation simplification
7. Senba-S. 11	translation limit	concentration-cancelation simplification
8. Espejo-Stevens-S. 12	quantization without blowup threshold	simultaneous blowup mass separation for systems

chemotaxis system mathematics of self-attractive Smoluchowski-Poisson equation

1. Nagai-Senba-Yoshida 97, Biler 98, Gajewski-Zacharias 98 global-in-time existence
2. Biler-Hilhorst-Nadieja 94, Nagai 95, Nagai 01, Senba-S. 02b blowup in finite time

critical mass - Neumann case

$$u_t = \nabla\cdot(\nabla u - u\nabla v)$$
$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \ \int_\Omega v = 0$$

Ohtsuka-Senba-S. 07

$$\|u_0\|_1 = 4\pi \Rightarrow T_{\max} = +\infty$$
$$t_k \uparrow +\infty, \ \lim_{k\to\infty} \|u(t_k)\|_\infty = +\infty$$
$$\Rightarrow$$
$$\exists \{t_k'\} \subset \{t_k\}, \ u(x, t+t_k')dx \rightharpoonup 4\pi\delta_{x(t)}dx$$
$$t \in (-\infty, +\infty) \mapsto x(t) \in \partial\Omega$$
$$\frac{dx}{dt} = 2\pi\nabla R(x)$$

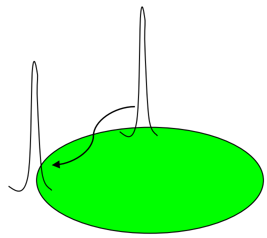
collapse formed in infinite time at a local maximum point of R on the boundary and then to that of a local minimum

Conjecture

Mass quantization of the blowup in infinite time:

$$T = +\infty, \ \lim_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = +\infty$$
$$\Rightarrow$$
$$\lambda \in 4\pi\mathbf{N}$$

Hamiltonian controls multiple collapse motion formed in infinite time



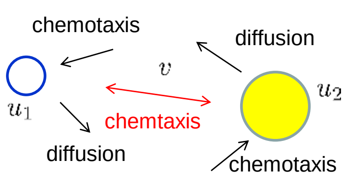
key pint of the proof for the Dirichlet case

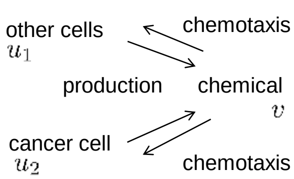
ε-regularity

Other Multi-Component Systems

DD model (hetero-separative, homo-aggregative type)
Kurokiba-Ogawa 03
Espejo-Stevens-Velazquez 10

$$u_{1t} = d\Delta u_1 - \chi\nabla\cdot u_1\nabla v$$
$$u_{2t} = d\Delta u_2 + \chi\nabla\cdot u_2\nabla v$$
$$-\Delta v = u_1 - u_2 \text{ in } \mathbf{R}^2\times(0,T)$$
$$u_i|_{t=0} = u_{i0}(x) \geq 0, \ i = 1,2$$



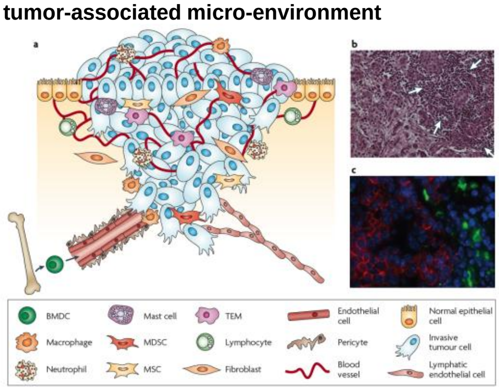


competitive system of chemotaxis (hetero-homo-aggregative)
Espejo-Stevens-Velazquez 09
Espejo-Stevens-S. 12

simultaneous blowup
collapse mass separation

$$u_{it} = d_i\Delta u_i - \chi_i\nabla\cdot u_i\nabla v \qquad -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$
$$d_i\frac{\partial u_i}{\partial \nu} - \chi_i u_i\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0 \qquad \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$
$$u_i|_{t=0} = u_{i0}(x) \geq 0 \qquad \int_\Omega v = 0, \ u = \sum_{i=1}^N u_i$$
$$i = 1, 2, \cdots, N$$

tumor-associated micro-environment



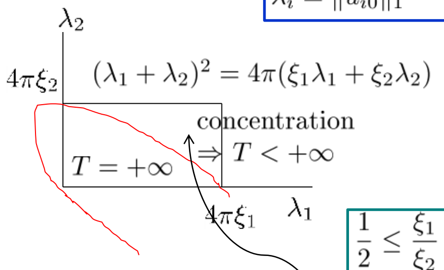
Competitive System

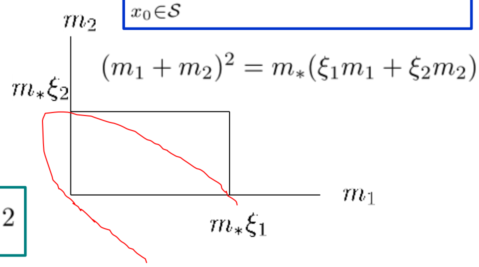
[blowup threshold]

$$\boxed{\begin{array}{l} N=2,\ \xi_i=d_i/\chi_i \\ m_*=m_*(x_0) \\ \lambda_i=\|u_{i0}\|_1 \end{array}}$$

[collapse mass]

$$\boxed{u_i(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m_i(x_0)\delta_{x_0}(dx)+f_i(x)dx}$$



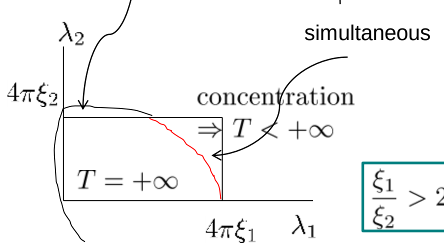


$$\boxed{\frac{1}{2}\le\frac{\xi_1}{\xi_2}\le 2}$$

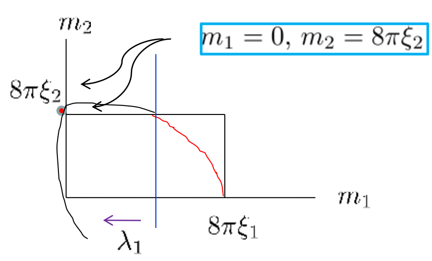
unknown except for radial case (blowup)

[mass separation for radial case]

simultaneous



$$\boxed{\frac{\xi_1}{\xi_2}>2}$$



cross chemotaxis sytstem

$$\partial_t u_1=d_1\Delta u_1-\chi_1\nabla\cdot u_1\nabla v_2$$

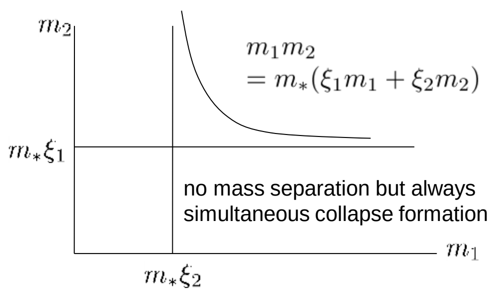
$$\partial_t u_2=d_2\Delta u_2-\chi_2\nabla\cdot u_2\nabla v_1$$

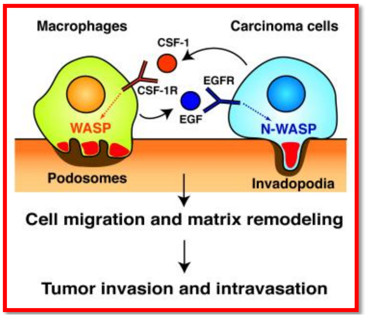
$$\left. d_1\frac{\partial u_1}{\partial\nu}-\chi_1 u_1\frac{\partial v_2}{\partial\nu}\right|_{\partial\Omega}=0$$

$$\left. d_2\frac{\partial u_2}{\partial\nu}-\chi_2 u_2\frac{\partial v_1}{\partial\nu}\right|_{\partial\Omega}=0$$

$$\left. u_i\right|_{t=0}=u_{i0}(x)\ge 0,\ \int_\Omega v_i\ dx=0$$

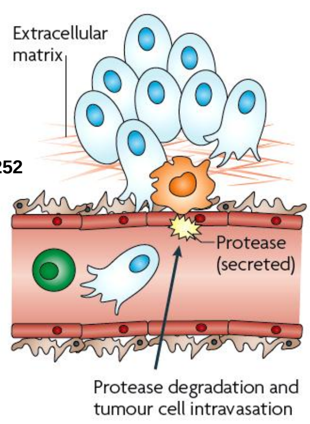
$$-\Delta v_i=u_i-\frac{1}{|\Omega|}\int_\Omega v_i,\ \left.\frac{\partial v_i}{\partial\nu}\right|_{\partial\Omega}=0,\ i=1,2$$





H. Yamaguchi et al. *Eur J Cell Biol* 85: 213-218 (2006)

J. Joyce, and J. Pollard. *Nat Rev Cancer* 9: 239-252 (2009)



Summary

1. Dirichlet boundary condition is used for the Poisson part in a model of statistical mechanics concerning the movement of self-interacting particles.

2. Here we studied Sire-Chavanis' model on self-gravitating Brownian particles in two-space dimension.

3. There is still a quantized blowup mechanism without collision because of the long-range interaction described by the Green's function of the Poisson part.

4. We have the formation of collapses with quantized mass and type II blowup rates as a result of the formation of sub-collapses, besides the exclusion of the boundary blowup.

5. A new argument guarantees the blowup threshold without the Trudinger-Moser inequality, that is, the use of two different weak limit equations, the scaling and translation.

6. Exclusion of boundary blowup, however, is available only to the competitive and cross chemotactic cases, for multi-component systems involved by the Dirichlet boundary condition in the Poisson part.

References

1. C. Sire and P.-H. Chavanis, Thermodynamics and collapse of self-gravitating Brownian particlse in \$D\$-dimensions, Phys. Rev. E 66 (2002) 046133

2. N.I. Kavallaris and P. Souplet, Grow-up rate and asymptotics for a two-dimensional Patlak-Keller-Segel model in a disc, SIAM J. Math. Anal. 41 (2009) 128-157

3. S., Exclusion of boundary blowup for 2D chemotaxis system provided with Dirichlet boundary condition for the Poisson part, preprint

4. E.E. Espejo, A. Stevens and S., Simultaneous blowup and mass separation during collapse in an interacting system of chemotaxis, Differential and Integral Equations 25 (2012) 251-288

5. T. Senba and S., Applied Analysis – Mathematical Methods in Natural Science, second edition, Imperial College Press, London, 2011

6. S. Mean Field Theories and Dual Variation, Atlantis Press, Amsterdam-Paris, 2008

7. S. Free Energy and Self-Interacting Particles, Birkhauser, Boston, 2005