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Abstract. The validity of singular perturbation expansion of the European option prices for a general fast mean-reverting stochastic volatility model is proved in the light of the Edgeworth expansion for ergodic diffusions. The Edgeworth expansion is validated for a broad class of triangular arrays of regenerative functionals. The validation procedure does not depend on the stationarity nor the geometric mixing property. The Heston model is treated as an example.

Keywords: fast mean reverting; Edgeworth expansion; implied volatility.

1. Introduction

The stochastic volatility model is a continuous-time limit of the ARCH model and a reasonable generalization of the Black-Scholes model. The plausible advantage of this model is the fact that it explains an empirical phenomenon known as the volatility smile or skew. See e.g., Heston[15], Hull and White [16], Renault and Touzi [25] for the detail. Unfortunately, no simple option pricing formula has been available in a general stochastic volatility model (see Heston [15], Nicolato and Venardos [22] for exceptional cases), which apparently causes difficulties in practical use. To overcome such a disadvantage, an asymptotic expansion is one of standard approaches. For example, Yoshida [27] established a small diffusion expansion and his method was utilized by Osajima [23] to validate an asymptotic expansion formula for the SABR model developed by Hagan, Kumar, Lesniewski and Woodward [13]. Masuda and Yoshida [21] applied the Edgeworth expansion for geometric mixing processes to a stochastic volatility model. Introducing a two time scales model, namely, the so-called fast mean-reverting stochastic volatility model, Fouque, Papanicolaou and Sircar [8] gave a singular perturbation expansion and its applications. Fouque, Papanicolaou, Sircar and Solna [10] extended this to a multiscale model. The validity of the singular perturbation expansion was proved by Fouque, Papanicolaou, Sircar and Solna [9], Conlon and Sullivan [5] for the European call option price in a model based on the Ornstein-Uhlenbeck process. Khasminskii and Yin [18] proved the validity in case that the volatility is an ergodic diffusion on a compact set. The aim of this article is to prove the validity for a broader class of ergodic diffusions. In particular, we incorporate the Heston model, which is beyond the preceding studies. Besides, we deal with the European option prices with a general payoff function including digital options.

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Our approach is quite different to the preceding studies; we exploit the Edgeworth expansion for ergodic diffusions.

The Edgeworth expansion is a refinement of the central limit theorem and has played an important role in statistics. There are three approaches to validate the Edgeworth expansion for ergodic continuous-time processes. Martingale approach and mixing approach developed by Yoshida [28] and Yoshida [29] respectively are widely applicable to general continuous-time processes. Regenerative approach developed by Fukasawa [12] is applicable only to strong Markov processes but requires a weaker condition of ergodicity. We exploit the last one because empirical studies such as Andersen, Bollerslev, Diebold and Labys [1] showed that the volatility is "very slowly mean-reverting", that is, the autocorrelation function decays slowly. As a result, we do not need to require that the volatility is geometrically mixing. This article extends the results of Fukasawa [12] to triangular arrays of additive functionals. This generalization is rather complicated but straightforward; therefore we defer this to Section 3.

2. Fast mean-reverting model

2.1. **Review.** Here we review an asymptotic theory for the stochastic volatility model developed by [8, 9]. The main result of this article is described in the next subsection. In [8], a family of the stochastic volatility models

$$\begin{cases} dS^\epsilon_t = rS^\epsilon_t dt + f(X^\epsilon_t)S^\epsilon_t dW^\rho_t, \\ dX^\epsilon_t = \left(\frac{1}{\epsilon}(m - X^\epsilon_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(X^\epsilon_t)\right)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}dW_t \end{cases} \tag{2.1}$$

is considered, where $W = (W_t)$ and $W^\rho = (W^\rho_t)$ are standard Brownian motions with correlation $\langle W, W^\rho\rangle_t = \rho t, \rho \in [-1, 1]$. For fixed $\epsilon > 0$, $S^\epsilon = (S^\epsilon_t)$ is supposed to be the asset price process under risk neutral measure and $r \in \mathbb{R}$ is risk-free rate. Other parameters m, ν are constants, f is a positive function, and Λ is associated with the market price of volatility risk. For a given payoff function h and maturity T^*, the European option price at time $t < T^*$

$$P^\epsilon(t, s, v) = e^{-r(T^*-t)}P[h(S^\epsilon_{T^*})|S^\epsilon_t = s, X^\epsilon_t = v] \tag{2.2}$$

satisfies

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P^\epsilon = 0, \quad P^\epsilon(T^*, s, v) = h(s) \tag{2.3}$$

where

$$\begin{aligned} \mathcal{L}_0 &= \nu^2\frac{\partial^2}{\partial v^2} + (m - v)\frac{\partial}{\partial v} \\ \mathcal{L}_1 &= \sqrt{2}\rho\nu s f(v)\frac{\partial^2}{\partial s\partial v} - \sqrt{2}\nu\Lambda(v)\frac{\partial}{\partial v}, \\ \mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2}f(v)^2 s^2\frac{\partial^2}{\partial s^2} + r(s\frac{\partial}{\partial s} - 1). \end{aligned} \tag{2.4}$$

Notice that $\mathcal{L}_0$ is the infinitesimal generator of the OU process

$$dX_t = (m - X_t)dt + \nu\sqrt{2}dW_t \tag{2.5}$$

and $\mathcal{L}_2$ is the Black-Scholes operator with volatility level $f(v)$. By a singular perturbation expansion around $\epsilon = 0$, one obtains

$$P^\epsilon = P_0 + \sqrt{\epsilon}P_1 + O(\epsilon), \tag{2.6}$$

where P_0 is the Black-Scholes price with constant volatility $\sqrt{\pi[f^2]}$, π is the ergodic distribution of the process $X = (X_t)$, and P_1 is represented in terms of derivatives of P_0 and several expectations with respect to π. More precisely, [8] gave an expression

$$P_0 + \sqrt{\epsilon}P_1 = P_0 - (T^* - t)\left(V_2 s^2 \frac{\partial^2 P_0}{\partial s^2} + V_3 s^3 \frac{\partial^3 P_0}{\partial s^3}\right), \tag{2.7}$$

where V_2 and V_3 are constants depending on ϵ. The validity of the expansion (2.6) is proved in [9] (see also [5, 18]) under the assumption that h is continuous and piecewise smooth, and f, $1/f$, Λ are bounded. In terms of modeling, however, this assumption seems too restrictive in that it rules out common stochastic volatility models such as the Heston model. Also, the continuity condition of h is not satisfied when considering digital options. The aim of the present article is to show that these assumptions of boundedness and smoothness are totally unnecessary. Our approach is quite different to the singular perturbation expansion. It sheds light on the relation between the option price and the underlying price distribution.

Before we state our result, it should be explained what is the intuition of $\epsilon \to 0$ and how we can apply the expansion (2.6) in practice. To fix ideas, let $\Lambda = 0$ for brevity. Then $\tilde{X}_t := X^\epsilon_{\epsilon t}$ satisfies

$$d\tilde{X}_t = (m - \tilde{X}_t)dt + \nu\sqrt{2}d\tilde{W}_t, \tag{2.8}$$

where $\tilde{W}_t = \epsilon^{-1/2}W_{\epsilon t}$ is a standard Brownian motion, and it holds

$$dS^\epsilon_t = rS^\epsilon_t dt + f(\tilde{X}_{t/\epsilon})S^\epsilon_t dW^\rho_t. \tag{2.9}$$

Hence ϵ stands for the volatility time scale. Note that

$$\langle \log(S^\epsilon)\rangle_t = \int_0^t f(\tilde{X}_{s/\epsilon})^2 ds = \epsilon\int_0^{t/\epsilon} f(\tilde{X}_s)^2 ds \sim \epsilon\int_0^{t/\epsilon} f(X_s)^2 ds \to \pi[f^2]t \tag{2.10}$$

by the law of large numbers for ergodic diffusion, where $X = (X_t)$ is a solution of (2.5). This convergence implies the log price $\log(S^\epsilon_t)$ is asymptotically normally distributed with mean $rt - \pi[f^2]t/2$ and variance $\pi[f^2]t$. This asymptotic argument is a quite plausible way of reducing a general stochastic volatility model to the Black-Scholes model by exploiting the ergodicity of the volatility process which is widely accepted. It should yield a valid approximation as long as the time to the expiration $T^* - t$ is large enough for the volatility process to fluctuate sufficiently.

As a practical application, in [8], the authors proposed its use in calibration problem. They derived an expansion of the Black-Scholes implied volatility (IV) from (2.6) of the form

$$\mathrm{IV} = a\frac{\log(K/S)}{T^* - t} + b + O(\epsilon), \tag{2.11}$$

where K is the strike price, S is the stock price, $T^* - t$ is the time to the maturity, a and b are constants connecting to V_2 and V_3 as

$$V_2 = \bar{\sigma}((\bar{\sigma} - b) - a(r + \frac{3}{2}\bar{\sigma}^2)), \quad V_3 = -a\bar{\sigma}^3, \quad \bar{\sigma}^2 = \pi[f^2]. \tag{2.12}$$

The methodology they introduced consists of (a) estimation of $\bar{\sigma}$ from historical stock returns, (b) estimation of a and b by fitting (2.11) to the implied volatility surface, and (c) pricing or hedging by using estimated $\bar{\sigma}$, a and b via (2.7) and (2.12). This approach captures the volatility skew as well as the term structure. It enables us to calibrate fast and stably due to parsimony of parameters; we have no more need to specify all the parameters in the underlying stochastic volatility model. As we show later, the fist step (a) can be eliminated and the validity of the expansion is assured in theory for a quite large class of stochastic volatility models.

2.2. **Main results.** Now, we describe a generalized model and the main results of this article. We consider the following stochastic differential equation;

$$\begin{cases} dZ_t = \mu(X_t)dt + \varphi(X_t)dW_t + \psi(X_t)dW'_t, \\ dX_t = b(X_t)dt + c(X_t)dW_t, \end{cases} \tag{2.13}$$

where $Z_t = \log(S_t)$ is the log price process, μ, φ, ψ, b and c are Borel functions, and (W, W') is a 2-dimensional standard Brownian motion. We introduce the volatility time scale parameter ϵ as

$$\begin{cases} dZ^\epsilon_t = \mu(X_{t/\epsilon})dt + \varphi(X_{t/\epsilon})dW^\epsilon_t + \psi(X_{t/\epsilon})dW'^\epsilon_t, \\ dX_t = b(X_t)dt + c(X_t)dW_t, \end{cases} \tag{2.14}$$

where $(W^\epsilon, W'^\epsilon)$ is a 2-dimensional standard Brownian motion defined as

$$W^\epsilon_t = \epsilon^{1/2}W_{t/\epsilon}, \quad W'^\epsilon_t = \epsilon^{1/2}W'_{t/\epsilon}. \tag{2.15}$$

Putting $X^\epsilon_t = X_{t/\epsilon}$, we have

$$\begin{cases} dZ^\epsilon_t = \mu(X^\epsilon_t)dt + \varphi(X^\epsilon_t)dW^\epsilon_t + \psi(X^\epsilon_t)dW'^\epsilon_t, \\ dX^\epsilon_t = \epsilon^{-1}b(X^\epsilon_t)dt + \epsilon^{-1/2}c(X^\epsilon_t)dW^\epsilon_t. \end{cases} \tag{2.16}$$

It is then natural to assume that (Z^ϵ, X^ϵ) satisfies under risk-neutral measure P^ϵ with constant risk-free rate r the following stochastic differential equation

$$\begin{aligned} dZ^\epsilon_t &= \left\{ r - \frac{\varphi(X^\epsilon_t)^2 + \psi(X^\epsilon_t)^2}{2} \right\} dt + \varphi(X^\epsilon_t)dW^\epsilon_t + \psi(X^\epsilon_t)dW'^\epsilon_t, \\ dX^\epsilon_t &= \epsilon^{-1}(b(X^\epsilon_t) - \Lambda_\epsilon(X^\epsilon_t))dt + \epsilon^{-1/2}c(X^\epsilon_t)dW^\epsilon_t, \end{aligned} \tag{2.17}$$

where Λ_ϵ represents the market price of volatility risk and is supposed to be $o(1)$ as $\epsilon \to 0$ in a sense specified later. In particular, we set $\Lambda_0 = 0$. Note that even if

$\Lambda_\epsilon(v) = \lambda(v) + o(1)$ for a Borel function λ, the following argument remains valid by incorporating λ in b. Notice that the case of

$$\begin{aligned} &\varphi(v) = \rho f(v), \quad \psi(v) = \sqrt{1-\rho^2} f(v), \\ &b(v) = m - v, \quad c(v) = \nu\sqrt{2}, \quad \Lambda_\epsilon(v) = \nu\sqrt{2\epsilon}\Lambda(v) \end{aligned} \tag{2.18}$$

in (2.17) is equivalent to the original fast mean-reverting model (2.1). By rewriting (2.17) for X with initial condition $Z_0^\epsilon = 0$, we have

$$\begin{aligned} Z_t^\epsilon =& rt - \frac{\epsilon}{2}\int_0^{t/\epsilon} \{\varphi(X_s)^2 + \psi(X_s)^2\} ds \\ &+ \sqrt{\epsilon}\int_0^{t/\epsilon} \varphi(X_s) dW_s + \sqrt{\epsilon}\int_0^{t/\epsilon} \psi(X_s) dW'_s \end{aligned} \tag{2.19}$$

and

$$dX_t = b_\epsilon(X_t)dt + c(X_t)dW_t, \tag{2.20}$$

where $b_\epsilon = b - \Lambda_\epsilon$.

Condition 2.1. *The Borel functions φ, ψ, b_ϵ, c are defined on $\mathbb{R}$, $\sup_{\epsilon\geq 0}|b_\epsilon|$ is locally integrable, and φ, ψ, c and $1/c$ are locally bounded.*

Under Condition 2.1, put

$$\gamma := -\limsup_{|v|\to\infty, \epsilon\to 0} \frac{v b_\epsilon(v)}{c(v)^2}. \tag{2.21}$$

For example, $\gamma = \infty$ in the OU case (2.5). Note that if

$$\liminf_{|v|\to\infty} |v|^{p-2} c(v)^2 > 0 \tag{2.22}$$

for $p \geq 0$ and $2\gamma > p - 1$, then the volatility process X is ergodic for each fixed ϵ with ergodic distribution

$$\pi_\epsilon(dv) = \frac{c_\epsilon}{c(v)^2} \exp\left\{2\int_0^v \frac{b_\epsilon(w)}{c(w)^2} dw\right\} dv, \tag{2.23}$$

where c_ϵ is a constant:

$$\frac{1}{c_\epsilon} = \int_{-\infty}^{\infty} \frac{1}{c(v)^2} \exp\left\{2\int_0^v \frac{b_\epsilon(w)}{c(w)^2} dw\right\} dv. \tag{2.24}$$

This is a simple consequence of a well-known fact that a one-dimensional diffusion is ergodic if its invariant distribution has finite total mass and if $s_\epsilon(\mathbb{R}) = \mathbb{R}$, where

$$s_\epsilon(v) = \frac{1}{c_\epsilon}\int_0^v \exp\left\{-2\int_0^u \frac{b_\epsilon(w)}{c(w)^2} dw\right\} du. \tag{2.25}$$

See e.g. [26] for the detail. We need the following stronger assumption of the ergodicity.

Condition 2.2. *Condition 2.1 is satisfied and there exists $p \geq 0$ such that*

$$2\gamma + 1 > 4p, \quad \limsup_{|v|\to\infty} \frac{1 + \varphi(v)^2 + \psi(v)^2}{|v|^{p-2} c(v)^2} < \infty. \tag{2.26}$$

If φ and ψ are of polynomial growth, we can see that the above assumption is satisfied by strongly dependent ergodic diffusions beyond exponentially mixing diffusions such as the OU process (see [12]). Put

$$F_\varphi(x) = \int_0^x \frac{\varphi(v)}{c(v)} dv, \quad \mathcal{L}_X^\epsilon = b_\epsilon(v)\frac{\partial}{\partial v} + \frac{1}{2}c(v)^2 \frac{\partial^2}{\partial v^2}. \tag{2.27}$$

Condition 2.3. *At least one of φ and ψ is not identically 0 and there exists a closed interval $I \subset \mathbb{R}$ such that*

1. *$F'_\varphi = \varphi/c$ is absolutely continuous on $\mathbb{R}$,*
2. *$\mathcal{L}_X^0 F_\varphi$ and φ^2 are continuous on I*
3. *if $\varphi \not\equiv 0$, then $1, \mathcal{L}_X^0 F_\varphi$ and $\varphi^2 + \psi^2$ are linearly independent on I,*
4. *$\mathcal{L}_X^\epsilon F_\varphi$ converges to $\mathcal{L}_X^0 F_\varphi$ uniformly on I as $\epsilon \to 0$,*
5. *if $\psi \not\equiv 0$, then $\inf_I \psi^2 > 0$ and ψ^2 is continuous on I.*

Theorem 2.1. *Under Conditions 2.2 and 2.3, the European option price (2.2) of a bounded payoff function h with $S^\epsilon = \exp Z^\epsilon$ satisfies*

$$P^\epsilon(t, s, v) = P^0(t, s) + \sqrt{\epsilon} P^1(t, s) + O(\epsilon) \tag{2.28}$$

for every t, s, v, where

$$P^0(t, s) = e^{-r(T^*-t)} \int h(s \exp((r - \sigma^2/2)(T^* - t) + \sigma\sqrt{T^* - t}z))\phi(z)dz \tag{2.29}$$

is the Black-Scholes price with volatility

$$\sigma^2 = \sigma_\epsilon^2 = \pi_\epsilon[\varphi^2] + \pi_\epsilon[\psi^2], \tag{2.30}$$

and

$$\begin{aligned} P^1(t, s) = e^{-r(T^*-t)} \int & h(s \exp((r - \sigma^2/2)(T^* - t) + \sigma\sqrt{T^* - t}z)) \\ & \phi(z) \left\{ -\frac{\delta}{\sigma^2}(z^2 - 1) + \frac{\delta}{\sigma^3\sqrt{T^* - t}}(z^3 - 3z) \right\} dz, \end{aligned} \tag{2.31}$$

where

$$\delta = -\int_{-\infty}^\infty \int_{-\infty}^x (\varphi(v)^2 + \psi(v)^2 - \sigma^2)\pi_\epsilon(dv) \frac{\varphi(x)}{c(x)} dx. \tag{2.32}$$

Proof Assume without loss of generality that $t = 0$. Fix the maturity T^* and put $T = T^*/\epsilon$. Define σ as (2.30). Let

$$K_t^T = \left(\int_0^t (\varphi^2(X_s) + \psi^2(X_s) - \sigma^2)ds, \int_0^t \varphi(X_s)dW_s + \int_0^t \psi(X_s)dW'_s \right) \tag{2.33}$$

and

$$A^T(x, y) = \sqrt{T^*}y - T^{-1/2}T^*x/2. \tag{2.34}$$

Observe that

$$
\begin{aligned}
\sqrt{T}A^T(K_T^T/T) &= \sqrt{\frac{T^*}{T}}\left(\int_0^T \varphi(X_t)dW_t + \int_0^T \psi(X_t)dW_t'\right) \\
&\quad - \frac{T*}{2T}\int_0^T (\varphi^2(X_t) + \psi^2(X_t) - \sigma^2)dt \\
&= \sqrt{\epsilon}\left(\int_0^{T^*/\epsilon} \varphi(X_t)dW_t + \int_0^{T^*/\epsilon} \psi(X_t)dW_t'\right) \\
&\quad - \frac{\epsilon}{2}\int_0^{T^*/\epsilon} (\varphi^2(X_t) + \psi^2(X_t) - \sigma^2)dt,
\end{aligned} \tag{2.35}
$$

so that from (2.19),

$$
h(S^\epsilon_{T^*}) = H(\sqrt{T}A^T(K_T^T/T)) \tag{2.36}
$$

with

$$
H(z) = h(s\exp((r - \sigma^2/2)T^* + z)). \tag{2.37}
$$

Now, let $[x_0, x_1] = I$, say, and $P_{x_0}[\cdot] = P^\epsilon[\cdot|X_0 = x_0]$. Define a sequence of stopping times $\{\tau_j\}$ as

$$
\tau_0 = 0, \quad \tau_{j+1} = \inf\left\{t > \tau_j; X_t = x_0 \sup_{s\in[\tau_j,t]} X_s \geq x_1\right\}. \tag{2.38}
$$

The proof exploits the fact that $K^T = \{K_t^T\}$ is decomposed by $\{\tau_j\}$ into a sum of independent variables due to the strong Markov property of X. We use also the following well-known fact; for every π_ϵ-integrable function g, it holds

$$
\pi_\epsilon[g] = \frac{1}{P_{x_0}[\tau_1]}P_{x_0}\left[\int_0^{\tau_1} g(X_t)dt\right], \quad P_{x_0}[\tau_1] = 2|s_\epsilon(x_1) - s_\epsilon(x_0)|. \tag{2.39}
$$

See e.g., [26] for the detail. Applying Theorem 3.1, Propositions 4.2, 4.4 and 4.7 below, we obtain the Edgeworth expansion

$$
P^\epsilon(t,s,v) = \int H(z)\left\{\phi(z;v^T) - T^*\delta\sqrt{\epsilon}\partial^3\phi(z;v^T)\right\}dz + O(\epsilon), \tag{2.40}
$$

where $\phi(\cdot;v^T)$ is the normal density with mean 0, variance v^T which admits

$$
\begin{aligned}
v^T &= -\frac{\sqrt{\epsilon}T^*}{P_{x_0}[\tau_1]}P_{x_0}\left[\int_0^{\tau_1}\varphi(X_t)dW_t\int_0^{\tau_1}(\varphi(X_t)^2 + \psi(X_t)^2 - \sigma^2)dt\right] \\
&\quad + \frac{T^*}{P_{x_0}[\tau_1]}P_{x_0}\left[\int_0^{\tau_1}(\varphi(X_t)^2 + \psi(X_t)^2)dt\right] + O(\epsilon)
\end{aligned} \tag{2.41}
$$

and

$$
\begin{aligned}
\delta =& \frac{1}{6P_{x_0}[\tau_1]}P_{x_0}\left[\left(\int_0^{\tau_1}\varphi(X_t)dW_t + \int_0^{\tau_1}\psi(X_t)dW_t'\right)^3\right] \\
&- \frac{1}{2P_{x_0}[\tau_1]^2}P_{x_0}\left[\tau_1\int_0^{\tau_1}\varphi(X_t)dW_t\right]P_{x_0}\left[\int_0^{\tau_1}(\varphi(X_t)^2 + \psi(X_t)^2)dt\right].
\end{aligned} \tag{2.42}
$$

Since

$$\sigma^2 = \frac{1}{P_{x_0}[\tau_1]} P_{x_0}\left[\int_0^{\tau_1} (\varphi(X_t)^2 + \psi(X_t)^2) dt\right], \tag{2.43}$$

we have

$$\delta = \frac{1}{2P_{x_0}[\tau_1]} P_{x_0}\left[\int_0^{\tau_1} \varphi(X_t) dW_t \int_0^{\tau_1} (\varphi(X_t)^2 + \psi(X_t)^2 - \sigma^2) dt\right] \tag{2.44}$$

by Itô's formula. Hence $v^T = \sigma^2 T^* - 2\delta T^* \sqrt{\epsilon} + O(\epsilon)$. In order to see (2.32) holds, use Itô's formula to have

$$\int_0^{\tau_1} (\varphi(X_t)^2 + \psi(X_t)^2 - \sigma^2) dt = -\int_0^{\tau_1} g(X_t) c(X_t) dW_t, \tag{2.45}$$

where

$$g(\xi) = 2(s_\epsilon)'(\xi) \int_{-\infty}^{\xi} (\varphi(\eta)^2 + \psi(\eta)^2 - \sigma^2) \pi_\epsilon(d\eta), \tag{2.46}$$

and then, use the Itô identity. Taylor expanding (2.40) with v^T around $\sigma^2 T^*$, we have the result. ////

In order to deal with φ, ψ and $1/c^2$ of exponential growth, it is however more suitable to work with the following condition. Put

$$\gamma_+ = -\limsup_{v\to\infty,\epsilon\to 0} \frac{b_\epsilon(v)}{c(v)^2}, \quad \gamma_- = \liminf_{v\to-\infty,\epsilon\to 0} \frac{b_\epsilon(v)}{c(v)^2}. \tag{2.47}$$

For example, $\gamma_+ = \gamma_- = \infty$ for the OU case (2.5). Note that if both γ_+ and γ_- are positive, then $\gamma = \infty$, so that the positivity of $\gamma_\pm$ implies that the diffusion is more strongly mean-reverting.

Condition 2.4. *Condition 2.1 is satisfied and there exist* $\kappa_+, \kappa_- > 0$ *such that*

$$\gamma_\pm > 2\kappa_\pm, \quad \limsup_{v\to\pm\infty} \frac{1 + \varphi(v)^2 + \psi(v)^2}{e^{\kappa_\pm |v|} c(v)^2} < \infty. \tag{2.48}$$

Theorem 2.2. *Under Conditions 2.3 and 2.4, the same conclusion holds as in Theorem 2.1.*

The proof of Theorem 2.2 is the same as that for Theorem 2.1. It is now straightforward to obtain the following corollary.

Corollary 2.3. *The put option price (2.2) with* $h(s) = (K - s)_+$ *admits the valid expansion*

$$P^\epsilon(t, s, v) = P^0(t, s; \sigma, K) - \frac{\delta\sqrt{\epsilon}}{\sigma^2} s\phi(d_1) d_2 + O(\epsilon), \tag{2.49}$$

where

$$P^0(t, s; \sigma, K) = K e^{-r(T^*-t)} N(-d_2) - sN(-d_1) \tag{2.50}$$

is the Black-Scholes price with volatility σ *of (2.30) and*

$$d_1 = \frac{\log(s/K) + (r + \sigma^2/2)(T^* - t)}{\sigma\sqrt{T^* - t}}, \quad d_2 = d_1 - \sigma\sqrt{T^* - t}. \tag{2.51}$$

In particular, the implied volatility IV *admits the valid expansion*

$$\mathrm{IV} = \sigma - \frac{\delta\sqrt{\epsilon}d_2}{\sigma^2\sqrt{T^*-t}} + O(\epsilon). \tag{2.52}$$

Note that the call option price is also expanded in the light of the well-known put-call parity. We assume only the boundedness on h, so that the digital options can be treated; it has not been proved so far that the digital option prices admit the singular perturbation expansion even in the OU case (2.1). Using (2.52), which has the form (2.11), one can calibrate the two parameters σ and $\delta\sqrt{\epsilon}$ from an observed volatility surface. This approach is an alternative to the methodology (a),(b),(c) described above; one can omit the step (a) requiring the use of historical data. In case that $b_\epsilon = b - \sqrt{\epsilon}\Lambda$ for a Borel function Λ with a suitable integrability condition, it could be deduced

$$\sigma_\epsilon = \bar{\sigma} + \sqrt{\epsilon}a + O(\epsilon) \tag{2.53}$$

with

$$\bar{\sigma}^2 = \pi_0[\varphi^2 + \psi^2], \quad a = \frac{1}{\bar{\sigma}}\int_{-\infty}^{\infty}\int_{-\infty}^{x}(\varphi(v)^2 + \psi(v)^2 - \bar{\sigma}^2)\pi_0(dv)\frac{\Lambda(x)}{c(x)^2}dx. \tag{2.54}$$

Hence Λ induces a volatility level correction as noted in [8]. It is now straightforward to see our expansion is consistent to [8] with $V_2 = 2V_3 - \bar{\sigma}a\sqrt{\epsilon}$ and $V_3 = -\delta\sqrt{\epsilon}$.

It should be noted that δ is proportional to the asymptotic skewness of the log return distribution, which is easily seen in the expression (2.31). In addition, we can see from (2.31) that δ controls also the fatness of the distribution tail. The relation among the return distribution, option prices and the parameters of the stochastic model was studied in [15] for the Heston model. Although the second-order expansion explains only the volatility skew as noted in [8], it is possible to incorporate the volatility smile by studying the next term of $O(\epsilon)$. The Edgeworth theory developed in the next section could be extended to admit the higher-order expansion given a suitable moment condition. However, we restrict ourselves to the second-order case in order to avoid tedious calculation.

Example 2.4. Consider the following fast mean-reverting Heston model

$$\begin{cases} dS_t^\epsilon = S_t^\epsilon(rdt + \sqrt{V_t^\epsilon}(\rho dW_t + \sqrt{1-\rho^2}dW_t')) \\ dV_t^\epsilon = -\epsilon^{-1}(aV_t^\epsilon - b)dt + \epsilon^{-1/2}c\sqrt{V_t^\epsilon}dW_t \end{cases} \tag{2.55}$$

for positive constants a, b, c. We assume $2b > c^2$ so that V^ϵ is ergodic for each $\epsilon > 0$. Since V^ϵ stays on the half line $(0, \infty)$, we cannot apply directly the preceding theorems. Put $X^\epsilon = \log(V^\epsilon)$ therefore. Use Itô's formula to have

$$dX_t^\epsilon = -\epsilon^{-1}(a - (b - c^2/2)e^{-X_t^\epsilon})dt + \epsilon^{-1/2}ce^{-X_t^\epsilon/2}dW_t, \tag{2.56}$$

or equivalently, $X_t = X_{\epsilon t}^\epsilon$ satisfies

$$dX_t = -(a - (b - c^2/2)e^{-X_t})dt + ce^{-X_t/2}dW_t. \tag{2.57}$$

Then, Condition 2.4 holds with

$$\gamma_+ = \infty, \;\; \gamma_- = \frac{b}{c^2} - \frac{1}{2}, \;\; \kappa_+ = 2, \;\; k_- = 0. \tag{2.58}$$

Applying Theorem 2.2, we assure the validity of the expansion formula (2.31) with

$$\sigma^2 = \frac{b}{a}, \;\; \delta = \frac{\rho bc}{2a^2}. \tag{2.59}$$

Here we used the fact that the the ergodic distribution of V^ϵ is the gamma distribution with scale $2a/c^2$ and shape $2b/c^2$; the calculation seems correct though (2.59) contradicts (5.10) of [19].

Example 2.5. Consider

$$\begin{cases} dS_t^\epsilon = S_t^\epsilon(rdt + \sqrt{V_t^\epsilon}(\rho dW_t + \sqrt{1-\rho^2}dW_t')) \\ dV_t^\epsilon = -\epsilon^{-1}(aV_t^\epsilon - b)dt + \epsilon^{-1/2}c|V_t^\epsilon|^\alpha dW_t \end{cases} \tag{2.60}$$

for positive constants a, b, c and $\alpha \in (1/2, 1]$. Since the scale function s^V of V^ϵ satisfies $s^V((0,\infty)) = \mathbb{R}$, we can apply Itô's formula to $X^\epsilon = \log(V^\epsilon)$ as in the preceding example to have

$$dX_t^\epsilon = -\epsilon^{-1}(a - be^{-X_t^\epsilon} + c^2e^{-2(1-\alpha)X_t^\epsilon}/2)dt + \epsilon^{-1/2}ce^{-(1-\alpha)X_t^\epsilon}dW_t. \tag{2.61}$$

If $\alpha < 1$, then Condition 2.4 holds with

$$\gamma_\pm = \infty, \;\; \kappa_+ = 3 - 2\alpha, \;\; k_- = 0. \tag{2.62}$$

If $\alpha = 1$, it holds with

$$\gamma_+ = \frac{a}{c^2} + \frac{1}{2}, \;\; \gamma_- = \infty \;\; \kappa_+ = 1, \;\; k_- = 0 \tag{2.63}$$

provided that $2a > 3c^2$.

3. Edgeworth expansion for regenerative functionals

3.1. **Conditions and a general result.**

In this section, we develop an Edgeworth theory for triangular arrays of general regenerative functionals, which extends [12]. Additive functionals of ergodic diffusions are treated in the next section as an application. Let $\mathbb{P}^T = (\Omega^T, \mathcal{F}^T, \{\mathbb{F}_t^T\}, P^T)$, $T > 0$ be a family of filtered probability spaces satisfying the usual assumptions and $K^T = (K_t^T)$ be an $\{\mathbb{F}_t^T\}$-adapted cadlag process defined on $\mathbb{P}^T$. Denote by $\mathbb{P}_t^T[\cdot]$ the conditional expectation operator given $\mathbb{F}_t^T$. For a given sequence of increasing $\{\mathbb{F}_t^T\}$-stopping times $\{\tau_j^T\}$ with $\tau_0^T = 0$, $\tau_j^T \to \infty$ as $j \to \infty$, put

$$\mathcal{K}_j^T = \left(\mathcal{K}_{j,t}^T\right)_{t\geq 0}, \;\; \mathcal{K}_{j,t}^T = K_{t+\tau_j^T}^T - K_{\tau_j^T}^T, \;\; l_j^T = \tau_{j+1}^T - \tau_j^T, \;\; j = 0, 1, 2, \dots \tag{3.1}$$

We say that K^T is a **regenerative functional** if there exists such a sequence $\{\tau_j^T\}$ that

(a) $(\mathcal{K}_j^T, l_j^T)$ is independent to $\mathbb{F}_{\tau_j^T}^T$ for each $j = 1, 2, \dots,$

(b) $(\mathcal{K}_j^T, l_j^T)$, $j = 1, 2, \dots$ are identically distributed.

In particular, $\bar{\mathcal{K}}_j^T := (\mathcal{K}_{j,l_j^T}^T, l_j^T)$, $j = 1, 2, \ldots$ are an iid sequence and independent to $\bar{\mathcal{K}}_0^T := (\mathcal{K}_{0,l_0^T}^T, l_0^T)$. In the following, we assume that K^T is an n-dimensional regenerative functional and that $\mathrm{Var}^T[\bar{\mathcal{K}}_j^T]$ exists and is of rank $n' + 1$ for $j \geq 1$. Without loss of generality, assume

$$\bar{\mathcal{K}}_j^T = (G_j^T, N_j^T, l_j^T), \quad j \geq 1, \tag{3.2}$$

where G_j^T is n'-dimensional and the variance matrix of (G_j^T, l_j^T) is of full rank. Put

$$\begin{aligned} &m_L^T = \mathbb{P}_0^T[l_1^T], \ \ m_G^T = \mathbb{P}_0^T[G_1^T], \ \ m_N^T = \mathbb{P}_0^T[N_1^T], \\ &\mu^T = (\mu_k^T) = (m_G^T, m_N^T)/m_L^T, \end{aligned} \tag{3.3}$$

and

$$\mathbb{K}_j^T = (\mathbb{G}_j^T, l_j^T), \ \ \mathbb{G}_j^T = G_j^T - l_j^T m_G^T/m_L^T. \tag{3.4}$$

Due to the definition, it is not difficult to see a law of large numbers holds:

$$K_t^T/t \to \mu^T \tag{3.5}$$

in probability as $t \to \infty$. Further, a central limit theorem

$$\sqrt{t}(K_t^T/t - \mu^T) \Rightarrow \mathcal{N}(0, V^T) \tag{3.6}$$

also holds, where V^T is the asymptotic variance matrix. Our aim in this section is to give a refinement of this central limit theorem. More precisely, for a given function $A^T : \mathbb{R}^n \to \mathbb{R}^d$, we present a valid approximation of the distribution of

$$\sqrt{T}(A^T(K_T^T/T) - A^T(\mu^T)) \tag{3.7}$$

up to $O(T^{-1})$ as $T \to \infty$. As far as considering this form, we can assume without loss of generality that $\mathbb{P}_0^T[|N_j^T|] = 0$ for all $j \geq 1$ in (3.2). Further, for notational simplicity, we concentrate on the case $d = 1$. Put

$$(\mu_{k,l}^T) = \mathrm{Var}[\mathbb{G}_1^T]/m_L^T, \ \ \rho^T = (\rho_k^T) = \mathrm{Cov}[\mathbb{G}_1^T, l_1^T] \tag{3.8}$$

and

$$\mu_{k,l,m}^T = (\kappa_{k,l,m}^T - \rho_k^T \mu_{l,m}^T - \rho_l^T \mu_{m,k}^T - \rho_m^T \mu_{k,l}^T)/m_L^T, \tag{3.9}$$

where $(\kappa_{k,l,m}^T)$ is the third moments of $\mathbb{G}_1^T$.

Condition 3.1. *Put $N = (n' + 2) \vee 4$. Then*

$$\limsup_{T\to\infty} \left\{ \mathbb{P}_0^T[|\bar{\mathcal{K}}_0^T|^2] + \mathbb{P}_0^T[|\mathbb{K}_1^T|^N] + \mathbb{P}_0^T\left[\int_{\tau_1^T}^{\tau_2^T} |K_t^T|^2 dt\right]\right\} < \infty. \tag{3.10}$$

Condition 3.2. *Let Ψ^T be the characteristic function of $\mathbb{K}_1^T$:*

$$\Psi^T(u) = \mathbb{P}_0^T[\exp\{iu \cdot \mathbb{K}_1^T\}]. \tag{3.11}$$

Then

$$\limsup_{T\to\infty} \sup_{|u|\geq b} |\Psi^T(u)| < 1 \tag{3.12}$$

for all $b > 0$ and there exists $p \geq 1$ such that

$$\limsup_{T\to\infty} \int_{\mathbb{R}^{n'+1}} |\Psi^T(u)|^p du < \infty. \tag{3.13}$$

Note that under Conditions 3.1 and 3.2, it holds

$$0 < \liminf_{T\to\infty} \inf_{|a|=1} \mathbb{P}_0^T[|a\cdot \mathbb{K}_1^T|^2] \leq \limsup_{T\to\infty} \sup_{|a|=1} \mathbb{P}_0^T[|a\cdot \mathbb{K}_1^T|^2] < \infty, \tag{3.14}$$

that is, the largest and smallest eigenvalues of the variance matrix of $\mathbb{K}_1^T$ is bounded and bounded away from 0 for sufficiently large T.

Condition 3.3.

$$\liminf_{T\to\infty} m_L^T > 0. \tag{3.15}$$

Under Conditions 3.1 and 3.3, the quantities μ^T, $(\mu_{k,l}^T)$, $(\mu_{k,l,m}^T)$ are bounded for sufficiently large T.

Condition 3.4. *There exist $\zeta > 0$ and $T_0 > 0$ such that for $B(\zeta) := \{x \in \mathbb{R}^n; |x| < \zeta\}$,*

1. *$A^T : \mathbb{R}^n \to \mathbb{R}$ is four times continuously differentiable on $\mu^T + B(\zeta)$ for $T \geq T_0$,*
2. *all the derivatives up to fourth order are uniformly bounded for $T \geq T_0$ on $\mu^T + B(\zeta)$,*
3.

$$0 < \liminf_{T\to\infty} v^T \leq \limsup_{T\to\infty} v^T < \infty, \tag{3.16}$$

where

$$v^T = \sum_{k,l=1}^{n'} \mu_{k,l}^T h_k^T a_l^T, \quad a_m^T = \partial_m A^T(\mu^T). \tag{3.17}$$

Put

$$a^T = (a_k^T) \in \mathbb{R}^n, \quad a_{k,l}^T = \partial_k \partial_l A^T(\mu^T). \tag{3.18}$$

Denote by ι the natural inclusion: $\mathbb{R}^{n'} \ni v \mapsto (v, 0, \ldots, 0) \in \mathbb{R}^n$.

Theorem 3.1. *Let $\mathcal{S}_M$ be the set of Borel functions on $\mathbb{R}$ which are uniformly bounded by M, $M > 0$. Under Conditions 3.1, 3.2, 3.3 and 3.4, it holds*

$$\mathbb{P}_0^T[H(\sqrt{T}(A^T(K_T^T/T) - A^T(\mu^T)))] = \int H(z)\mathcal{P}_T(z)dz + O(T^{-1}) \tag{3.19}$$

uniformly in $H \in \mathcal{S}_M$, where $\mathcal{P}_T$ is defined by

$$\mathcal{P}_T(z) = \phi(z; v^T) + T^{-1/2}\left\{A_1^T p_1(z; v^T) + \frac{A_3^T}{6} p_3(z; v^T)\right\}, \tag{3.20}$$

$\phi(z; v^T)$ is the normal density with mean 0 and variance v^T,

$$p_1(z; v^T) = -\partial\phi(z; v^T), \quad p_3(z; v^T) = -\partial^3\phi(z; v^T), \tag{3.21}$$

and

$$
\begin{aligned}
A_1^T &= \frac{1}{2}\sum_{k,l=1}^{n'} a_{k,l}^T \mu_{k,l}^T + a^T \cdot \left\{ \mathbb{P}_0^T[K_{\tau_1^T}^T] + \frac{1}{m_L^T}\mathbb{P}_0^T\left[\int_{\tau_1^T}^{\tau_2^T} K_t^T dt\right] - \frac{\iota(\rho^T)}{m_L^T} \right\}, \\
A_3^T &= \sum_{k,l,m=1}^{n'} a_k^T a_l^T a_m^T \mu_{k,l,m}^T + 3\sum_{j,k,l,m=1}^{n'} a_j^T a_k^T a_{l,m}^T \mu_{j,l}^T \mu_{k,m}^T.
\end{aligned} \tag{3.22}
$$

3.2. **Proof of Theorem 3.1.** The proof is essentially the same as that of Theorem 1 of [12]. As in [12], we assume without loss of generality that $\mu^T = 0$ and $A^T(0) = 0$. We start with a general lemma.

Lemma 3.2. *Let X_j^n be a triangular array of k-dimensional independent random variables having a mean* 0*, $k \in \mathbb{N}$. Assume that $X_j^n \sim X_1^n$ for all j and that*

$$
\limsup_{n\to\infty} E[|X_1^n|^s] < \infty \tag{3.23}
$$

for an integer $s \geq 4$,

$$
\limsup_{n\to\infty} \sup_{|u|\geq b} |\Psi^n(u)| < 1, \quad \limsup_{n\to\infty} \int_{\mathbb{R}^k} |\Psi^n(u)|^p du < \infty, \tag{3.24}
$$

for all $b > 0$ and for some $p \geq 1$ respectively, where

$$
\Psi^n(u) = E[\exp\{iu \cdot X_1^n\}]. \tag{3.25}
$$

Then, $S_m^n = m^{-1/2}\sum_{j=1}^m X_j^n$ has a bounded density p_m^n for sufficiently large m and n. Further, it holds

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}^k}(1+|x|^s)\Bigg| p_m^n(x) - \phi(x;0,v^n) \\
&\qquad + \frac{1}{6\sqrt{m}}\sum_{a,b,c=1}^{k} \kappa_{abc}^n \partial_a\partial_b\partial_c\phi(x;0,v^n)\Bigg| = O(m^{-1})
\end{aligned} \tag{3.26}
$$

uniformly in $n \geq n_0$, for a sufficiently large n_0, where v^n is the variance matrix of X_1^n, κ_{abc}^n are the third moments of X_1^n.

Proof This lemma is a variant of Theorem 19.2 of [3]. Although the distribution of X_1^n depends on n, the assertion is proved in a similar manner with the aid of Theorem 9.10 of [3], due to our assumptions. For example, we have (and use)

$$
0 < \liminf_{n\to\infty} \inf_{|u|=1} E[|u \cdot X_1^n|^2] \leq \limsup_{n\to\infty} \sup_{|u|=1} E[|u\cdot X_1^n|^2] < \infty. \tag{3.27}
$$

////

Step 1 [The regenerative method]: Let $\bar{\pi} : \mathbb{R}^n \to \mathbb{R}^{n'}$ be the projection onto the first n'-dimensional subspace. Put $\alpha = m_L^T$,

$$
U_m^T = \frac{1}{\sqrt{m}}\sum_{j=1}^m \bar{\pi}(\mathbb{G}_j^T), \quad V_m^T = \frac{1}{\sqrt{m}}\sum_{j=1}^m (l_j^T - \alpha), \quad b_m^T = \frac{T - \alpha m}{\sqrt{m}} \tag{3.28}
$$

for $m \geq 1$ and $U_0^T = V_0^T = b_0^T = 0$. Notice that (U_m^T, V_m^T) is a sum of iid variables and independent to $(K_{\tau_1^T}^T, \tau_1^T)$. Put $M_T = \max\{M \geq 0; \sum_{m=0}^M l_m^T \leq T\}$. By Chebyshev's inequality and (3.14), we have $\mathbb{P}_0^T[|T - \alpha M_T| \geq \delta T] = O(T^{-1})$ for all $\delta \in (0, 1)$. Let $\delta \in (0, 1/2)$ be fixed. By decomposing

$$K_T^T = K_{\tau_1^T}^T + \sqrt{m}\iota(U_m^T) + R_m^T \tag{3.29}$$

on the set $\{M_T = m\}$, we have

$$\begin{aligned} &\mathbb{P}_0^T[H(\sqrt{T}A^T(K_T^T/T))] \\ &= \sum_{m;|T-\alpha m|<\delta T} \mathbb{P}_0^T[\psi_{m,1}^T(K_{\tau_1^T}^T, U_m^T, R_m^T)\psi_{m,2}^T(\tau_1^T, V_m^T, l_{m+1}^T)] + O(T^{-1}), \end{aligned} \tag{3.30}$$

where

$$R_m^T = R_m^T(\tau_1^T, V_m), \quad R_m^T(l,\eta) = K_T^T - K_{T_m(l,\eta)}^T, \quad T_m(l,\eta) = (l + \sqrt{m}\eta + \alpha m) \wedge T \tag{3.31}$$

for $m \geq 0$,

$$\psi_{m,1}^T(f, \xi, r) = H(\sqrt{T}A^T((f + \sqrt{m}\iota(\xi) + r)/T)) \tag{3.32}$$

and $\psi_{m,2}^T$ is the indicator function of the set

$$\{(l, \eta, t) \in \mathbb{R}^3; 0 \leq \sqrt{m}(b_m^T - \eta) - l < t\}. \tag{3.33}$$

Since H is bounded by the assumption, the estimate (3.30) is uniform in $H \in \mathcal{S}_M$. Hereafter, we always drop "uniformly in $H \in \mathcal{S}_M$" for short whenever stating identity with $O(T^{-1})$. Besides, we write $\sum_{m;T}$ to mean $\sum_{m;|T-\alpha m|<\delta T}$. Remind that α depends on T but is bounded and bounded away from 0 for sufficiently large T by the assumptions. By Lemma 3.2, there exists a bounded density $p_m^T(\xi, \eta)$ of (U_m^T, V_m^T), so that

$$\begin{aligned} &\mathbb{P}_0^T[\psi_{m,1}^T(K_{\tau_1^T}^T, U_m^T, R_m^T)\psi_{m,2}^T(\tau_1^T, V_m^T, l_{m+1}^T)] = \\ &\int \psi_{m,1}^T(f, \xi, r)\psi_{m,2}^T(l, \eta, t)Q_R^T(dr, dt; T - T_m(l, \eta))Q_F^T(df, dl)p_m^T(\xi, \eta)d\xi d\eta, \end{aligned} \tag{3.34}$$

where $Q_F^T(\cdot)$ and $Q_R^T(\cdot; t)$ is the distribution of $\bar{\mathcal{K}}_0^T$ and $(\mathcal{K}_{1,t}^T, l_1^T)$ respectively. Further, applying Lemma 3.2 again with $s = N$, with the aid of Lemma 3.3 below, we have

$$\begin{aligned} &\mathbb{P}_0^T[\psi(\sqrt{T}A^T(K_T^T/T))] \\ &= \sum_{m;T} \int \psi_{m,1}^T(f, \xi, r)\psi_{m,2}^T(l, \eta, t)Q_R^T(dr, dt; T - T_m(l, \eta))Q_F^T(df, dl) \\ &\qquad \times \phi^T(\xi, \eta)\left\{1 + m^{-1/2}p^T(\xi, \eta)\right\} d\xi d\eta + O(T^{-1}), \end{aligned} \tag{3.35}$$

where ϕ^T is a normal density and p^T is a polynomial, which are determined by the variance matrix and the third cumulants of $\mathbb{K}_1^T$.

Lemma 3.3. *It holds*

$$\sum_{m;T} \int \frac{\psi^T_{m,2}(l,\eta,t)}{m(1+|\eta|^2)} Q^T_R(dr,dt;T-T_m(l,\eta))Q^T_F(df,dl)d\eta = O(T^{-1}) \tag{3.36}$$

Proof By changing variable: $\eta = (v-\alpha m - l)/\sqrt{m}$, the summand is equal to

$$m^{-3/2} \int \left\{ 1 + \left| \frac{v-\alpha m - l}{\sqrt{m}} \right|^2 \right\}^{-1} 1_{\{0 \le T-v<t\}} Q^T_R(dr,dt;T-v)Q^T_F(df,dl)dv. \tag{3.37}$$

Observe that

$$\begin{aligned} &\sum_{m;T} m^{-3/2} \left\{ 1 + \left| \frac{v-\alpha m - l}{\sqrt{m}} \right|^2 \right\}^{-1} \\ &\le ((1-\delta)T/\alpha)^{-3/2} \left\{ 2 + \int_{-\infty}^{\infty} \left\{ 1 + \left| \frac{v-\alpha u - l}{\sqrt{(1+\delta)T/\alpha}} \right|^2 \right\}^{-1} du \right\} \\ &= ((1-\delta)T/\alpha)^{-3/2} \left\{ 2 + \pi \sqrt{(1+\delta)T/\alpha^3} \right\} = O(T^{-1}). \end{aligned} \tag{3.38}$$

////

Step 2 [Calculation of the sum over m**] :** Here we deal with the sum over m by changing variable and using Taylor's expansion. Put

$$\psi^T(u;f,r) = H\left(\sqrt{T} A^T \left(\frac{\iota(u)}{\sqrt{T}} + \frac{f+r}{T} \right) \right) \tag{3.39}$$

for $f, r \in \mathbb{R}^n$, and

$$\phi^{T,m} = \phi^T \left\{ 1 + m^{-1/2} p^T \right\}, \ \ \phi^{T,m}_\xi = (\partial_j \phi^{T,m})^{n'}_{j=1}, \ \ \phi^{T,m}_\eta = \partial_{n'+1} \phi^{T,m}. \tag{3.40}$$

Changing variables: $\xi = u\sqrt{T/m}$, $\eta = (v-\alpha m - l)/\sqrt{m}$ and using Taylor's expansion, the summand of (3.35) turns to

$$\begin{aligned} &\frac{1}{\sqrt{m}} \left(\frac{T}{m} \right)^{n'/2} \int \psi^T(u;f,r) 1_{\{0 \le T-v<t\}} \\ &\times \left\{ \phi^{T,m}(\sqrt{\alpha}u, b^T_m) + \phi^{T,m}_\xi(\sqrt{\alpha}u, b^T_m) \cdot \theta^m_\xi + \phi^{T,m}_\eta(\sqrt{\alpha}u, b^T_m)\theta^m_\eta + A^m \right\} \\ &\times Q^T_R(dr,dt;T-v)Q^T_F(df,dl)dudv, \end{aligned} \tag{3.41}$$

where

$$\theta^m_\xi = (\sqrt{T/m} - \sqrt{\alpha})u, \ \ \theta^m_\eta = (v-l-T)/\sqrt{m} \tag{3.42}$$

and A^m is a negligible reminder term as we see in the following lemma.

Lemma 3.4. *It holds*

$$\begin{aligned} &\sum_{m;T} \frac{1}{\sqrt{m}} \left(\frac{T}{m} \right)^{n'/2} \int |A^m| 1_{\{0 \le T-v<t\}} \\ &\qquad Q^T_R(dr,dt;T-v)Q^T_F(df,dl)dudv = O(T^{-1}). \end{aligned} \tag{3.43}$$

Proof We can put $A^m = A^m_{2,0} + A^m_{1,1} + A^m_{0,2}$ where

$$\begin{aligned} A^m_{2,0} &= \sum_{i,j=1}^{n'} \int_0^1 (1-s)\partial_i\partial_j\phi^{T,m}(\sqrt{\alpha}u + s\theta^m_\xi, b^T_m + s\theta^m_\eta)ds\{\theta^m_\xi\}_i\{\theta^m_\xi\}_j\ , \\ A^m_{1,1} &= \sum_{j=1}^{n'} \int_0^1 2(1-s)\partial_j\partial_{n'+1}\phi^{T,m}(\sqrt{\alpha}u + s\theta^m_\xi, b^T_m + s\theta^m_\eta)ds\{\theta^m_\xi\}_j\theta^m_\eta\ , \\ A^m_{0,2} &= \int_0^1 (1-s)\partial^2_{n'+1}\phi^{T,m}(\sqrt{\alpha}u + s\theta^m_\xi, b^T_m + s\theta^m_\eta)ds\{\theta^m_\eta\}^2. \end{aligned} \tag{3.44}$$

We shall show that these $A^m_{i,j}$'s are negligible up to $O(T^{-1})$. Notice that

$$0 \le \frac{\sqrt{m}}{a^T_m}\left(\sqrt{\frac{T}{\alpha m}} - 1\right) = (\alpha)^{-1}\left(\sqrt{\frac{T}{\alpha m}} + 1\right)^{-1} < \frac{1}{\alpha(1 + (1+\delta)^{-1/2})}, \tag{3.45}$$

so that $|\theta^m_\xi| \le m^{-1/2}C|b^T_m u|$ for m with $|T - \alpha m| < \delta T$ and some constant C. In the following, we use ϵ and C as generic positive constants independent of T and m. It also holds $|\sqrt{\alpha}u + s\theta^m_\xi|^2 \ge \alpha u^2/(1+\delta)$ for all $s \in [0,1]$, so that

$$\int |A^m_{2,0}|du \le \frac{C|b^T_m|^2}{m}\int_0^1 \exp(-\epsilon|b^T_m + s\theta^m_\eta|^2)ds. \tag{3.46}$$

Since

$$\begin{aligned} &\sum_{m;T} m^{-3/2}|b^T_m|^2 \exp(-\epsilon|b^T_m + s\theta^m_\eta|^2) \\ &\le CT^{-3/2}\sum_{m=-\infty}^{\infty} \frac{|T-\alpha m|^2}{T}\exp\left\{-\frac{\epsilon\alpha(T - \alpha m - s(T - v + l))^2}{T(1+\delta)}\right\} \\ &\le CT^{-3/2}\Bigg\{1 + |T - v + l|^2/\sqrt{T} + \\ &\int_{-\infty}^{\infty} \frac{|T - \alpha z - s(T-v+l)|^2}{T}\exp\left\{-\frac{\epsilon\alpha(T - \alpha z - s(T-v+l))^2}{T(1+\delta)}\right\}dz\Bigg\} \\ &\le CT^{-1}(1 + |T - v + l|^2/T) \end{aligned} \tag{3.47}$$

uniformly in $s \in [0,1]$, we conclude

$$\begin{aligned} &\sum_{m;T} \frac{1}{\sqrt{m}}\left(\frac{T}{m}\right)^{n'/2}\int |A^m_{2,0}|du 1_{\{0 \le T - v < t\}} \\ &\qquad Q^T_R(dr, dt; T - v)Q^T_F(df, dl)dv = O(T^{-1}). \end{aligned} \tag{3.48}$$

In the same manner, we obtain

$$\begin{aligned} &\sum_{m;T} \frac{1}{\sqrt{m}}\left(\frac{T}{m}\right)^{n'/2}\int |A^m_{0,2}|du 1_{\{0 \le T - v < t\}} \\ &\qquad \le CT^{-1}|T - v + l|^2 1_{\{0 \le T - v < t\}} \end{aligned} \tag{3.49}$$

and

$$\sum_{m;T} \frac{1}{\sqrt{m}} \left(\frac{T}{m}\right)^{n'/2} \int |A^m_{1,1}| du 1_{\{0 \le T - v < t\}} \le CT^{-1} |T - v + l| (1 + |T - v + l| / \sqrt{T}) 1_{\{0 \le T - v < t\}}, \tag{3.50}$$

hence $A_{1,1}$ and $A_{0,2}$ also are negligible. ////

It suffices then to cope with the expectations of the following terms with respect to $Q(dr, dt, df, dl, dv) := 1_{\{0 \le T - v < t\}} Q^T_R(dr, dt; T - v) Q^T_F(df, dl) dv$;

$$\begin{aligned} T_1 &= \sum_{m;T} \frac{1}{\sqrt{m}} \left(\frac{T}{m}\right)^{n'/2} \int \psi^T(u; f, r) \phi^{T,m}(\sqrt{\alpha} u, b^T_m) du, \\ T_2 &= \sum_{m;T} \frac{1}{\sqrt{m}} \left(\frac{T}{m}\right)^{n'/2} \int \psi^T(u; f, r) \phi^{T,m}_\xi(\sqrt{\alpha} u, b^T_m) du \cdot \theta^m_\xi, \\ T_3 &= \sum_{m;T} \frac{1}{\sqrt{m}} \left(\frac{T}{m}\right)^{n'/2} \int \psi^T(u; f, r) \phi^{T,m}_\eta(\sqrt{\alpha} u, b^T_m) du \theta^m_\eta. \end{aligned} \tag{3.51}$$

In the sequel, we write $S_1 \equiv S_2$ to mean

$$\int (S_1(r, t, f, l, v) - S_2(r, t, f, l, v)) Q(dr, dt, df, dl, dv) = O(T^{-1}). \tag{3.52}$$

Following Malinovskii [20] (see also Section 4.2 of [12]), we obtain

$$T_1 \equiv \int \psi^T(u; f, r) \int_{-\infty}^{\infty} \alpha^{n'/2-1} \phi^T(\sqrt{\alpha} u, \lambda) \left\{1 + \sqrt{\frac{\alpha}{T}} \left(p^T(\sqrt{\alpha} u, \lambda) + \frac{\lambda(n'-1)}{2\alpha}\right)\right\} d\lambda du. \tag{3.53}$$

In the same manner, we can prove $T_3 \equiv 0$ and

$$T_2 \equiv \frac{\alpha^{n'/2-1}}{2\sqrt{T}} \iint_{-\infty}^{\infty} \psi^T(u; f, r) u \cdot \partial_\xi \phi^T(\sqrt{\alpha} u, \lambda) \lambda d\lambda du. \tag{3.54}$$

Further, integrating in λ, we have

$$T_1 + T_2 \equiv \frac{1}{\alpha} \int \psi^T(u; f, r) \phi_\mu(u) \left\{1 - \frac{\rho^T \cdot q^\mu(u)}{\alpha \sqrt{T}} + \frac{1}{6\sqrt{T}} \sum_{i,j,k=1}^{n'} \mu^T_{i,j,k} p^\mu_{i,j,k}(u)\right\} du, \tag{3.55}$$

where ϕ_μ is the n'-dimensional normal density with mean 0, covariance matrix $(\mu^T_{i,j})$,

$$q^\mu(u) = -\phi_\mu(u)^{-1} \partial_u \phi_\mu(u), \quad p^\mu_{i,j,k}(u) = -\phi_\mu(u)^{-1} \partial_i \partial_j \partial_k \phi_\mu(u). \tag{3.56}$$

Step 3 [Transformation by A^T and calculation of the coefficients]: This is the last step of the proof. In order to deal with the transformation by A^T, we need a variant of Lemma 2.1 of Bhattacharya and Ghosh [2]; unfortunately, we cannot directly apply it because A^T depends on T and the additional terms f and r are involved, which derive from the first and last blocks in the decomposition of K^T. Put $A^{T,\zeta}(u) = A^T(\iota(u) + \zeta) - A^T(\zeta)$ for $u \in \mathbb{R}^{n'}$ and $\zeta \in \mathbb{R}^n$. (Recall that we are assuming $\mu^T = 0$ and $A^T(0) = 0$.) Let $B_0 \subset \mathbb{R}^{n'}$ be a neighborhood of 0 and $\zeta_0 > 0$, $T_0 > 0$ be constants such that

(1) $A^{T,\zeta}$ is four times continuously differentiable in u on B_0 for every $T \geq T_0$ and $|\zeta| \leq \zeta_0$,
(2) all the derivatives up to fourth order are uniformly bounded on B_0, $T \geq T_0$ and $|\zeta| \leq \zeta_0$,
(3) $\inf_{|\zeta|\leq\zeta_0, T\geq T_0} v^{T,\zeta} > 0$, where

$$v^{T,\zeta} = \sum_{k,l=1}^{n'} \mu^T_{k,l} h^{T,\zeta}_k a^{T,\zeta}_l, \quad a^{T,\zeta}_m = \partial_m A^{T,\zeta}(0). \tag{3.57}$$

Put

$$a^{T,\zeta} = (a^{T,\zeta}_k) \in \mathbb{R}^{n'}, \quad a^{T,\zeta}_{k,l} = \partial_k \partial_l A^{T,\zeta}(0) \tag{3.58}$$

and

$$\psi^{T,\zeta}(u) = H(\sqrt{T} A^{T,\zeta}(u/\sqrt{T}) + \sqrt{T} A^T(\zeta)). \tag{3.59}$$

Note that $\psi^{T,\zeta}(u) = \psi^T(u; f, r)$ if $\zeta = (f + r)/T$. By the argument of Lemma 2.1 of Bhattacharya and Ghosh [2] (see also Theorem 2.2 of Hall [14]), we have

$$\begin{aligned} &\int \psi^{T,\zeta}(u)\phi_\mu(u) \left\{ 1 - \frac{\rho^T \cdot q^\mu(u)}{\alpha\sqrt{T}} + \frac{1}{6\sqrt{T}} \sum_{i,j,k=1}^{n'} \mu^T_{i,j,k} p^\mu_{i,j,k}(u) \right\} du \\ &= \int H(z + \sqrt{T} A^T(\zeta)) \mathcal{P}_{T,\zeta}(z) dz + O(T^{-1}) \end{aligned} \tag{3.60}$$

uniformly in $\zeta \in \bar{B}(\zeta_0)$, where $\mathcal{P}_{T,\zeta}$ is defined by

$$\mathcal{P}_{T,\zeta}(z) = \phi(z; v^{T,\zeta}) + T^{-1/2} \left\{ A^{T,\zeta}_1 p_1(z; v^{T,\zeta}) + \frac{A^{T,\zeta}_3}{6} p_3(z; v^{T,\zeta}) \right\}, \tag{3.61}$$

$\phi(z; v^{T,\zeta})$ is the normal density with mean 0 and variance $v^{T,\zeta}$, and

$$\begin{aligned} p_1(z; v^{T,\zeta}) &= -\partial\phi(z; v^{T,\zeta}), \\ p_3(z; v^{T,\zeta}) &= -\partial^3\phi(z; v^{T,\zeta}), \\ A^{T,\zeta}_1 &= -\frac{1}{\alpha} \sum_{j=1}^{n'} a^{T,\zeta}_j \rho^T_j + \frac{1}{2} \sum_{k,l=1}^{n'} a^{T,\zeta}_{k,l} \mu^T_{k,l}, \\ A^{T,\zeta}_3 &= \sum_{k,l,m=1}^{n'} a^{T,\zeta}_k a^{T,\zeta}_l a^{T,\zeta}_m \mu^T_{k,l,m} + 3 \sum_{j,k,l,m=1}^{n'} a^{T,\zeta}_j a^{T,\zeta}_k a^{T,\zeta}_{l,m} \mu^T_{j,l} \mu^T_{k,m}. \end{aligned} \tag{3.62}$$

Now, since

$$\int 1_{\{|f+r|>\zeta_0 T\}} Q(dr, dt, df, dl, dv) = O(T^{-1}), \tag{3.63}$$

we can take $\zeta = (f+r)/T$ and change variable $w = z + \sqrt{T}A^T(\zeta)$ to have

$$\begin{aligned}(3.64)\quad &\int \psi^T(u; f, r)\phi_\mu(u)\left\{1 - \frac{\rho^T \cdot q^\mu(u)}{\alpha\sqrt{T}} + \frac{1}{6\sqrt{T}}\sum_{i,j,k=1}^{n'} \mu^T_{i,j,k} p^\mu_{i,j,k}(u)\right\} du \\ &\equiv \int \psi(w)\left\{\phi(w; v^T) + \frac{1}{\sqrt{T}}\left(\hat{A}^T_1 p_1(w; v^T) + \frac{A^T_3}{6} p_3(w; v^T)\right)\right\} dz,\end{aligned}$$

where $\hat{A}_1 = a^T \cdot (f + r - \iota(\rho^T)/\alpha) + \sum_{i,j} a^T_{i,j}\mu^T_{i,j}/2$. Finally, taking the expectation with respect to Q, we obtain (3.22).

4. Edgeworth formula for ergodic diffusions

4.1. **Notation and assumptions.** Here we treat a family of Itô-diffusions

$$(4.1)\qquad dX^T_t = b^T(X^T_t)dt + c^T(X^T_t)dW^{0T}_t$$

defined on a filtered probability space $(\Omega^T, \mathcal{F}^T, \{\mathbb{F}^T_t\}, P^T)$ and regenerative functionals K^T_t of the form

$$(4.2)\qquad \Big(\int_0^t f^T(X^T_s)ds, \int_0^t g^T_1(X^T_s)dW^{1T}_s, \dots, \int_0^t g^T_{n_2}(X^T_s)dW^{n_2T}_s, F^T(X^T_t) - F^T(X^T_0)\Big),$$

where f^T, $g^T = (g^T_1, \dots, g^T_{n_2})$, F^T are n_1, n_2, n_3-dimensional Borel functions on $\mathbb{R}$ respectively, $(W^{0T}, W^{1T}, \dots, W^{n_2T})$ is an n_2+1-dimensional standard $\{\mathbb{F}^T_t\}$-Brownian motion, and b^T, c^T are one-dimensional Borel functions on $\mathbb{R}$. Our aim here is to give sufficient conditions for Theorem 3.1 to hold. Let $\mathcal{L}_I$ and $\mathcal{L}_B$ be the sets of the locally integrable functions and the locally bounded Borel functions respectively. We assume the following condition throughout this section.

Condition 4.1. *There exist $T_0 > 0$, $\underline{b}$, $\overline{b} \in \mathcal{L}_I$, $a \in \mathcal{L}_B$ such that for all $T \geq T_0$,*

$$(4.3)\qquad \underline{b} \leq b^T/|c^T|^2 \leq \overline{b}, \quad (|F^T| \vee |f^T| \vee |g^T|^2 \vee 1)/|c^T|^2 \leq a,$$

and

$$(4.4)\qquad \int_0^\infty s^\dagger(v)dv = \int_{-\infty}^0 s^\dagger(v)dv = \infty, \quad \int_{\mathbb{R}} \frac{a(v)}{s^\dagger(v)}dv < \infty,$$

where

$$(4.5)\qquad s^\dagger(v) = \begin{cases}\exp\left\{-2\int_0^v \overline{b}(w)dw\right\} & v \geq 0,\\ \exp\left\{2\int_v^0 \underline{b}(w)dw\right\} & v < 0.\end{cases}$$

Define the scale function s^T_0 and the speed measure π^T_0 corresponding to (4.1) as

$$(4.6)\qquad s^T_0(v) = \int_0^v \exp\left\{-2\int_0^u \frac{b^T(w)}{c^T(w)^2}dw\right\}du, \quad \pi^T_0(dv) = \frac{dv}{c^T(v)^2(s^T_0)'(v)},$$

where $(s_0^T)'$ is the first derivative of s_0^T. Note that

$$|s_0^T(B)| \geq \int_B s^\dagger(v)dv, \quad \pi_0^T(B) \leq \int_B \frac{a(v)}{s^\dagger(v)}dv \tag{4.7}$$

for every Borel set $B \subset \mathbb{R}$, so that

$$s_0^T(\mathbb{R}) = \mathbb{R}, \quad M^T := \pi_0^T(\mathbb{R}) < \infty. \tag{4.8}$$

Hence the process X^T is ergodic for each T (see e.g., [26]) and K^T of (4.2) is actually a regenerative functional with respect to $\{\tau_j^T\}$ given by

$$\tau_{j+1}^T = \inf\{t > \tau_j^T; X_t^T = x \text{ and there exists } s \in (\tau_j^T, t) \text{ such that } X_s^T = y\} \tag{4.9}$$

with $\tau_0^T = 0$, where x and y are arbitrarily fixed two distinct points. This fact is due to the strong Markov property of X^T. Put

$$s^T(v) = M^T s_0^T(v), \quad \pi^T(dv) = \pi_0^T(dv)/M^T. \tag{4.10}$$

Then, as is well-known, π^T is the ergodic distribution of X^T, so that

$$K_t^T/t \to \mu^T = \left(\pi^T[f^T], 0\right) \tag{4.11}$$

as $t \to \infty$ for each $T \geq T_0$. It also holds

$$P^T\left[\int_{\tau_1^T}^{\tau_2^T} \varphi(X_t^T)dt\right] = P^T[\tau_2^T - \tau_1^T]\pi^T[\varphi] \tag{4.12}$$

for every positive $\varphi \in \mathcal{L}_B$. Hereafter, we often omit the upper script T for short;

$$X = X^T, W^j = W^{jT}, f = f^T, \tau_j = \tau_j^T, c = c^T, s = s^T, \dots \tag{4.13}$$

Note that $s(X)$ is a local martingale by Itô's formula, so that there exists a standard Brownian motion B such that $X = s^{-1}(B_{\langle s(X)\rangle})$ (see e.g., [17]). In particular, we have

$$\int_0^\tau \varphi(X_t)dt = \int_0^{\langle s(X)\rangle_\tau} \frac{\varphi(s^{-1}(B_t))}{\tilde{c}(B_t)^2}dt \tag{4.14}$$

for every stopping time τ and $\varphi \in \mathcal{L}_B$, where

$$\tilde{c}(v) = \tilde{c}^T(v) = c(s^{-1}(v))s'(s^{-1}(v)). \tag{4.15}$$

When considering a hitting time

$$\tau(z) = \inf\{t > 0; X_t = z\} \tag{4.16}$$

for $z \in \mathbb{R}$, we have

$$\langle s(X)\rangle_{\tau(z)} = \tilde{\tau}(z), \tag{4.17}$$

where

$$\tilde{\tau}(z) = \inf\{s > 0; B_s = s(z)\}. \tag{4.18}$$

Hence,

$$\int_0^{\tau(z)} \varphi(X_t)dt = \int_0^{\tilde{\tau}(z)} \frac{\varphi(s^{-1}(B_t))}{\tilde{c}(B_t)^2}dt. \tag{4.19}$$

This identity plays an essential role in the following argument.

4.2. **Smoothness without independent components.** We first consider the case $n_2 = 0$ in (4.2) to give sufficient conditions for Condition 3.2 to be satisfied. More precisely, we treat

$$K_t^T = \left(\int_0^t f(X_s)ds, F(X_t) - F(X_0) \right), \tag{4.20}$$

where

$$f = (f_1, \ldots, f_{n_1}) \;\; F = (F_1, \cdots, F_{n_3}). \tag{4.21}$$

What we want to show is that the conditional distribution of $K_{\tau_1}^T - \mu^T \tau_1$ given $X_0 = x$ is smooth, so that we can assume $\pi[f] = 0$ without loss of generality. Here we take $n = n_1 + n_3$ and $n' = n_1$. The following lemma is a slight generalization of Borizov's lemma [4].

Lemma 4.1. *Let $x < y$ be fixed and suppose $X_0 = x$. Let $\varphi = \varphi^T \in \mathcal{L}_B$ and suppose*

$$M_\varphi := \sup_{v \in [2x-y, y], T \geq T_0} \frac{|\varphi(v)|}{c(v)^2 s'(v)^2} < \infty. \tag{4.22}$$

Assume there exists an interval $I \subset (x, y)$ such that

$$m_\varphi := \inf_{v \in I, T \geq T_0} \frac{|\varphi(v)|}{c(v)^2 s'(v)^2} > 0. \tag{4.23}$$

Then the distribution of

$$\int_0^{\tau(y)} \varphi(X_t)dt \tag{4.24}$$

is infinite divisible with Lévy measure L^φ satisfying

$$-c_1 + \frac{c_2}{\sqrt{z}} \leq L^\varphi((z, \infty)) \leq c_3 + \frac{c_4}{\sqrt{z}} \tag{4.25}$$

for $z > 0$ or

$$-c_1 + \frac{c_2}{\sqrt{|z|}} \leq L^\varphi((-\infty, z]) \leq c_3 + \frac{c_4}{\sqrt{|z|}} \tag{4.26}$$

for $z < 0$, where $c_i, i = 1, 2, 3, 4$ are positive constants depending only on

$$M_\varphi, \;\; m_\varphi, \;\; |s(I)|, \;\; |s(y) - s(x)|. \tag{4.27}$$

Moreover, denoting by Ψ its characteristic function, there exists a constant $\epsilon > 0$ depending only on $c_i, i = 1, 2, 3, 4$ such that for every $u \in \mathbb{R}$ with $|u| \geq 1/\epsilon$, it holds

$$|\Psi(u)| \leq e^{-\epsilon \sqrt{|u|}}. \tag{4.28}$$

Proof Use the same argument as [4], Lemmas 1, 2 and 3 with the aid of (4.19). ////

Condition 4.2. *There exists an interval $[x_0, x_1] \subset \mathbb{R}$ such that*

(1) *$f = f^T$ converges to $f^\infty = (f_1^\infty, \ldots, f_{n_1}^\infty)$ uniformly on $[x_0, x_1]$.*
(2) *f^∞ is continuous on $[x_0, x_1]$.*
(3) *$1, f_1^\infty, \ldots, f_{n_1}^\infty$ are linearly independent on $[x_0, x_1]$.*

Under this condition, we take $(x, y) = (x_0, x_1)$ in (4.9) and consider the decomposition (3.2) with

$$G_j^T = \int_{\tau_j}^{\tau_{j+1}} f(X_t)dt, \quad N_j^T = F(X_{\tau_{j+1}}) - F(X_{\tau_j}) \tag{4.29}$$

because $F(X_{\tau_j}) = F(x_0)$ for all j. It suffices therefore to deal with the characteristic function Ψ^T of

$$\mathbb{K}^T = \left(\int_0^{\tau_1} f(X_s)ds, \tau_1 \right) \tag{4.30}$$

with initial condition $X_0 = x_0$. Denote by $\Psi_0^T(u)$ the characteristic function of

$$\left(\int_0^{\tau(x_1)} f(X_t)dt, \tau(x_1) \right) \tag{4.31}$$

with initial condition $X_0 = x_0$. Since

$$|\Psi^T(u)| \leq |\Psi_0^T(u)| \tag{4.32}$$

by the strong Markov property, it suffices to deal with Ψ_0^T instead of Ψ^T.

Proposition 4.2. *Under Conditions 4.1 and 4.2, there exists $\epsilon > 0$ such that it holds*

$$|\Psi_0^T(u)| \leq e^{-\epsilon\sqrt{|u|}} \tag{4.33}$$

for $T \geq 1/\epsilon$ and $|u| \geq 1/\epsilon$. In particular, Condition 3.2 hold for (4.20).

Proof Let $S^{n'}$ be the n'-dimensional unit surface and

$$\bar{f}^T(s, v) = \sum_{j=1}^{n'} s_j f_j(v) + s_{n'+1}, \tag{4.34}$$

for $s \in S^{n'}$. Note that for all $s \in S^{n'}$, there exist $T_s > 0$ and an interval $I_s \subset (x_0, x_1)$ such that

$$k(s) := \inf_{T \geq T_s, v \in I_s} |\bar{f}^T(s, v)| > 0 \tag{4.35}$$

by the assumptions. Then, there exists $\epsilon_s > 0$ such that for $t \geq 1/\epsilon_s$,

$$|\Psi_0^T(ts)| \leq e^{-\epsilon_s\sqrt{t}} \tag{4.36}$$

by Lemma 4.1. Put $k(t, s) = \inf_{T \geq T_s, v \in I_s} \bar{f}^T(t, v)$ for $t \in S^{n'}$. Since the open covering

$$\bigcup_{s \in S^{n'}} \left\{ t \in S^{n'}; k(t, s) > k(s)/2 \right\} \tag{4.37}$$

of the compact set $S^{n'}$ has a finite subcovering, we can conclude (4.33). To show the last assertion, use Petrov's lemma (see [24], p.10.). ////

4.3. **Smoothness with independent components.** This subsection treats the general case $n_2 \geq 1$. The following lemma is a variant of Lemma 4.1;

Lemma 4.3. *Let $x < y$ be fixed and suppose $X_0 = x$. Let $\varphi_1 = \varphi_1^T, \varphi_2 = \varphi_2^T \in \mathcal{L}_B$ and suppose*

$$M_\varphi := \sup_{v\in[2x-y,y],T\geq T_0} \frac{|\varphi_1(v)| \vee \varphi_2(v)^2}{c(v)^2 s'(v)^2} < \infty \tag{4.38}$$

Assume there exists an interval $I \subset (x, y)$ such that

$$m_\varphi := \inf_{v\in I,T\geq T_0} \frac{|\varphi_1(v)| \wedge \varphi_2(v)^2}{c(v)^2 s'(v)^2} > 0. \tag{4.39}$$

Then the distribution of

$$K_\varphi(u_1, u_2) = u_1 \int_0^{\tau(y)} \varphi_1(X_t)dt + u_2 \int_0^{\tau(y)} \varphi_2(X_t)dW_t^1 \tag{4.40}$$

is infinite divisible with Lévy measure L^φ satisfying at least one of the following two conditions.

(1) *It holds for $z > 0$ that*

$$-c_1 - \frac{c_2\sqrt{|u_1|}}{\sqrt{z}} + \frac{c_3|u_2|}{z} \leq L^\varphi((z,\infty)) \leq c_4 + \frac{c_5\sqrt{|u_1|}}{\sqrt{z}} + \frac{c_6|u_2|}{z} \tag{4.41}$$

and that

$$-c_1' + \frac{c_2'\sqrt{|u_1|}}{\sqrt{z}} - \frac{c_3'|u_2|}{z} \leq L^\varphi((z,\infty)) \leq c_4 + \frac{c_5\sqrt{|u_1|}}{\sqrt{z}} + \frac{c_6|u_2|}{z}. \tag{4.42}$$

(2) *It holds for $z < 0$ that*

$$-c_1 - \frac{c_2\sqrt{|u_1|}}{\sqrt{|z|}} + \frac{c_3|u_2|}{|z|} \leq L^\varphi((-\infty,z]) \leq c_4 + \frac{c_5\sqrt{|u_1|}}{\sqrt{|z|}} + \frac{c_6|u_2|}{|z|} \tag{4.43}$$

and that

$$-c_1' + \frac{c_2'\sqrt{|u_1|}}{\sqrt{|z|}} - \frac{c_3'|u_2|}{|z|} \leq L^\varphi((-\infty,z]) \leq c_4 + \frac{c_5\sqrt{|u_1|}}{\sqrt{|z|}} + \frac{c_6|u_2|}{|z|} \tag{4.44}$$

where c_i, c_i' are positive constants depending only on

$$M_\varphi, \ \ m_\varphi, \ \ |s(I)|, \ \ |s(y) - s(x)|. \tag{4.45}$$

Moreover, denoting by Ψ the characteristic function of

$$\left(\int_0^{\tau(y)} \varphi_1(X_t)dt, \int_0^{\tau(y)} \varphi_2(X_t)dW_t^1\right), \tag{4.46}$$

there exists a constant $\epsilon > 0$ depending only on $c_i, c_i', i = 1, 2, 3, 4$ such that for every $u \in \mathbb{R}^2$ with $|u| \geq 1/\epsilon$, it holds

$$|\Psi(u)| \leq e^{-\epsilon\sqrt{|u|}}. \tag{4.47}$$

Proof We will use (4.19) repeatedly without any notice. Put $\alpha = s(y) - s(x)$. Let $\tau_{i/m}$, $i = 1, \dots, m$ be the times at which X first attains the levels $s^{-1}(s(x) + i\alpha/m)$. Let $\tilde{\tau}_{i/m}$, $i = 1, \dots, m$ be the times at which B first attains the levels $s(x) + i\alpha/m$. Putting

$$K_\varphi^{mi} = u_1 \int_{\tau_{(i-1)/m}}^{\tau_{i/m}} \varphi_1(X_t)dt + u_2 \int_{\tau_{(i-1)/m}}^{\tau_{i/m}} \varphi_2(X_t)dW_t^1, \tag{4.48}$$

we have the expression

$$K_\varphi(u_1, u_2) = \sum_{i=1}^m K_\varphi^{mi} \tag{4.49}$$

for every $m \in \mathbb{N}$. Note that K_φ^{mi}, $i = 1, \dots, m$ are independent by the strong Markov property of X. Besides, $\{K_\varphi^{mi}\}$ is a null array since

$$\sup_{1 \le i \le m} P[|K_\varphi^{mi}| > \epsilon] \le P[M_\varphi |u_1| \tilde{\tau}_{1/m} > \epsilon/2] + P[M_\varphi \tilde{\tau}_{1/m} u_2^2 \mathcal{N}^2 > \epsilon^2/4], \tag{4.50}$$

which converges to 0 as $m \to \infty$, where $\mathcal{N}$ is a standard normal variable independent of B. Here we use the fact that K_φ^{mi} has the same distribution as

$$u_1 K_\varphi^{mi,1} + \sqrt{u_2^2 K_\varphi^{mi,2}}\mathcal{N}, \tag{4.51}$$

where

$$K_\varphi^{mi,1} = \int_{\tilde{\tau}_{(i-1)/m}}^{\tilde{\tau}_{i/m}} \frac{\varphi_1(s^{-1}(B_t))}{\tilde{c}(B_t)^2}dt, \quad K_\varphi^{mi,2} = \int_{\tilde{\tau}_{(i-1)/m}}^{\tilde{\tau}_{i/m}} \frac{\varphi_2(s^{-1}(B_t))^2}{\tilde{c}(B_t)^2}dt. \tag{4.52}$$

Hence, $K_\varphi(u_1, u_2)$ is infinitely divisible. (see e.g., [6], XVII.7.) To obtain the inequalities for L^φ, as in [4], we use the following fact; for every continuity point $z > 0$ of L^φ,

$$\lim_{m \to \infty} \sum_{i=1}^m P[K_\varphi^{mi} > z] = L^\varphi((z, \infty)) \tag{4.53}$$

and for every continuity point $z < 0$ of L^φ,

$$\lim_{m \to \infty} \sum_{i=1}^m P[K_\varphi^{mi} \le z] = L^\varphi((-\infty, z]). \tag{4.54}$$

(see e.g., [6], XVII.7.) Observe that for $z > 0$

$$\begin{aligned} P\left[\sqrt{u_2^2 K_\varphi^{mi,2}}\mathcal{N} > z\right] &\le \int_0^\infty \int_{z/\sqrt{u_2^2 M_\varphi t}}^\infty \phi(y;1)dy \frac{\alpha}{m\sqrt{2\pi t^3}} \exp\left\{-\frac{\alpha^2}{2tm^2}\right\} dt \\ &= \int_0^\infty \int_{z/\sqrt{u_2^2 M_\varphi}}^\infty \frac{\alpha}{2\pi m t^2} \exp\left\{-\frac{\alpha^2 + m^2 u^2}{2tm^2}\right\} dudt \\ &= \int_{z/\sqrt{u_2^2 M_\varphi}}^\infty \int_0^\infty \frac{\alpha}{2\pi m} \exp\left\{-\frac{s}{2}\left\{u^2 + \frac{\alpha^2}{m^2}\right\}\right\} dsdu \\ &\le \frac{\alpha |u_2| \sqrt{M_\varphi}}{m\pi z}, \end{aligned} \tag{4.55}$$

as well as for $z < 0$,

$$P\left[\sqrt{u_2^2 K_\varphi^{mi,2}} \mathcal{N} \le z \right] \le \frac{\alpha |u_2| \sqrt{M_\varphi}}{m\pi |z|}. \tag{4.56}$$

On the other hand, by Lemma 4.1, it holds

$$-c_1 + \frac{c_2 \sqrt{|u_1|}}{\sqrt{z}} \le \lim_{m\to\infty} \sum_{i=1}^m P[u_1 K_\varphi^{mi,1} > z] \le c_3 + \frac{c_4 \sqrt{|u_1|}}{\sqrt{z}} \tag{4.57}$$

for $z > 0$ or

$$-c_1 + \frac{c_2 \sqrt{|u_1|}}{\sqrt{|z|}} \le \lim_{m\to\infty} \sum_{i=1}^m P[u_1 K_\varphi^{mi,1} \le z] \le c_3 + \frac{c_4 \sqrt{|u_1|}}{\sqrt{|z|}} \tag{4.58}$$

for $z < 0$, where $c_i, i = 1, 2, 3, 4$ are positive constants depending only on

$$M_\varphi, \ \ m_\varphi, \ \ |s(I)|, \ \ |s(y) - s(x)|. \tag{4.59}$$

Hence, it is straightforward to obtain the result by using e.g.,

$$\begin{aligned} P[K_\varphi^{mi} > z] &\ge P[u_1 K_\varphi^{mi,1} > 2z] - P\left[\sqrt{u_2^2 K_\varphi^{mi,2}} \mathcal{N} \le -z \right], \\ P[K_\varphi^{mi} > z] &\ge P\left[\sqrt{u_2^2 K_\varphi^{mi,2}} \mathcal{N} > 2z \right] - P[u_1 K_\varphi^{mi,1} \le -z], \\ P[K_\varphi^{mi} > z] &\le P\left[\sqrt{u_2^2 K_\varphi^{mi,2}} \mathcal{N} > z/2 \right] + P[u_1 K_\varphi^{mi,1} > z/2]. \end{aligned} \tag{4.60}$$

To show the last inequality for Ψ, consider the distribution of $K_\varphi(u_1, u_2)$ for $u = (u_1, u_2) \in S^1$. Let us treat the case (4.41) and (4.42) hold for example. Note that by the Lévy-Khinchin expression, there exists a constant $\sigma^2 \ge 0$ such that

$$\mathrm{Re}\, \Psi(tu) = -\sigma^2 t^2/2 - 2 \int_{\mathbb{R}} \sin^2(zt/2) L^\varphi(dz) \tag{4.61}$$

for $t \ge 0$, where $u \in S^1$ is fixed and L^φ corresponds to $K_\varphi(u_1, u_2)$. Take $z_0, z_1 > 0$ such that $z_0 < z_1$, $z \mapsto \sin(z)$ is increasing on $[z_0/2, z_1/2]$, $c_3/z_0 > c_6/z_1$ and $c_2'/\sqrt{z_0} > c_5/\sqrt{z_1}$. Then we have

$$\begin{aligned} &\int_{\mathbb{R}} \sin^2(zt/2) L^\varphi(dz) \\ &\ge \int_{(z_0/|t|, z_1/|t|]} \sin^2(zt/2) L^\varphi(dz) \\ &\ge \sin^2(z_0/2) L^\varphi((z_0/|t|, z_1/|t|]) \\ &\ge \left(\frac{c_3}{z_0} - \frac{c_6}{z_1} \right) |u_2||t| - \left(\frac{c_2}{\sqrt{z_0}} + \frac{c_6}{\sqrt{z_1}} \right) \sqrt{|u_1||t|} - c_1 - c_4, \end{aligned} \tag{4.62}$$

as well as

$$\ge \left(\frac{c_2'}{\sqrt{z_0}} - \frac{c_5}{\sqrt{z_1}} \right) \sqrt{|u_1||t|} - \left(\frac{c_3'}{z_0} + \frac{c_6}{z_1} \right) |u_2||t| - c_1' - c_4. \tag{4.63}$$

Using (4.62) for the case $|u_2|\sqrt{|t|} \geq 1$ and (4.63) for the case $|u_2|\sqrt{|t|} < 1$, we conclude the last assertion. ////

Let us show that Condition 3.2 is satisfied also in the case that $n_2 \geq 1$. Here we consider $n = n_1 + n_2 + n_3$ and $n' = n_1 + n_2$. We again assume $\pi[f] = 0$ and Condition 4.2. Then, take $(x, y) = (x_0, x_1)$ in (4.9) as before. By the same reason as in the preceding subsection, it suffices to deal with the characteristic function Ψ_0^T of

$$\left(\int_0^{\tau(x_1)} f(X_t)dt, \int_0^{\tau(x_1)} g_1(X_t)dW_t^1, \dots, \int_0^{\tau(x_1)} g_{n_2}(X_t)dW_t^{n_2}, \tau(x_1)\right) \tag{4.64}$$

with initial condition $X_0 = x_0$. We will assume the following in addition;

Condition 4.3.

$$c_g := \inf_{v\in[x_0,x_1], i=1,\dots,n_2, T\geq T_0} |g_i(v)|^2 > 0. \tag{4.65}$$

Proposition 4.4. *Under Conditions 4.1, 4.2 and 4.3, there exists $\epsilon > 0$ such that it holds*

$$|\Psi_0^T(u)| \leq e^{-\epsilon\sqrt{|u|}} \tag{4.66}$$

for $T \geq 1/\epsilon$ and $|u| \geq 1/\epsilon$. In particular, the conditions of Condition 3.2 hold for (4.2).

Proof For $u = (u_1, u_2, u_3) \in \mathbb{R}^{n'+1}$, $u_1 \in \mathbb{R}^{n_1}$, $u_2 \in \mathbb{R}^{n_2}$, $u_3 \in \mathbb{R}$, put $v_1 = (u_1, u_3) \in \mathbb{R}^{n_1+1}$, $v_2 = u_2 \in \mathbb{R}^{n_2}$. It suffices to treat the characteristic function of

$$|v_1|\int_0^{\tau(x_1)} \bar{f}^T(v_1/|v_1|, X_t)dt + |v_2|\int_0^{\tau(x_1)} g(v_2/|v_2|, X_t)dW_t^1, \tag{4.67}$$

where

$$\bar{f}^T(s, x) = \sum_{j=1}^{n'} s_j f_j(x) + s_{n_1+1}, \tag{4.68}$$

for $s \in S^{n_1}$ and

$$g(s, x) = \sqrt{\sum_{j=1}^{n_2} s_j^2 g_j(x)^2} \geq \sqrt{c_g} \tag{4.69}$$

for $s \in S^{n_2-1}$. Note that for all $s \in S^{n_1}$, there exist $T_s > 0$ and an interval $I_s \subset (x_0, x_1)$ such that

$$k(s) := \inf_{T\geq T_s, v\in I_s} |\bar{f}^T(s, v)| > 0 \tag{4.70}$$

by the assumptions. Then, there exists $\epsilon_s > 0$ such that for $t \geq 1/\epsilon_s$,

$$|\Psi_0^T(ts)| \leq e^{-\epsilon_s\sqrt{t}} \tag{4.71}$$

by Lemma 4.3. Put $k(t, s) = \inf_{T\geq T_s, v\in I_s} \bar{f}^T(t, v)$ for $t \in S^{n_1}$. Since the open covering

$$\bigcup_{s\in S^{n_1}} \{t \in S^{n_1}; k(t, s) > k(s)/2\} \tag{4.72}$$

of the compact set S^{n_1} has a finite subcovering, we conclude (4.66). To show the last assertion, use Petrov's lemma (see [24], p.10.). ////

4.4. **On the moment conditions.** Here we consider Condition 3.1 in the case that K^T has the form

$$K_t^T = \left(\int_0^t f(X_s)ds, \int_0^t \varphi(X_s)dW_s^0, \int_0^t \psi(X_s)dW_s^1\right), \tag{4.73}$$

where $f^T, \varphi^T, \psi^T \in \mathcal{L}_B$; it will be straightforward to extend the results here to the cases of higher dimensional regenerative functionals such as (4.2). Put

$$\begin{aligned} \gamma &= -\limsup_{|x|\to\infty, T\to\infty} \frac{xb(x)}{c(x)^2} \\ \gamma_+ &= -\limsup_{x\to\infty, T\to\infty} \frac{b(x)}{c(x)^2}, \\ \gamma_- &= \liminf_{x\to-\infty, T\to\infty} \frac{b(x)}{c(x)^2}. \end{aligned} \tag{4.74}$$

Here we abuse the notation a little with $\gamma, \gamma_\pm$, which have been already used in Section 3.2. The following lemma is a variant of Theorem 2 of [12].

Lemma 4.5. *If there exist $p \geq 0$, $N \in \mathbb{N}$ such that*

$$2\gamma + 1 > Np, \quad \limsup_{|x|\to\infty, T\to\infty} \frac{1 + |f(x)|}{|x|^{p-2}c(x)^2} < \infty, \tag{4.75}$$

then, for every $x_0, x_1 \in \mathbb{R}$,

$$\limsup_{T\to\infty} P^T\left[\left|\int_0^{\tau(x_1)} f(X_t)dt\right|^N \middle| X_0 = x_0\right] < \infty. \tag{4.76}$$

Proof The fact that the process $X = X^T$ and $f = f^T$ depend on T is beyond Theorem 2 of [12]. It is however easy to see the same reasoning is valid here under the condition (4.75). ////

Lemma 4.6. *If there exist $\kappa_\pm \geq 0$, $N \in \mathbb{N}$ such that*

$$2\gamma_\pm > N\kappa_\pm, \quad \limsup_{x\to\pm\infty, T\to\infty} \frac{1 + |f(x)|}{e^{\kappa_\pm |x|}c(x)^2} < \infty, \tag{4.77}$$

then, for every $x_0, x_1 \in \mathbb{R}$,

$$\limsup_{T\to\infty} P^T\left[\left|\int_0^{\tau(x_1)} f(X_t)dt\right|^N \middle| X_0 = x_0\right] < \infty. \tag{4.78}$$

Proof Define $P_x[\cdot] = P^T[\cdot|X_0 = x]$ and

$$G_f^k(x) = P_x\left[\int_0^{\tau(x_1)} f^T(X_t^T)G_f^{k-1}(X_t)dt\right] \tag{4.79}$$

recursively for $k \in \mathbb{N}$, where $G^0_f(x) \equiv 1$ and x_1 is fixed. As in the proof of Theorem 2 of [12], we use Kac's moment formula [7]

$$P_x\left[\left|\int_0^{\tau(x_1)} |f(X_t)|dt\right|^k\right] = k! G^k_{|f|}(x) \tag{4.80}$$

and an explicit expression

$$G^1_{|f|}(x) = 2\int_x^{x_1} (s(x_1) - s(z))|f(z)|\pi(dz) + 2(s(x_1) - s(x))\int_{-\infty}^x |f(z)|\pi(dz) \tag{4.81}$$

in case $x \leq x_1$, and

$$G^1_{|f|}(x) = 2(s(x) - s(x_1))\int_x^{\infty} |f(z)|\pi(dz) + 2\int_{x_1}^x (s(z) - s(x_1))|f(z)|\pi(dz) \tag{4.82}$$

in case $x > x_1$. Denote by π' the density of π. Take $\tilde{\gamma}_\pm$ such that $\gamma_\pm > \tilde{\gamma}_\pm > N\kappa_\pm/2$. Then there exists $A > 0$ such that if $|w| > A$, $T > A$ and $(z - w)(w - A) > 0$, then

$$s'(w)|f(z)|\pi'(z) \leq e^{\kappa_\pm |z|} \exp\left\{2\int_w^z \frac{b(v)}{c(v)^2}dv\right\} \leq e^{\kappa_\pm|z| - 2\tilde{\gamma}_\pm|z-w|} \tag{4.83}$$

where $\pm$ coincides with the signature of w. This inequality implies that

$$\limsup_{x\to\pm\infty, T\to\infty} e^{-\kappa_\pm|x|} G_{|f|}(x) < \infty \tag{4.84}$$

as long as f satisfies (4.77). The result then follows the iterative use of (4.80). ////

Proposition 4.7. *Assume that (4.75) or (4.77) holds for f with $N = 4$ and for $\varphi^2 \vee \psi^2$ with $N = 2$ instead of f. Then Condition 3.1 holds for K^T of (4.73) and τ^T_j of (4.9) for every $x, y \in \mathbb{R}$.*

Proof Use the strong Markov property of X, the preceding two lemmas, the Cauchy-Schwarz and Burkholder-Davis-Gundy inequalities. ////

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