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# The Effects of a Debt Financing Constraint in a Real Options Model

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**Abstract**

This paper derives the firm value and the investment strategy (investment timing, debt financing, leverage, and endogenous default) when an entrepreneur makes a real investment with debt financing both in monopoly and in duopoly. In particular, we clarify the effects of the entrepreneur's financing constraint where a part of the investment cost must be financed by debt. The leverage and the credit spread of the constrained entrepreneur are higher than those of the unconstrained one. The investment timing of the constrained entrepreneur is later, which is consistent with the standard underinvestment theory. The financial restriction binds more tightly in duopoly than in monopoly. Surprisingly, however, in duopoly the financing constraint plays a role in moderating the preemptive competition and improving the firm value in equilibrium.

**Keywords:** real options, debt financing, financing constraint, internal funds, underinvestment, stopping game

# 1 Introduction

The real options approach has become an increasingly standard framework for the investment timing decision in corporate finance (see [2]). Although the early literature on real options investigated the monopolist's investment decisions, recent studies have investigated the problem of several firms competing in the same market from a game theoretic approach (e.g., [5, 7, 15, 19]).

From a different perspective, one of the most important problems in corporate finance is the derivation of the optimal capital structure. The theory of the optimal capital struc-

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ture obtained through the trade-off between tax advantages and default costs, proposed by [14], has subsequently been developed in more recent work, including [4, 11].

Naturally, studies on the capital structure and financing have a deep connection with those concerning the investment timing decision. In [12, 17, 18], firm value, investment timing, debt financing, and endogenous bankruptcy have been simultaneously studied in a model where a firm undertakes real investment alongside the issuance of debt.[1] In this context, this paper investigates the effects of the entrepreneur's financial constraint where a part of the investment cost must be financed by debt. This work is motivated by the observation that few entrepreneurs have enough internal funds to cover the investment cost. Not to mention, from various aspects, a number of researchers have clarified the effects of financing constraints in determining investment. In the real options context, papers [1, 9] investigated the relationship between financial restrictions and investment timing. However, this paper differs substantially from them in that we rigorously model the entrepreneur's debt financing for investment.

We theoretically show that the entrepreneur with less internal funds makes an investment with the issuance of debt at a higher coupon payment than the unrestricted firm. It is also shown that the leverage and the credit spread are higher. We numerically check that the financing constraint delays the investment timing. This is consistent with the standard theoretical and empirical results of underinvestment for a more financially constrained firm (e.g., [3, 6])[2].

Furthermore, we extend the analysis to duopoly where two entrepreneurs attempt to preempt each other for the first mover's advantage. We derive the entrepreneurs' strategy and value in the equilibrium of the stopping game. The hastened timing through the preemptive competition makes the constraint bind more tightly, which means a higher leverage and a larger credit spread than in monopoly. We can also show that the preemptive investment timing in duopoly of the constrained entrepreneurs is later than in the unconstrained duopoly, which is consistent with the standard underinvestment theory.

In addition to the plausible results mentioned above, we obtain two surprising results as follows. First, the financing constraint moderates the preemptive competition and therefore improves the firm value in equilibrium. This means that the financial restriction, which has only a negative effect on a monopolist, brings about a positive effect on firms under competition. Second, an infinitesimal difference between the financial restrictions of the entrepreneurs makes a significant change in the outcome.

The paper is organized as follows. Section 2 introduces the setup and the preliminary results in the case of no financial restriction. Section 3 provides our main results, which include the investment strategy and the firm value of the constrained entrepreneur both

[1] The earlier work of [13] also investigated simultaneous financing and investment decisions in a dynamic model from a different approach.

[2] It should be noted that some recent studies including [8] have challenged the positive monotonic relationship between investment and financing constraints.

in monopoly and in duopoly. In this section, we also give comparative statistics and economic implications using numerical examples. Section 4 concludes the paper.

# 2 Preliminaries

## 2.1 Setup

We consider a risk-neutral entrepreneurial firm that has an option to start a new project. The entrepreneur can choose the investment time by observing market demand $X(t)$ at time $t$. The firm collects a profit flow $QX(t)$ by paying a sunk cost $I$ and initiating the project, where $Q$ and $I$ are positive constants. Assume that the firm faces a constant tax rate $\tau \in (0,1)$. For simplicity, we assume that $X(t)$ obeys the following geometric Brownian motion:

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}B(t), X(0) = x(> 0),$$

where $\mu(< r)$ ($r$ denotes the risk-free interest rate) and $\sigma(> 0)$ are constants and $B(t)$ represents the one-dimensional standard Brownian motion. The initial value $X(0) = x$ is a sufficiently small constant so that the entrepreneur has to wait for its entry condition to be met.

## 2.2 Monopolist without financing constraint

As a benchmark, we consider an entrepreneur without issuance of debt, which means that he/she either has sufficient internal funds; i.e., available money or equity financing, to start the project. The entrepreneur's problem of finding the optimal investment time is expressed as follows:

$$\begin{aligned} V_{ae}(x) &= \sup_{T\in\mathcal{T}} \mathbb{E}[\int_T^{+\infty} \mathrm{e}^{-rt}(1-\tau)QX(t)\mathrm{d}t - \mathrm{e}^{-rT}I], \\ &= \sup_{T\in\mathcal{T}} \mathbb{E}[\mathrm{e}^{-rT}(\Pi(X(T)) - I)], \end{aligned} \tag{1}$$

where $\mathcal{T}$ is a set of all $\mathcal{F}_t$ stopping times, ($\{\mathcal{F}_t\}$ is the usual filtration generated by $B(t)$) and $\Pi(X(T))$ is defined by:

$$\Pi(X(T)) = \frac{1-\tau}{r-\mu} QX(T). \tag{2}$$

It is easily shown (see, for example, [2]) that the optimal investment time $T_{ae}^i$ of problem (1) is:

$$T_{ae}^i = \inf\left\{t > 0 \mid X(t) \geq x_{ae}^i = \frac{\beta I}{(\beta-1)\Pi(1)}\right\},$$

and the firm value $V_{ae}(x)$ is:

$$V_{ae}(x) = \left(\frac{x}{x_{ae}^i}\right)^{\beta} (\Pi(x_{ae}^i) - I), \tag{3}$$

where $\beta$ is a positive characteristic root defined by:

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \ (> 1).$$

Now, we explain the investment strategy and the firm value of the firm that has the optimal capital structure as presented in [17]. In the usual manner, we solve the problem backward. Assume that the entrepreneur has already invested at time $s$ with market demand $X(s)$ along with issuing debt with coupon $c$. [3] The entrepreneur has an incentive to default after the debt is in place. He/she chooses the default time $T^d$ so as to maximize his/her own value, which can be regarded as the equity value[4], as follows:

$$\begin{aligned} &E(X(s), c) \\ &= \operatorname*{ess\,sup}_{\substack{T \in \mathcal{T} \\ T \geq s}} \mathbb{E}[\int_s^T \mathrm{e}^{-r(t-s)}(1-\tau)(QX(t) - c)\mathrm{d}t \mid \mathcal{F}_s]. \end{aligned}$$

The optimal default time $T^d$ is $T^d = \inf\{t \geq s \mid X(t) \leq x^d(c)\}$, where the default trigger $x^d(c)$ is the function defined by:

$$x^d(c) = \frac{\gamma}{\gamma - 1} \frac{r - \mu}{r} \frac{c}{Q}. \tag{4}$$

Here $\gamma$ denotes a negative characteristic root defined by:

$$\gamma = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \ (< 0).$$

Then, at time $s$ the equity value $E(X(s), c)$, the debt value $D(X(s), c)$, and the firm value $V(X(s), c) = E(X(s), c) + D(X(s), c)$ are expressed as:

$$E(X(s), c) = \Pi(X(s)) - \frac{(1-\tau)c}{r} - \left(\Pi(x^d(c)) - \frac{(1-\tau)c}{r}\right)\left(\frac{X(s)}{x^d(c)}\right)^\gamma,$$

$$D(X(s), c) = \frac{c}{r} - \left(\frac{c}{r} - (1-\alpha)\Pi(x^d(c))\right)\left(\frac{X(s)}{x^d(c)}\right)^\gamma, \tag{5}$$

$$V(X(s), c) = \Pi(X(s)) + \frac{\tau c}{r} - \left(\alpha\Pi(x^d(c)) + \frac{\tau c}{r}\right)\left(\frac{X(s)}{x^d(c)}\right)^\gamma, \tag{6}$$

for $X(s) \geq x^d(c)$. If $X(s) < x^d(c)$, $E(X(s), c) = 0$ and $V(X(s), c) = D(X(s), c) = (1-\alpha)\Pi(X(s))$. Here, $\alpha \in [0, 1]$ is a given constant representing the default cost. Note that the bondholders collect the entire default value; i.e., $(1-\alpha)\Pi(x^d(c))$. It should be noted that the above equity value, debt value, and firm value are essentially the same as those in [11].

[3]This paper considers a bond with infinite maturity (i.e., the bondholders receive coupon payments until the firm's bankruptcy) as in the existing literature such as [11, 4, 17, 18], so that we can deduce the analytical results.

[4]Throughout this paper, we use the terminology "equity value" by convention. However, the model neither distinguishes between the entrepreneur and equityholders, nor requires the issuance of the equity. The entrepreneur uses internal financing and debt financing for the project.

Throughout this paper, we assume that the bondholders behave competitively, and we also assume that there is no asymmetric information between the entrepreneur and the bondholders. That is, the entrepreneur can borrow the whole debt value (5) from the bondholders. Then, the entrepreneur chooses the investment time $T^i$ and coupon $c$ to maximize the firm value (6), which equals his/her gain from the project. The entrepreneur's problem is the following:

$$V_{de}(x) = \sup_{\substack{T \in \mathcal{T} \\ c:\mathcal{F}_T \text{measurable}}} \mathbb{E}[\mathrm{e}^{-rT}(V(X(T), c) - I)]. \tag{7}$$

Note that $\arg\max_{c \geq 0} V(X(s), c)$ becomes:

$$c(X(s)) = \frac{r}{r-\mu} \frac{\gamma - 1}{\gamma} \frac{QX(s)}{h} \ (>0), \tag{8}$$

for $X(s) > 0$, where $h$ is a constant given by:

$$h = \left[1 - \gamma\left(1 - \alpha + \frac{\alpha}{\tau}\right)\right]^{-\frac{1}{\gamma}} \ (>1). \tag{9}$$

Substituting (8) into (6), we can show:

$$V(X(s), c(X(s))) = \psi^{-1}\Pi(X(s)), \tag{10}$$

where the function $\Pi(\cdot)$ is defined by (2) and $\psi$ is a positive constant defined by:

$$\psi = \left[1 + \frac{\tau}{(1-\tau)h}\right]^{-1} \ (<1).$$

Thus, problem (7) can be rewritten as:

$$V_{de}(x) = \sup_{T \in \mathcal{T}} \mathbb{E}[\mathrm{e}^{-rT}(\psi^{-1}\Pi(X(T)) - I)].$$

The optimal investment time $T^i$ is $T^i = \inf\{t > 0 \mid X(t) \geq x^i\}$, and the optimal coupon $c^i$ is $c^i = c(x^i)(>0)$, where the function $c(\cdot)$ is defined by (8) and the investment trigger $x^i$ is given by:

$$x^i = \psi x^i_{ae}. \tag{11}$$

It follows from $\psi < 1$ that $x^i < x^i_{ae}$; that is, investment takes place earlier than in the case of no debt issuance. From (4) and (8), we have the default trigger:

$$x^d(c(x^i)) = \frac{x^i}{h}. \tag{12}$$

The firm value $V_{de}(x)$ at the initial time can be calculated as:

$$V_{de}(x) = \left(\frac{x}{x^i}\right)^{\beta} (\psi^{-1}\Pi(x^i) - I) = \psi^{-\beta} V_{ae}(x). \tag{13}$$

That is, the firm value of the entrepreneur with the optimal capital structure becomes $\psi^{-\beta}(>1)$ times the firm value (3) in the no-debt case.

The leverage $LV$ and the credit spread $CS$ at the time of investment, $T^i$, are:

$$\begin{aligned} LV &= \frac{D(x^i, c(x^i))}{V(x^i, c(x^i))} \\ &= \frac{\gamma - 1}{\gamma} \frac{\psi(1-\xi)}{h(1-\tau)}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} CS &= \frac{c(x^i)}{D(x^i, c(x^i))} - r \\ &= r\frac{\xi}{1-\xi}, \end{aligned} \tag{15}$$

respectively, where $\xi$ is a positive constant defined by:

$$\xi = \left(1 - (1-\alpha)(1-\tau)\frac{\gamma}{\gamma - 1}\right) h^{\gamma} (< 1).$$

Note that neither (14) nor (15) depends on the investment trigger $x^i$.

# 3 Main Results

## 3.1 Monopoly

Assume that the entrepreneur faces an imposed financial restriction whereby $qI$ must be financed by debt. The parameter $q \in (0, 1]$ represents how difficult it is for the entrepreneur to collect the investment cost by self- and equity financing. A larger $q$ means an entrepreneur with less internal funds for the investment project. Under the financing constraint, at the time of investment, $T$, the demand $X(T)$ and the coupon $c$ must satisfy:

$$D(X(T), c) \geq qI. \tag{16}$$

Namely, we consider problem (7) with the financing constraint (16).[5] The purpose of this study is to clarify the effects of the debt financing constraint. We obtain the following proposition, where the notations with the superscript $*$ denote the corresponding quantities of the financially constrained firm.

**Proposition 1** There exist a unique $x_2 > 0$ satisfying $D(x_2, c(x_2)) = qI$ and a unique $x_1 \in (0, x_2]$ satisfying $\max_{c \geq 0} D(x_1, c) = qI$. The outcomes are classified into the following two cases. Note that $x^i$ is the investment trigger of the unconstrained entrepreneur,

[5] There might be some arguments against our setting where the financing constraint is taken into consideration only at the time of investment and is not considered at the time of bankruptcy (i.e., the default timing can always be optimized). This can be justified as follows. At the time of investment, the entrepreneur needs to finance the massive investment cost. It is hard for the entrepreneur who has not initiated the project to raise the required funding by means of self- and equity financing. In contrast, it is not so difficult for the firm that is in business to make a coupon payment, which is significantly lower than the investment cost, out of profits of the project, or by means of equity financing until the firm's endogenous (optimal) bankruptcy.

defined by (11).

Case (1-a): $x^i \geq x_2$

The investment strategy and the firm value coincide with those of the unrestricted entrepreneur.

Case (1-b): $x^i < x_2$

The entrepreneur invests at:

$$T^{i*} = \inf\{t > 0 \mid X(t) \geq x^*\}, \tag{17}$$

along with issuing debt with coupon $c^*(x^*)$, and then defaults at $T^{d*} = \inf\{t > T^{i*} \mid X(t) \leq x^d(c^*(x^*))\}$. The investment trigger $x^*$, the coupon $c^*(x^*)$, the leverage $LV^*$, and the credit spread $CS^*$ at the time of investment satisfy the inequalities:

$$x_1 \leq x^* \leq x_2,\ c^*(x^*) \geq c(x^*),\ LV^* \geq LV,\ CS^* \geq CS,$$

respectively. The firm value $V_{de}^*(x)$ at the initial time satisfies $V_{de}^*(x) \leq V_{de}(x)$.

**(Proof)** The function $D(\cdot, c(\cdot))$ (see (5), (8)) is strictly monotonically increasing and continuous. Then, from $\lim_{x\downarrow 0} D(x, c(x)) = 0$, and $\lim_{x\to+\infty} D(x, c(x)) = +\infty$, there exists a unique $x_2 > 0$ satisfying $D(x_2, c(x_2)) = qI$.

On the other hand, for a fixed $X(s) > 0$, the function $D(X(s), \cdot)$ is concave. From the first-order optimality condition, we have $\arg\max_{c\geq 0} D(X(s), c) = \tilde{c}(X(s))$, where the function $\tilde{c}(X(s))$ is defined by:

$$\tilde{c}(X(s)) = \frac{r}{r-\mu}\frac{\gamma-1}{\gamma}\frac{QX(s)}{\tilde{h}}, \tag{18}$$

$$\tilde{h} = [1 - \gamma(1-(1-\alpha)(1-\tau))]^{-\frac{1}{\gamma}}. \tag{19}$$

It follows from (19) and (9) that $1 < \tilde{h} < h$. Then, by (18) and (8), we have:[6]

$$\tilde{c}(X(s)) > c(X(s)) \quad (X(s) > 0). \tag{20}$$

Note that the function $D(\cdot, \tilde{c}(\cdot))$ is strictly monotonically increasing and continuous. Because $\lim_{x\downarrow 0} D(x, \tilde{c}(x)) = 0$ and $D(x_2, \tilde{c}(x_2)) \geq D(x_2, c(x_2)) = qI$, there exists a unique $x_1 \in (0, x_2]$ satisfying $D(x_1, \tilde{c}(x_1)) = qI$.

Case (1-a): $x^i \geq x_2$

This case is obvious because the optimal investment strategy of the unrestricted entrepreneur can also be available for the restricted entrepreneur; i.e., $D(x^i, c(x^i)) \geq qI$.

Case (1-b): $x^i < x_2$

Under the assumption that $x = X(0)$ is small enough, we can restrict the investment time to the form of (17) (called "trigger strategy" ) without loss of generality, because the problem is time-homogeneous Markovian. Let us check $x^* \in [x_1, x_2]$. The entrepreneur cannot raise enough investment funds by means of debt if the demand is less than $x_1$.

[6]This relationship is also shown in [11].

Because $x^i < x_2$, the investment at trigger $x_2$ alongside the issuance of debt with coupon $c(x_2)$ generates a higher expected profit than any investment at trigger $x'(> x_2)$ alongside the issuance of debt with coupon $c(x')$. Thus, the investment trigger $x^*$ must lie in the interval $[x_1, x_2]$.

We define a function $c^*(\cdot)$ by:

$$c^*(X(s)) = \arg \max_{c \geq 0, D(X(s),c) \geq qI} V(X(s), c) \quad (X(s) > 0). \tag{21}$$

By (20) and the concavity of the functions $D(X(s), \cdot)$ and $V(X(s), \cdot)$, we have $c^*(X(s)) \in [c(X(s)), \tilde{c}(X(s))]$ and $D(X(s), c^*(X(s))) = qI$ for $X(s) \in [x_1, x_2]$. Therefore, the optimal coupon $c^*(x^*)$ at the investment trigger $x^*$ satisfies $c^*(x^*) \geq c(x^*)$.

We can show the inequality for the leverages as follows:

$$\begin{aligned} LV &= \frac{D(x^i, c(x^i))}{V(x^i, c(x^i))} \\ &= \frac{D(x^*, c(x^*))}{V(x^*, c(x^*))} \\ &\leq \frac{qI}{V(x^*, c^*(x^*))} \qquad (22) \\ &= LV^*, \end{aligned}$$

where (22) follows from $D(x^*, c(x^*)) \leq qI$ and $V(x^*, c(x^*)) \geq V(x^*, c^*(x^*))$. As for the credit spreads, we can calculate them as follows:

$$\begin{aligned} CS &= \frac{c(x^i)}{D(x^i, c(x^i))} - r \\ &= \frac{c(x^*)}{D(x^*, c(x^*))} - r \\ &= \frac{r}{1 - \left(1 - (1-\alpha)(1-\tau)\frac{\gamma}{\gamma-1}\right)\left(\frac{x^*}{x^d(c(x^*))}\right)^\gamma} - r \\ &\leq \frac{r}{1 - \left(1 - (1-\alpha)(1-\tau)\frac{\gamma}{\gamma-1}\right)\left(\frac{x^*}{x^d(c^*(x^*))}\right)^\gamma} - r \qquad (23) \\ &= CS^*, \end{aligned}$$

where (23) follows from $x^d(c^*(x^*)) \geq x^d(c(x^*))$. The inequality $V^*_{de}(x) \leq V_{de}(x)$ is obvious. □

In Proposition 1, the thresholds $x_1$ and $x_2$ mean the lowest demand with which the project can be financed and the threshold determining whether the financing restriction binds, respectively. Note that $x_1$ and $x_2$ monotonically increase for $q$. If Case (1-b) is satisfied for $q = 1$, there is a turning point $\hat{q} \in (0, 1)$ between Case (1-a) and Case (1-b).

In Case (1-a) the entrepreneur does not suffer any loss from the financing constraint. On the other hand, in Case (1-b) he/she uses costly[7] debt financing because of the insufficient internal funds and therefore suffers the loss from the restriction. In this case,

[7]Although we assume the competitive bondholders and symmetric information between the entrepreneur and the bondholders, the high cost of debt financing is due to the default costs dominant over the tax effects.

the coupon and the leverage of the constrained entrepreneur are higher than those of the unconstrained one. Then, the bankruptcy probability and credit spread of the restricted firm also become higher than the optimal levels, which means that the rating of the bond becomes low.

Of course, the firm value $V^*_{de}(x)$ monotonically decreases for $q$. The firm value $V^*_{de}(x)$ is not always larger than the unleveraged firm value (3). It depends on the trade-off between the tax advantages and the financial restriction.

In Case (1-b), the investment trigger $x^*$ is usually larger[8] than $x^i$ and therefore coupon $c^*(x^*)$ becomes larger than $c^i = c(x^i)$. This is because the restricted entrepreneur attempts to reduce the loss because of the financial restriction by waiting for a greater demand. The inequality $x^* > x^i$ is consistent with the standard results of theoretical and empirical studies, which show that a firm with less internal funds invests less or later than a firm with sufficient internal funds (e.g., [3, 6]).

Finally, we show comparative statics with respect to volatility $\sigma$. We examine the sensitivity numerically because an analytical discussion is impossible because of the non-analytical investment trigger $x^*$. Figure 1[9] depicts the firm value $V^*_{de}(x)$, the investment trigger $x^*$, and the leverage $LV^*$ of the restricted entrepreneur (in Case (1-a) $V^*_{de} = V_{de}, x^* = x^i, LV^* = LV$). For most parameter values, $V^*_{de}(x)$ and $x^*$ (resp. $LV^*$) monotonically increase (resp. decreases) with $\sigma$. This is because a higher uncertainty increases both the value of deferring the investment and the default probability of the firm.

It depends on the parameter values whether Case (1-a) or (1-b) is satisfied. For some parameter values, Case (1-a) holds for every $\sigma$. If we find a $\sigma$ satisfying Case (1-b), Case (1-b) holds for lower $\sigma$s, as seen in Figure 1. The interpretation is as follows. The lower the volatility, the lower the optimal investment trigger $x^i$. Because $D(\cdot, c(\cdot))$ is a monotonically increasing function, the debt value $D(x^i, c(x^i))$ under the optimal capital structure becomes lower. Then, the debt value under the optimal capital structure is likely to be short, and the entrepreneur must take a higher leverage to raise the investment funds. As a consequence, a lower uncertainty makes the financing constraint bind more tightly.

### 3.2 Duopoly

This section considers the competition between two symmetric restricted entrepreneurs with complete information. Assume that each firm receives a cash flow $Q_2X(t)$ when both firms are active in the market. We assume that $Q_2 = 0$[10] so that we can analytically

[8] Although we have no theoretical proof, $x^* > x^i$ held for all numerical examples we tried.

[9] We fix the parameter values as $r = 0.07, \mu = 0, I = 2, Q = 0.1, \tau = 0.4, \alpha = 1, q = 1, x = X(0) = 1$.

[10] This assumption is also made in several papers including [19, 10]. It means that the market is small enough to be supplied by a single firm. This paper considers such a simple situation where the first mover's advantage is very strong, but in the subsequent research, we will relax the assumption and focus on the second mover's advantage of choosing the coupon after the leader's choice of the coupon level.

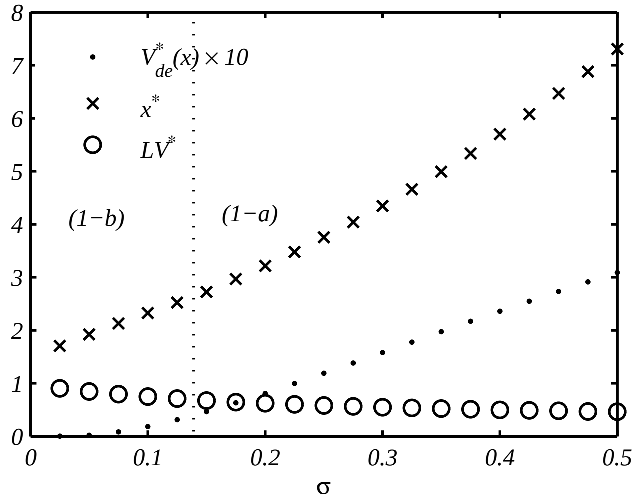


Figure 1: $V^*_{de}(x), x^*$, and $LV^*$ for various $\sigma$.

capture interesting features of strategic investment of the entrepreneurs. In this section, we would like to verify whether the financing constraint which is not binding in monopoly becomes binding because of the preemptive competition in duopoly. For this reason, this section assumes that Case (1-a) holds in Proposition 1; i.e., $x^i \geq x_2$.

As usual (cf. [5, 7, 19]), we begin where one of the entrepreneurs (called the leader) has already invested at $X(s)$ along with the issuance of debt with coupon $c$. The leader's firm value $L(X(s), c)$ at time $s$ can be expressed as:

$$L(X(s), c) = V(X(s), c) - I. \tag{24}$$

On the other hand, the firm value, denoted by $F(X(s), c)$, of the other entrepreneur (called the follower) responding optimally against the leader can be written as:

$$F(X(s), c) = \left(\frac{X(s)}{x^d(c)}\right)^\gamma V_{de}(x^d(c)), \tag{25}$$

when $X(s) \geq x^d(c)$. Eq. (25) is the value of the option to invest after the leader's bankruptcy. Note that the follower chooses the same investment trigger (of course, *not at the same time*), coupon, default trigger, leverage, and credit spread as those of the unrestricted monopolist; i.e., $x^i$, $c(x^i)$, $x^d(c(x^i))$, $LV$, and $CS$ because of the assumption of Case (1-a); i.e., $x^i \geq x_2$ in this section.

Using (24) and (25), we formulate a stopping game where both entrepreneurs try to preempt each other for the first mover's advantage. We consider the duopoly of two symmetric unrestricted entrepreneurs, prior to the duopoly of two symmetric entrepreneurs with the financing constraint. Define the action space of the entrepreneur without a financial constraint as:

$$\mathcal{A} = \{(T, c) \mid T \in \mathcal{T}, c : \mathcal{F}_T \text{measurable}\}. \tag{26}$$

Recall that $\mathcal{T}$ denotes the set of all stopping times. If one of the entrepreneurs (denoted by entrepreneur 1) chooses the strategy $(T_1, c_1) \in \mathcal{A}$ and the other (denoted by entrepreneur 2) chooses the strategy $(T_2, c_2) \in \mathcal{A}$, entrepreneurs 1 and 2 expect to obtain:

$$\pi_1(T_1, c_1, T_2, c_2) = \mathbb{E}[1_{\{T_1<T_2\}}\mathrm{e}^{-rT_1}L(X(T_1), c_1) + 1_{\{T_1>T_2\}}\mathrm{e}^{-rT_2}F(X(T_2), c_2)$$

$$+1_{\{T_1=T_2\}}\mathrm{e}^{-rT_1}\frac{L(X(T_1), c_1) + F(X(T_2), c_2)}{2}],$$

and

$$\pi_2(T_1, c_1, T_2, c_2) = \mathbb{E}[1_{\{T_1<T_2\}}\mathrm{e}^{-rT_1}F(X(T_1), c_1) + 1_{\{T_1>T_2\}}\mathrm{e}^{-rT_2}L(X(T_2), c_2)$$

$$+1_{\{T_1=T_2\}}\mathrm{e}^{-rT_1}\frac{F(X(T_1), c_1) + L(X(T_2), c_2)}{2}],$$

respectively.[11]

Let us derive an equilibrium of the stopping game (denoted by game $G$) by two symmetric unrestricted entrepreneurs with strategy space $\mathcal{A}$. More precisely, we find $(\tilde{T_1}, \tilde{c_1}, \tilde{T_2}, \tilde{c_2}) \in \mathcal{A} \times \mathcal{A}$ satisfying both:

$$\pi_1(\tilde{T_1}, \tilde{c_1}, \tilde{T_2}, \tilde{c_2}) = \max_{(T_1,c_1)\in\mathcal{A}} \pi_1(T_1, c_1, \tilde{T_2}, \tilde{c_2}),$$

and

$$\pi_2(\tilde{T_1}, \tilde{c_1}, \tilde{T_2}, \tilde{c_2}) = \max_{(T_2,c_2)\in\mathcal{A}} \pi_2(\tilde{T_1}, \tilde{c_1}, T_2, c_2).$$

We can show the following proposition.

**Proposition 2** There exists a unique solution $x_P$ satisfying:

$$\psi^{-1}\Pi(x_P) - I = h^{\gamma-\beta}\left(\frac{x_P}{x^i}\right)^{\beta}(\psi^{-1}\Pi(x^i) - I), \tag{27}$$

in the interval $(\psi x_{NPV}, x^i)$. An equilibrium of game $G$ is $(T_P, c(x_P), T_P, c(x_P))$, where $c(\cdot)$ is defined by (8) and:

$$T_P = \inf\{t > 0 \mid X(t) \geq x_P\}.$$

The firm value of each entrepreneur in equilibrium is:

$$h^{\gamma-\beta}V_{de}(x). \tag{28}$$

**(Proof)** The function $\Pi(\cdot)$ is linear by its definition (2), and the function $(\cdot)^{\beta}$ is convex from $\beta > 1$. The function $h^{\gamma-\beta}\left(\cdot/x^i\right)^{\beta}(\psi^{-1}\Pi(x^i) - I) - (\psi^{-1}\Pi(\cdot) - I)$ is then convex. Therefore, from:

$$h^{\gamma-\beta}(\frac{\psi x_{NPV}}{x^i})^{\beta}(\psi^{-1}\Pi(x^i) - I) > \psi^{-1}\Pi(\psi x_{NPV}) - I = 0$$

[11] Following many studies including [5, 19], this paper assumes that one of the entrepreneurs is chosen as a leader with probability 1/2 when both entrepreneurs attempt to invest at the same time; i.e., $T_1 = T_2$.

and

$$h^{\gamma-\beta}\left(\frac{x^i}{x^i}\right)^{\beta}(\psi^{-1}\Pi(x^i)-I)<\psi^{-1}\Pi(x^i)-I,$$

there exists a unique $x_P$ satisfying (27) in the interval $(\psi x_{NPV}, x^i)$.

From (10) we have:

$$L(X(s),c(X(s)))=\psi^{-1}\Pi(X(s))-I. \tag{29}$$

It follows from (25) and (12) that:

$$\begin{aligned}F(X(s),c(X(s))) &= \left(\frac{X(s)}{x^d(c(X(s)))}\right)^{\gamma}V_{de}(x^d(c(X(s))))\\ &= h^{\gamma}V_{de}(X(s)/h)\\ &= h^{\gamma-\beta}\left(\frac{X(s)}{x^i}\right)^{\beta}(\psi^{-1}\Pi(x^i)-I),\end{aligned} \tag{30}$$

for $X(s)\le x^i$. By (29), (30) and the definition of $x_P$, we have the relationship:

$$L(X(s),c(X(s)))-F(X(s),c(X(s)))\begin{cases}<0 & (0<X(s)<x_P)\\ =0 & (X(s)=x_P)\\ >0 & (x_P<X(s)\le x^i).\end{cases} \tag{31}$$

Let us now check:

$$\pi_1(T_P,c(x_P),T_P,c(x_P))=\max_{(T_1,c_1)\in\mathcal{A}}\pi(T_1,c_1,T_P,c(x_P)). \tag{32}$$

For an arbitrary $(T_1,c_1)\in\mathcal{A}$, we can calculate as follows:

$$\begin{aligned}&\pi_1(T_1,c_1,T_P,c(x_P))\\ =\;&\mathbb{E}[1_{\{T_1<T_P\}}\mathrm{e}^{-rT_1}L(X(T_1),c_1)+1_{\{T_1>T_P\}}\mathrm{e}^{-rT_P}F(x_P,c(x_P))\\ &+1_{\{T_1=T_P\}}\mathrm{e}^{-rT_1}\frac{L(x_P,c_1)+F(x_P,c(x_P))}{2}]\\ \le\;&\mathbb{E}[1_{\{T_1<T_P\}}\mathrm{e}^{-rT_1}L(X(T_1),c(X(T_1)))+1_{\{T_1\ge T_P\}}\mathrm{e}^{-rT_P}L(x_P,c(x_P))] \qquad (33)\\ =\;&\mathbb{E}[1_{\{T_1<T_P\}}\mathrm{e}^{-rT_1}(\psi^{-1}\Pi(X(T_1))-I)+1_{\{T_1\ge T_P\}}\mathrm{e}^{-rT_P}(\psi^{-1}\Pi(x_P)-I)] \qquad (34)\\ \le\;&\sup_{T\in\mathcal{T}}\mathbb{E}[1_{\{T<T_P\}}\mathrm{e}^{-rT_1}(\psi^{-1}\Pi(X(T))-I)+1_{\{T\ge T_P\}}\mathrm{e}^{-rT_P}(\psi^{-1}\Pi(x_P)-I)]\\ =\;&\mathbb{E}[\mathrm{e}^{-rT_P}(\psi^{-1}\Pi(x_P)-I)]. \qquad (35)\end{aligned}$$

Here, (33) follows from (31) and the optimality of $c(\cdot)$, i.e., $L(X(s),c(X(s)))=\max_{c\ge 0}L(X(s),c)$, while (34) follows from (29). We can show (35) by:

$$\left(\frac{x}{y}\right)^{\beta}(\psi^{-1}\Pi(y)-I)\le\left(\frac{x}{x_P}\right)^{\beta}(\psi^{-1}\Pi(x_P)-I)\quad(x\le y\le x_P)$$

which results from $x_P<x^i$.

On the other hand, we obtain:

$$\begin{aligned}\pi_1(T_P,c(x_P),T_P,c(x_P)) &= \mathbb{E}[\mathrm{e}^{-rT_P}\frac{L(x_P,c(x_P))+F(x_P,c(x_P))}{2}]\\ &= \mathbb{E}[\mathrm{e}^{-rT_P}L(x_P,c(x_P))]\\ &= \mathbb{E}[\mathrm{e}^{-rT_P}\psi^{-1}(\Pi(x_P)-I)].\end{aligned} \tag{36}$$

Thus, we obtain (32) from (35) and (36). Taking account of the symmetry, we also obtain:

$$\pi_2(T_P, c(x_P), T_P, c(x_P)) = \max_{(T_2,c_2)\in\mathcal{A}} \pi(T_P, c(x_P), T_2, c_2). \tag{37}$$

By (32) and (37), $(T_P, c(x_P), T_P, c(x_P))$ is an equilibrium of game $G$.

Finally, we can calculate as follows:

$$\begin{aligned}
\pi_i(T_P, c(x_P), T_P, c(x_P)) &= \mathbb{E}[\mathrm{e}^{-rT_P}\frac{L(x_P, c(x_P)) + F(x_P, c(x_P))}{2}] \\
&= \mathbb{E}[\mathrm{e}^{-rT_P}F(x_P, c(x_P))] \\
&= \left(\frac{x}{x_P}\right)^{\beta} h^{\gamma-\beta}\left(\frac{x_P}{x^i}\right)^{\beta}(\psi^{-1}\Pi(x^i) - I) \qquad (38) \\
&= h^{\gamma-\beta}\left(\frac{x}{x^i}\right)^{\beta}(\psi^{-1}\Pi(x^i) - I) \\
&= h^{\gamma-\beta}V_{de}(x), \qquad (39)
\end{aligned}$$

for $i = 1, 2$, where (38) and (39) follow from (30) and (13), respectively. □

The outcome in Proposition 2 can be interpreted as follows. Each entrepreneur applies for the investment project at the preemptive timing $T_P = \inf\{t > 0 \mid X(t) \geq x_P\}$, and one of the entrepreneurs is allowed to execute the investment as a leader at $T_P$, along with issuing debt with coupon $c(x_P)$. The leader then defaults at $T_L^d = \inf\{t > T_P \mid X(t) \leq x_P/h\}$. After the leader's bankruptcy, the other, as a follower, invests at $T_F^i = \inf\{t > T_L^d \mid X(t) \geq x^i\}$ along with issuing debt with coupon $c(x^i)$, and then defaults at $T_F^d = \inf\{t > T_F^i \mid X(t) \leq x^i/h\}$.

The preemptive trigger $x_P$ in Proposition 2 may be smaller than the unleveraged firm's zero-NPV point $x_{NPV}$, though of course it is larger than in the leveraged case, $\psi x_{NPV}$. The leader has a smaller investment trigger, coupon, and default trigger than the monopolist (or follower); i.e., $x_P < x^i$, $c(x_P) < c(x^i)$ and, $x^d(c(x_P)) = x_P/h < x^d(c(x^i)) = x^i/h$. Both firms' leverage and credit spread at the investment time remain unchanged from those of the monopolist, i.e., (14) and (15), respectively, because the entrepreneurs can optimize their capital structure. The firm value (28) is $h^{\gamma-\beta}(< 1)$ times the monopolist's value (13) because of the preemptive competition.

Note that a firm's endogenous default decision generates a positive firm value in spite of the assumption $Q_2 = 0$. This feature contrasts with several results in the previous literature. In [19] and [16] a leader does not always obtain a profit from the market because it takes a random development term from investment until the completion of the project. The random development term generates a positive value under competition. In [10] incomplete information about the rival firm's strategy plays a role in generating a positive value under competition.

Next, let us restrict the strategy space $\mathcal{A}$ defined by (26) to:

$$\mathcal{A}^* = \{(T, c) \in \mathcal{A} \mid D(X(T), c) \geq qI\}.$$

Recall that $D(X(T), c) \geq qI$ represents the financial constraint of the entrepreneurs (see (16)). Let $G^*$ denote the game between two symmetric restricted entrepreneurs with action space $\mathcal{A}^*$. We will find an equilibrium of game $G^*$, that is, $(\tilde{T_1}, \tilde{c_1}, \tilde{T_2}, \tilde{c_2}) \in \mathcal{A}^* \times \mathcal{A}^*$ satisfying:

$$\pi_1(\tilde{T_1}, \tilde{c_1}, \tilde{T_2}, \tilde{c_2}) = \max_{(T_1, c_1) \in \mathcal{A}^*} \pi_1(T_1, c_1, \tilde{T_2}, \tilde{c_2}),$$

and

$$\pi_2(\tilde{T_1}, \tilde{c_1}, \tilde{T_2}, \tilde{c_2}) = \max_{(T_2, c_2) \in \mathcal{A}^*} \pi_2(\tilde{T_1}, \tilde{c_1}, T_2, c_2).$$

The following proposition shows an equilibrium of game $G^*$ and the firm value in equilibrium.

**Proposition 3** The outcomes are classified into the following three cases. Note that $x_i$ $(i = 1, 2)$ and $x_P$ are defined in Propositions 1 and 2, respectively.

Case (2-a) $x_2 \leq x_P$

An equilibrium of game $G^*$ and the firm value in equilibrium are the same as those of the unconstrained case in Proposition 2.

Case (2-b-1) $x_2 > x_P$ and there exists $x_P^* \in [x_1, x_2)$ such that:

$$L(x_P^*, c^*(x_P^*)) = F(x_P^*, c^*(x_P^*)).$$

Recall that the function $c^*(\cdot)$ is defined by (21). Assume that:[12]

$$\left(\frac{x}{y}\right)^\beta L(y, c^*(y)) \leq \left(\frac{x}{x_P^*}\right)^\beta L(y, c^*(x_P^*)) \quad (x \leq y \leq x_P^*). \tag{40}$$

An equilibrium of game $G^*$ is $(T_P^*, c^*(x_P^*), T_P^*, c^*(x_P^*))$, where:

$$T_P^* = \inf\{t > 0 \mid X(t) \geq x_P^*\}.$$

The firm value of each entrepreneur in equilibrium is:

$$h^{*\gamma-\beta} V_{de}(x), \tag{41}$$

where $h^*$ is a constant in the interval $[\tilde{h}, h)$. Recall that $h$ and $\tilde{h}$ are defined by (9) and (19), respectively.

Case (2-b-2) $x_2 > x_P$ and there exists no $x_P^* \in [x_1, x_2)$ such that:

$$L(x_P^*, c^*(x_P^*)) = F(x_P^*, c^*(x_P^*)).$$

An equilibrium of game $G^*$ is $(T_1^*, \tilde{c}(x_1), T_1^*, \tilde{c}(x_1))$, where the function $\tilde{c}(\cdot)$ is defined by (18) and:

$$T_1^* = \inf\{t > 0 \mid X(t) \geq x_1\}.$$

The firm value of each entrepreneur in equilibrium is:

$$\frac{\left(\frac{x}{x_1}\right)^\beta L(x_1, \tilde{c}(x_1)) + \tilde{h}^{\gamma-\beta} V_{de}(x)}{2}. \tag{42}$$

[12]This assumption is not strong because in most cases $(x/y)^\beta L(y, c^*(y))$ monotonically increases with $y \in [x, x^i]$.

**(Proof)**

Case (2-a)

Using (32) shown in Proposition 2, we have:

$$
\begin{aligned}
\pi_1(T_P, c(x_P), T_P, c(x_P)) &= \max_{(T_1,c_1)\in\mathcal{A}} \pi_1(T_1, c_1, T_P, c(x_P)) \\
&\geq \max_{(T_1,c_1)\in\mathcal{A}^*} \pi_1(T_1, c_1, T_P, c(x_P)), \qquad (43)
\end{aligned}
$$

where (43) holds because $\mathcal{A}^* \subset \mathcal{A}$. From $(T_P, c(x_P)) \in \mathcal{A}^*$ in Case (2-a), we obtain:

$$
\pi_1(T_P, c(x_P), T_P, c(x_P)) = \max_{(T_1,c_1)\in\mathcal{A}^*} \pi_1(T_1, c_1, T_P, c(x_P)).
$$

The same is true with respect to $\pi_2$. Therefore, $(T_P, c(x_P), T_P, c(x_P))$ is an equilibrium of game $G^*$. Then the firm value equals (28).

Case (2-b-1)

Under assumption (40), it can be checked in the same way as the proof of Proposition 2 that $(T_P^*, c^*(x_P^*), T_P^*, c^*(x_P^*))$ is an equilibrium of game $G^*$. The firm value in equilibrium can be calculated as:

$$
\begin{aligned}
\pi_i(T_P^*, c^*(x_P^*), T_P^*, c^*(x_P^*)) &= \mathbb{E}[\mathrm{e}^{-rT_P^*} \frac{L(x_P^*, c^*(x_P^*)) + F(x_P^*, c^*(x_P^*))}{2}] \\
&= \mathbb{E}[\mathrm{e}^{-rT_P^*} F(x_P^*, c^*(x_P^*))] \\
&= \left(\frac{x}{x_P^*}\right)^{\beta} \left(\frac{x_P^*}{x^d(c^*(x_P^*))}\right)^{\gamma} V_{de}(x^d(c^*(x_P^*))) \qquad (44) \\
&= \left(\frac{x}{x_P^*}\right)^{\beta} h^{*\gamma} \left(\frac{x_P^*}{h^* x^i}\right)^{\beta} (\psi^{-1}\Pi(x^i) - I) \qquad (45) \\
&= h^{*\gamma-\beta} V_{de}(x).
\end{aligned}
$$

Here, (44) follows from (25). We obtain (45) as follows. As shown in the proof of Proposition 1, $c^*(x_P^*)$ lies in the interval $(c(x_P^*), \tilde{c}(x_P^*)]$, and hence, $c^*(x_P^*)$ can be expressed as:

$$
c^*(x_P^*) = \frac{r}{r-\mu} \frac{\gamma - 1}{\gamma} \frac{Q x_P^*}{h^*}, \qquad (46)
$$

for some constant $h^* \in [\tilde{h}, h)$. Substituting (46) into (4), we have $x^d(c^*(x_P^*)) = x_P^*/h^*$, which implies (45).

Case (2-b-2)

Note that $c^*(x_2) = c(x_2)$ by definition of $x_2$ in Proposition 1. Then, from $x_2 > x_P$ and (31), we have:

$$
L(x_2, c^*(x_2)) > F(x_2, c^*(x_2)). \qquad (47)
$$

From (47), $c^*(x_1) = \tilde{c}(x_1)$, the continuity of the functions $L(\cdot, c^*(\cdot))$ and $F(\cdot, c^*(\cdot))$, and the definition of Case (2-b-2), we obtain:

$$
L(x_1, \tilde{c}(x_1)) > F(x_1, \tilde{c}(x_1)). \qquad (48)
$$

Using $x^d(\tilde{c}(x_1)) = x_1/\tilde{h}$, we can show that:

$$
F(x_1, \tilde{c}(x_1)) = \tilde{h}^{\gamma-\beta} V_{de}(x). \qquad (49)
$$

We can check by (48) and (49) that $(T_1^*, \tilde{c}(x_1), T_1^*, \tilde{c}(x_1))$ is an equilibrium of game $G^*$ as follows. For an arbitrary strategy $(T_1, c_1) \in \mathcal{A}^*$, we have:

$$\begin{aligned} & \pi_1(T_1, c_1, T_1^*, \tilde{c}(x_1)) \\ = \; & \mathbb{E}[1_{\{T_1 > T_1^*\}} \mathrm{e}^{-rT_1^*} F(x_1, \tilde{c}(x_1)) + 1_{\{T_1 = T_1^*\}} \mathrm{e}^{-rT_1^*} \frac{L(x_1, \tilde{c}(x_1)) + F(x_1, \tilde{c}(x_1))}{2}] \quad (50) \\ \leq \; & \mathbb{E}[\mathrm{e}^{-rT_1^*} \frac{L(x_1, \tilde{c}(x_1)) + F(x_1, \tilde{c}(x_1))}{2}] = \pi_i(T_1^*, \tilde{c}(x_1), T_1^*, \tilde{c}(x_1)) \quad (51) \\ = \; & \frac{\left(\frac{x}{x_1}\right)^{\beta} L(x_1, \tilde{c}(x_1)) + \tilde{h}^{\gamma-\beta} V_{de}(x)}{2}, \quad (52) \end{aligned}$$

where (51) and (52) result from (48) and (49), respectively. We have (50) because either $T_1(\omega) > T_1^*(\omega)$ or $(T_1(\omega), c_1(\omega)) = (T_1^*(\omega), \tilde{c}(x_1))$ is satisfied for an arbitrary sample pass $\omega$ if $(T_1, c_1) \in \mathcal{A}^*$. □

Case (2-a) remains unchanged from the outcome in the case of no financial restriction in Proposition 2 because the financial restriction of the entrepreneurs is not binding under the preemptive competition. Case (2-b) is the case where the financing constraint becomes binding because of the timing hastened (note that $x_P < x^i$) by the preemptive competition between the entrepreneurs. Recall that the financing constraint is not binding in monopoly under the assumption of $x^i \geq x_2$ in this section.

Case (2-b-1) can be interpreted as follows. Both entrepreneurs apply for the investment project when the market demand $X(t)$ reaches the preemptive trigger $x_P^*$. One of the entrepreneurs (the leader) is chosen with probability 1/2 to execute the investment alongside the issuance of debt with coupon $c^*(x_P^*)$. The leader defaults when the demand falls to the default trigger $x^d(c^*(x_P^*)) = x_P^*/h^*$. After the leader's bankruptcy, the follower invests by issuing debt with coupon $c(x^i)$ when the demand reincreases to the investment trigger $x^i$. Note that the leader and the follower have the same firm value (41). An example of Case (2-b-1) is Figure 2.[13]

Case (2-b-2) can be interpreted as follows. Both entrepreneurs apply for the investment project when the market demand $X(t)$ reaches the lowest demand $x_1$ with which the project can be financed. One of the entrepreneurs (the leader) is chosen with probability 1/2 to execute the investment alongside the issuance of debt with coupon $\tilde{c}(x_1)$. The leader defaults at the default trigger $x^d(\tilde{c}(x_1)) = x_1/\tilde{h}$. After the leader's bankruptcy, the follower invests by issuing debt with coupon $c(x^i)$ when the demand reincreases to the investment trigger $x^i$. In this case, the leader's value $(x/x_1)^{\beta} L(x_1, \tilde{c}(x_1))$ is higher than the follower's $\tilde{h}^{\gamma-\beta} V_{de}(x)$. In the example in Figure 3,[14] the leader's firm value is almost 1.5 times as high as the follower's value.

[13] We set the parameter values as $r = 0.07, \mu = 0.04, \sigma = 0.2, I = 2, Q = 0.1, \tau = 0.4, \alpha = 0.2, q = 1$. We have $x_1 = 0.84, x_2 = 0.908, x_P^* = 0.842$. Note that Case (1-a) holds for monopoly.

[14] We took $\sigma = 0.1$ with the other parameters unchanged from those in Figure 2. We have $x_1 = 0.731, x_2 = 0.754$ and no $x_P^*$. Note that Case (1-a) holds for monopoly.

We note a surprising effect of the financial restriction in a duopoly. Taking account of $\gamma - \beta < 0$, $\tilde{h} \leq h^* < h$, and $(x/x_1)^\beta L(x_1, \tilde{c}(x_1)) > \tilde{h}^{\gamma-\beta} V_{de}(x)$ in Case (2-b-2), the firm values (41) and (42) in the restricted duopoly are higher than (28) in the unrestricted duopoly in Proposition 2. This implies that in duopoly, the financing constraint, unlike in monopoly, plays a positive role in moderating the preemptive competition and increasing the firm value. Actually, in the example of Figure 2, the firm value in equilibrium approximately doubles from the effect of the financing constraint.

**Corollary 1** The following statements hold with respect to the leader's investment strategy. Note that in equilibrium in Proposition 2, one of the entrepreneurs is chosen as a leader with probability 1/2.

Case (2-a)

The leader's investment strategy remains unchanged from that in the unconstrained duopoly.

Case (2-b-1)

The leader's investment trigger $x_P^*$, coupon $c^*(x_P^*)$, leverage $LV_P^*$, and credit spread $CS_P^*$ satisfy the following inequalities:

$$x_P < x_P^* < x_2,\ c^*(x_P^*) > c(x_P^*) > c(x_P),\ LV_P^* > LV,\ CS_P^* > CS.$$

Case (2-b-2)

The leader's investment trigger $x_1$, coupon $c^*(x_1)$, leverage $LV_1^*$, and credit spread $CS_1^*$ satisfy the following inequalities:

$$x_P < x_1 < x_2,\ c^*(x_1) > c(x_1) > c(x_P),\ LV_1^* > LV,\ CS_1^* > CS.$$

**(Proof)**

Case (2-a)

The statement immediately follows from Proposition 3.

Case (2-b-1)

The inequalities:

$$L(X(s), c^*(X(s))) < L(X(s), c(X(s))) \leq F(X(s), c(X(s))) < F(X(s), c^*(X(s)))$$

hold for $X(s) \in (0, x_P]$, where the last inequality results from $h^* < h$. Then, we have $x_P < x_P^*$ by definition of $x_P^*$. The inequalities $c^*(x_P^*) > c(x_P^*) > c(x_P)$ result from $x_P < x_P^*$ and the definitions of $c^*(\cdot)$ and $c(\cdot)$; i.e., (21) and (8). Using $c^*(x_P^*) > c(x_P^*)$, we calculate in a similar way to the proof of Proposition 1 as follows:

$$\begin{aligned} LV &= \frac{D(x^i, c(x^i))}{V(x^i, c(x^i))} \\ &= \frac{D(x_P^*, c(x_P^*))}{V(x_P^*, c(x_P^*))} \\ &< \frac{qI}{V(x^*, c^*(x_P^*))} \\ &= LV_P^*, \end{aligned}$$

$$
\begin{aligned}
CS &= \frac{c(x^i)}{D(x^i, c(x^i))} - r \\
&= \frac{c(x_P^*)}{D(x_P^*, c(x_P^*))} - r \\
&= \frac{r}{1 - \left(1 - (1-\alpha)(1-\tau)\frac{\gamma}{\gamma-1}\right)\left(\frac{x_P^*}{x^d(c(x_P^*))}\right)^{\gamma}} - r \\
&< \frac{r}{1 - \left(1 - (1-\alpha)(1-\tau)\frac{\gamma}{\gamma-1}\right)\left(\frac{x_P^*}{x^d(c^*(x_P^*))}\right)^{\gamma}} - r \\
&= CS_P^*.
\end{aligned}
$$

Case (2-b-2)

The proof is done in the same way as Case (2-b-1). □

In Case (2-b), the leader invests at the later timing along with issuing debt with a higher coupon than in the unrestricted case. The leader's leverage and credit spread become higher than the optimal level of the unconstrained entrepreneur. The inequalities $x_P^* > x_P$ in Case (2-b-1) and $x_1 > x_P$ in Case (2-b-2) are consistent with the standard theoretical and empirical results of underinvestment for a constrained firm (e.g., [3, 6]).

We must recall that in the unrestricted case in Proposition 2 the preemptive competition has no influence upon the leverage and the credit spread. In contrast, the leader in Case (2-b) bears the higher leverage and credit spread because the leader must invest given a low market demand $x_P^*(< x_2)$ or $x_1(< x_2)$, where the financing constraint is binding. Accordingly, we can say that the leader's bankruptcy probability and rating drop in the competitive situation. In the example of Figure 2, we have the leader's leverage $LV_P^* = 0.891(> LV = 0.803)$ and credit spread $CS_P^* = 0.025(> CS = 0.013)$, respectively.

Figure 4[15] illustrates the firm value, the leader's investment trigger, the leader's leverage in equilibrium, i.e., $((41), x_P^*, LV_P^*)$, and $((42), x_1^*, LV_1^*)$ in Cases (2-b-1) and (2-b-2), respectively. Likewise in monopoly, the value and the investment trigger monotonically increase with $\sigma$, while the leverage monotonically decreases with $\sigma$. For most parameter values, the financial restriction which is not binding in monopoly becomes tightly binding because of the investment trigger hastened by the preemptive competition. As in Figure 4, Case (2-b-1) is likely to hold for a higher volatility, while Case (2-b-2) holds for a lower volatility. This is reasonable because in the no-uncertainty setting, both entrepreneurs attempt to invest as long as the investment cost can be financed (and the project value is positive).

Finally, let us consider the situation where there is a minute difference $\epsilon(> 0)$ between the financial constraints of the entrepreneurs. Let entrepreneurs 1 and 2 denote the firms with the financing constraints characterized by parameters $q - \epsilon$ and $q$, respectively. Note that the outcome in Case (2-a) does not change because the financial restrictions are not

[15] We use the same parameter value as Figures 2 and 3. We take the initial value $x = X(0) = 0.5$. Note that Case (1-a) is satisfied in monopoly.

binding for any entrepreneurs.

In Case (2-b), entrepreneur 1 always invests as a leader and obtains a higher profit than entrepreneur 2. In Case (2-b-1), because of $L(x_P^*, c^*(x_P^*)) = F(x_P^*, c^*(x_P^*))$, the difference between the firm values of entrepreneurs 1 and 2 converges to 0 on letting $\epsilon \downarrow 0$. In contrast, in Case (2-b-2), the difference converges to $(x/x_1)^\beta (L(x_1, \tilde{c}(x_1)) - \tilde{h}^{\gamma-\beta} V_{de}(x)(> 0)$ on letting $\epsilon \downarrow 0$. It means that the difference in the financing constraint has a great influence on the outcome even if it is infinitesimal. In fact, the leader's firm value is approximately 1.5 times as large as the follower's in the example of Figure 3.

# 4 Conclusion

This paper has investigated the firm value and the investment strategy (investment, coupon, leverage, and endogenous default) when an entrepreneur makes a real investment with debt financing. We clarified the effects of the entrepreneur's financing constraint where a part of the investment cost must be financed by debt, both in monopoly and in duopoly. The results obtained in this paper can be summarized as follows. The restricted entrepreneur issues debt with a higher coupon and makes investment at a later time, which is consistent with the standard theory of underinvestment, than the unrestricted one. The leverage and the credit spread of the constrained entrepreneur become higher than the optimal level. The hastened timing through the strategic preemption in duopoly makes the financing constraint bind more tightly than that in monopoly. To our surprise, in duopoly, the financial restriction plays a role in moderating the preemptive competition and improving the firm value in equilibrium. A small difference between the financial restrictions of the firms may significantly influence the outcome, especially in the case of low uncertainty. From the viewpoint of an entrepreneur, more available internal funds than the rival's makes him/her the winner of the preemptive competition, but the competition by firms with sufficient inner funds may decrease the excess gain from the project.

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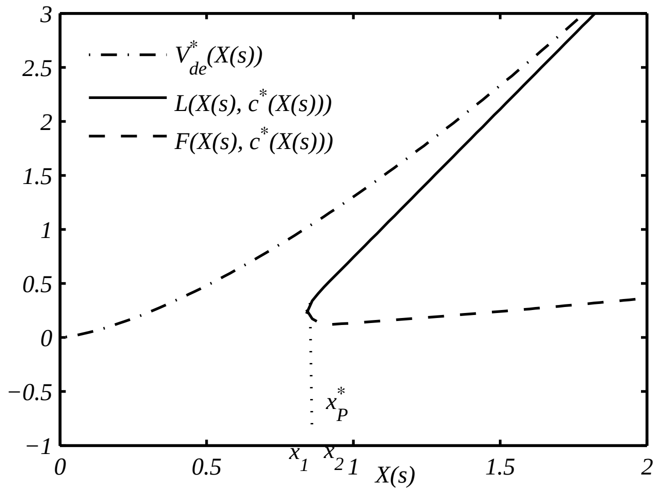


Figure 2: Case (2-b-1).

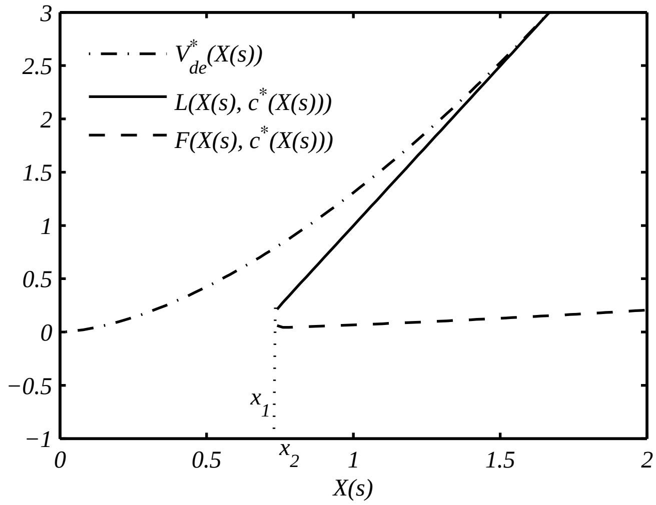


Figure 3: Case (2-b-2).

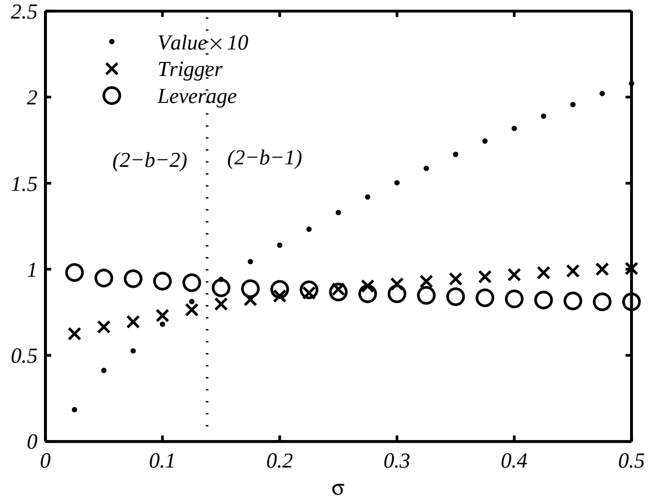


Figure 4: The firm value, the leader's investment trigger and leverage for various $\sigma$.