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**American Step-Up and Step-Down Credit Default
Swaps under Lévy Models**

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AMERICAN STEP-UP AND STEP-DOWN CREDIT DEFAULT SWAPS UNDER LÉVY MODELS

TIM S.T. LEUNG[†] AND KAZUTOSHI YAMAZAKI[‡]

ABSTRACT. This paper studies the valuation of a class of credit default swaps (CDSs) with the embedded option to switch to a different premium and notional principal anytime prior to a credit event. These are early exercisable contracts that give the protection buyer or seller the right to step-up, step-down, or cancel the CDS position. The pricing problem is formulated under a structural credit risk model based on Lévy processes. This leads to the analytic and numerical studies of several optimal stopping problems subject to early termination due to default. In a general spectrally negative Lévy model, we rigorously derive the optimal exercise strategy. This allows for instant computation of the credit spread under various specifications. Numerical examples are provided to examine the impacts of default risk and contractual features on the credit spread and exercise strategy.

Keywords: optimal stopping; credit default swaps; step-up and step-down options; Lévy processes; scale functions

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1. INTRODUCTION

Credit default swaps (CDSs) are among the most liquid and widely used instruments for managing and transferring credit risks. Despite the recent market turbulence, their market size still exceeds US\$30 trillions¹. In a standard single-name CDS, the protection buyer pays a pre-specified periodic premium (the CDS spread) to the protection seller to cover the loss of the face value of an asset if the reference entity defaults before expiration. The contract stipulates that both the buyer and seller have to commit to their respective positions until the default time or expiration date. To modify the initial CDS exposure in the future, one common way is to acquire appropriate positions later from the market, but it is subject to credit spread fluctuations and market illiquidity, especially during adverse market conditions.

To provide additional flexibility to investors, credit default swaptions and other derivatives on CDSs have emerged. For instance, the payer (receiver) default swaption is a European option that gives the holder the right to buy (sell) protection at a pre-specified strike spread at expiry, given that default has not occurred. Otherwise, the swaption is knocked out. See, for example, [22]. By appropriately combining a default swaption with a vanilla CDS position, one can create a callable or puttable default swap. A callable (puttable) CDS allows the protection buyer (seller) to terminate the contract at some fixed future date. Hence, as described here, the callable/puttable CDSs are in fact *cancellable* CDSs. Typically, the callable feature is paid for through incremental premium on top of the standard CDS spread, so selling a callable CDS can enhance the yield from the seller's perspective.

In this paper, we consider a class of CDSs embedded with an option for the investor (protection buyer or seller) to adjust the premium and notional amount once for a pre-specified fee prior to default. Specifically, these contracts equip the standard default swaps with the early exercisable rights such as (i) the *step-up option* that allows the investor to increase the protection and premium at exercise, and (ii) the *step-down option* to reduce the protection and premium. By definition, these contracts are indeed generalized versions of the callable and puttable CDSs mentioned above, and thus are more flexible credit risk management tools. Henceforth, we shall use the more general meaning of the terminology callable and puttable CDSs, rather than limiting them to cancellable CDSs.

The main contribution of our paper is to determine the credit spread for these CDSs under a Lévy model, and analyze the optimal strategy for the buyer or seller to exercise the step-up/down option. Specifically, we model the default time as the *first passage time* of a Lévy process representing some underlying asset value. We decompose the CDS with step-up/down option into a combination of an American-style credit default swaption and a vanilla CDS. From the investor's perspective, this gives rise to an optimal stopping problem subject to possible sudden early termination from default risk. Our formulation is based on a general Lévy process, and then

¹According to the ISDA Market Survey, the total CDS outstanding volume in 2009 is US\$30,428 billions.

we solve analytically for a general *spectrally negative* Lévy process. By employing the *scale function* and other properties of Lévy processes, we derive analytic characterization for the optimal exercising strategy. This in turn allows for a highly efficient computation of the credit spread for these CDS contracts. We provide a series of numerical examples to illustrate the credit spread behavior and optimal exercising strategy under various contract specifications and scenarios.

We adopt a Lévy-based structural credit risk model that extends the original approach introduced by Black and Cox [10] where the asset value follows a geometric Brownian motion. Other structural default models based on Lévy and other jump processes can also be found in [12, 20, 42]. To our best knowledge, the valuation of American step-up and step-down CDSs has not been studied elsewhere. For Lévy-based pricing models for other credit derivatives, such as European credit default swaptions and collateralized debt obligations (CDOs), we highlight [3, 15, 25], among others.

Lévy processes have been widely applied in derivatives pricing. Some well-known examples of Lévy pricing models include the variance gamma (VG) model [33], the normal inverse Gaussian (NIG) model [7], the CGMY model [14] as well as a number of jump diffusion models (see [27, 34]). In this paper, instead of focusing on a particular type of Lévy process, we consider a general class of Lévy processes with only negative jumps. This is called the spectrally negative Lévy process and has been drawing much attention recently, as a generalization of the classical Cramér-Lundberg and other compound-Poisson type processes. A number of fluctuation identities can be expressed in terms of the scale function and are used in a number of applications. We refer the reader to [1, 5] for derivative pricing, [30] for optimal capital structure, [8, 9] for stochastic games, [6, 29, 32] for optimal dividend problem, and [16] for optimal timing of capital reinforcement. For a comprehensive account, see [28].

A major component of our pricing problem involves a non-standard American option subject to default risk (see Proposition 2.1). This is related to some existing work on perpetual early exercisable options under various Lévy models, for example [2, 5, 11, 35]. The infinite horizon nature of these problems provides significant convenience for analysis and sometimes permits an explicit solution. For finite maturity American options, the solution can be obtained by solving the underlying partial integral differential equation (PIDE) or by other approximation methods; see for example [4, 21, 23]. In our model, the solution is analytic for a general spectrally negative Lévy process. For numerical examples, we select the phase-type (and hyperexponential) fitting approach by Egami and Yamazaki [17] to illustrate the cases when the process is a mixture of diffusion and a compound Poisson process with Pareto-distributed jumps.

The rest of the paper is organized as follows. In Section 2, we formulate the default swap valuation problems under a general Lévy model. In Section 3, we focus on the spectrally negative Lévy model and provide a complete solution and detailed analysis. Section 4 provides the numerical results along with a discussion on numerical

approximation of scale functions. Section 5 concludes the paper and presents some extensions of our model. Most proofs are deferred to the Appendix.

2. PROBLEM OVERVIEW

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where \mathbb{P} is the *risk-neutral* measure used for pricing. We assume there exists a Lévy process $X = \{X_t; t \geq 0\}$, and denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the filtration generated by X . The value of the reference entity (a company stock or other assets) is assumed to evolve according to an *exponential Lévy process* $S_t = e^{X_t}$, $t \geq 0$. Following the Black-Cox [10] structural approach, the default event is triggered by S crossing a lower level D , so the default time is given by the first passage time: $\theta_D := \inf\{t \geq 0 : X_t \leq \log D\}$. Without loss of generality, we can take $\log D = 0$ by shifting the initial value x . Henceforth, we shall work with the default time:

$$\theta := \inf\{t \geq 0 : X_t \leq 0\},$$

where we assume $\inf \emptyset = \infty$. Throughout this paper, we denote by \mathbb{P}^x the probability law and \mathbb{E}^x the expectation under which $X_0 = x$.

2.1. Credit Default Swaps and Swaptions. In preparation for CDS with step-up/down options, let us start with the basic concepts of credit default swaps and swaptions. Under a T -year credit default swap (CDS) on a unit face value, the protection buyer pays a constant premium payment $\$p$ continuously over time until default time θ or maturity T , whichever comes first. If default occurs before T , the buyer will receive the default payment $\alpha := 1 - R$ at time θ , where R is the assumed constant recovery rate (typically 40%). From the buyer's perspective, the expected discounted payoff is given by

$$\bar{C}(x; p, \alpha, T) := \mathbb{E}^x \left[- \int_0^{\theta \wedge T} e^{-rt} p \, dt + \alpha e^{-r\theta} \mathbf{1}_{\{\theta \leq T\}} \right],$$

where $r > 0$ is the positive constant risk-free interest rate. The quantity $\bar{C}(x; p, \alpha, T)$ can be viewed as the market price for the buyer to enter (or long) a CDS with an agreed premium p , default payment α and maturity T . On the opposite side of the trade, the protection seller's expected cash flow is $-\bar{C}(x; p, \alpha, T) = \bar{C}(x; -p, -\alpha, T) \in \mathbb{R}$.

In standard practice, the CDS spread \bar{p} is determined at inception such that $\bar{C}(x; \bar{p}, \alpha, T) = 0$, yielding zero expected cash flows for both parties. Direct calculations show that the credit spread can be expressed as

$$(2.1) \quad \bar{p}(x; \alpha, T) = \frac{\alpha r \zeta_T(x)}{1 - \zeta_T(x) - e^{-rT} \mathbb{P}^x\{\theta > T\}}, \quad \text{where} \quad \zeta_T(x) := \mathbb{E}^x \left[e^{-r\theta} \mathbf{1}_{\{\theta \leq T\}} \right].$$

For most Lévy models, due to the lack of explicit formulas, the computation of the CDS spread is based on simulation or other approximation methods (see, for example, [12]). Alternatively, one can consider the perpetual

case as an approximation. This is a popular approach adopted for equity derivatives, especially American options, for which the finite-maturity contracts do not admit closed-form solutions while the perpetual versions often do.

To illustrate, we set $T = +\infty$ and express the buyer's CDS price as

$$(2.2) \quad \begin{aligned} C(x; p, \alpha) &:= \mathbb{E}^x \left[- \int_0^\theta e^{-rt} p \, dt + \alpha e^{-r\theta} \right] \\ &= \left(\frac{p}{r} + \alpha \right) \zeta(x) - \frac{p}{r}, \end{aligned}$$

where

$$(2.3) \quad \zeta(x) := \mathbb{E}^x \left[e^{-r\theta} \right].$$

The seller's CDS price is $-C(x; p, \alpha) = C(x; -p, -\alpha) \in \mathbb{R}$. Solving $C(x; p, \alpha) = 0$ yields the credit spread:

$$(2.4) \quad p(x; \alpha) = \frac{\alpha r \zeta(x)}{1 - \zeta(x)}.$$

Therefore, the credit spread calculation reduces to computing the Laplace transform $\zeta(x)$, which admits an explicit analytic formula under some well-known Lévy models (see (3.4) below for the spectrally negative case). It is clear from (2.4) that the CDS spread scales linearly in α : $p(x; \alpha) = \alpha p(x; 1)$.

Next, we introduce a *perpetual American payer* and *receiver default swaptions*, which give the holder the right to, respectively, buy and sell protection on a perpetual CDS with default payment a at a pre-specified spread κ for the strike price K upon exercise. If default occurs prior to exercise, then the swaption is knocked out and becomes worthless. The payer and receiver swaption holder is required to pay an upfront fee, which is given by respectively

$$(2.5) \quad v(x; \kappa, a, K) := \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} (C(X_\tau; \kappa, a) - K)^+ 1_{\{\tau < \theta\}} \right], \quad \text{and}$$

$$(2.6) \quad u(x; \kappa, a, K) := \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} (-C(X_\tau; \kappa, a) - K)^+ 1_{\{\tau < \theta\}} \right],$$

where $\mathcal{S} := \{ \tau \in \mathbb{F} : \tau \leq \theta \text{ a.s.} \}$ is the set of all stopping times *smaller than or equal to* the default time. The two price functions are related by

$$(2.7) \quad v(x; \kappa, a, K) = u(x; -\kappa, -a, K).$$

In summary, $v(x; \kappa, a, K)$ is the payer default swaption price when $\kappa, a \geq 0$, and it is the receiver default swaption price when $\kappa, a \leq 0$.

2.2. American Callable Step-Up and Step-Down CDS. Next, we consider a CDS contract with an embedded option that permits the protection buyer to change the face value and premium once for a fee. Beginning from initiation, the buyer pays a premium p for a protection of a unit face value. At any time prior to default, the buyer can select a time τ to switch to a new contract with a new premium \hat{p} and face value q for a fee γ . The default payment then changes from α to $\hat{\alpha} = q\alpha$ after the exercise time τ . Here, $p, \alpha, \gamma, \hat{\alpha}$, and \hat{p} are constant non-negative parameters pre-specified at time zero. The buyer's maximal expected cash flow is given by

$$(2.8) \quad V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) := \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[- \int_0^\tau e^{-rt} p \, dt + 1_{\{\tau < \infty\}} \left(- \int_\tau^\theta e^{-rt} \hat{p} \, dt - e^{-r\tau} \gamma 1_{\{\tau < \theta\}} + e^{-r\theta} (\hat{\alpha} 1_{\{\tau < \theta\}} + \alpha 1_{\{\tau = \theta\}}) \right) \right].$$

This formulation covers default swaps with the following provisions:

- (1) *Step-up Option*: if $\hat{p} > p$ and $\hat{\alpha} > \alpha$, then the buyer is allowed to increase the coverage once from α to $\hat{\alpha}$ by paying the fee γ and a higher premium \hat{p} thereafter.
- (2) *Step-down Option*: when $\hat{p} < p$ and $\hat{\alpha} < \alpha$, then the buyer can reduce the coverage once from α to $\hat{\alpha}$ by paying the fee γ and a reduced premium \hat{p} thereafter.
- (3) *Cancellation Right*: as a special case of the step-down option with $\hat{p} = \hat{\alpha} = 0$, the resulting contract allows the buyer to terminate the CDS at time τ .

In addition, the perpetual vanilla CDS corresponds to the case with $\gamma = 0, p = \hat{p}$ and $\alpha = \hat{\alpha}$, and the CDS spread is given by (2.4). We ignore the contract specifications with $(\hat{p} - p)(\hat{\alpha} - \alpha) \leq 0$ since they would mean paying more (less) premium in exchange for a reduced (increased) protection after exercise. In summary, we study the valuation of the (perpetual) *American callable step-up/down CDS*. For any fixed parameters $(p, \hat{p}, \alpha, \hat{\alpha}, \gamma)$, the value $V(x)$ is referred to as the buyer's price, so the seller's price is $-V(x)$. The credit spread p^* is determined from the equation $V(x; p^*, \hat{p}, \alpha, \hat{\alpha}, \gamma) = 0$ so that no cash transaction occurs at inception.

In preparation for our solution procedure, we first provide a useful representation of the buyer's value V . Define

$$\tilde{\alpha} := \alpha - \hat{\alpha}, \quad \text{and} \quad \tilde{p} := p - \hat{p}.$$

Here, $\tilde{\alpha} > 0$ and $\tilde{p} > 0$ hold for a step-down CDS and $\tilde{\alpha} < 0$ and $\tilde{p} < 0$ for a step-up CDS.

Proposition 2.1. *The perpetual American callable step-up/down CDS can be decomposed into a vanilla CDS plus a perpetual American payer/receiver swaption. Precisely, we have*

$$(2.9) \quad V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) = C(x; p, \alpha) + v(x; -\tilde{p}, -\tilde{\alpha}, \gamma),$$

where $C(\cdot)$ and $v(\cdot)$ are given in (2.2) and (2.5) respectively.

Proof. First, by a rearrangement of integrals, the expression inside the expectation in (2.8) becomes

$$\begin{aligned} & 1_{\{\tau < \infty\}} \left(\int_{\tau}^{\theta} e^{-rt} \tilde{p} dt - \int_0^{\theta} e^{-rt} p dt - e^{-r\tau} \gamma 1_{\{\tau < \theta\}} - e^{-r\theta} \tilde{\alpha} 1_{\{\tau < \theta\}} + e^{-r\theta} \alpha \right) + 1_{\{\tau = \infty\}} \left(- \int_0^{\infty} e^{-rt} p dt \right) \\ &= 1_{\{\tau < \infty\}} \left(\int_{\tau}^{\theta} e^{-rt} \tilde{p} dt - e^{-r\tau} \gamma 1_{\{\tau < \theta\}} - e^{-r\theta} \tilde{\alpha} 1_{\{\tau < \theta\}} \right) - \int_0^{\theta} e^{-rt} p dt + e^{-r\theta} \alpha \end{aligned}$$

since $\tau = \infty$ implies $\theta = \infty$ by the definition of \mathcal{S} . Because the last two terms do not depend on τ , we can rewrite the buyer's value function as

$$V(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[\underbrace{1_{\{\tau < \infty\}} \left(\int_{\tau}^{\theta} e^{-rt} \tilde{p} dt - e^{-r\tau} \gamma 1_{\{\tau < \theta\}} - e^{-r\theta} \tilde{\alpha} 1_{\{\tau < \theta\}} \right)}_{=: f(x)} \right] - \mathbb{E}^x \left[\int_0^{\theta} e^{-rt} p dt \right] + \alpha \mathbb{E}^x \left[e^{-r\theta} \right].$$

Here, the last two terms in fact constitute $C(x; p, \alpha)$. Next, using the fact $\{\tau < \theta, \tau < \infty\} = \{X_{\tau} > 0, \tau < \infty\}$ for every $\tau \in \mathcal{S}$ and the strong Markov property of X at time τ , we rewrite the first term as

$$(2.10) \quad f(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} h(X_{\tau}) 1_{\{\tau < \infty\}} \right],$$

with

$$(2.11) \quad h(x) := 1_{\{x > 0\}} \left(\mathbb{E}^x \left[\int_0^{\theta} e^{-rt} \tilde{p} dt - e^{-r\theta} \tilde{\alpha} \right] - \gamma \right) = 1_{\{x > 0\}} (C(x; -\tilde{p}, -\tilde{\alpha}) - \gamma).$$

Since $\theta \in \mathcal{S}$ and $h(X_{\theta}) = 0$ a.s. on $\{\theta < \infty\}$, it follows from (2.10) that $f(x) \geq 0$. Therefore, it is never optimal to exercise at any τ if $h(X_{\tau}) < 0$. Consequently, we can replace $h(x)$ with $(h(x))^+$ in (2.10). As a result, with $-\tilde{p}, -\tilde{\alpha} > (<) 0$, the function $f(x)$ is indeed the price of a perpetual *American payer (receiver) default swaption* written on the buyer's (seller's) CDS price with strike $\gamma \geq 0$. This implies that $f(x) = v(x; -\tilde{p}, -\tilde{\alpha}, \gamma) \forall x \in \mathbb{R}$, and therefore (2.9) follows. \square

The decomposition (2.9) in Proposition 2.1 yields a *static replication* of the American callable step-up/down CDS. To this end, one may also verify the result by a no-arbitrage argument. We summarize the buyer's and seller's positions in the American callable step-up/down CDS in Table 1.

Proposition 2.1 also provides some insight on the protection buyer's exercise timing. To illustrate, let us consider an example where the premium and protection are doubled after exercise, i.e. $\hat{p} = 2p$ and $\hat{\alpha} = 2\alpha$. For any candidate exercise time τ , the observable market prevailing vanilla CDS spread is given by $p(X_{\tau}; \alpha)$ in (2.4), and $C(X_{\tau}; p(X_{\tau}; \alpha), \alpha) = 0$ by definition. Hence, if $p(X_{\tau}; \alpha) \leq -\tilde{p} = p$ at τ , then $h(X_{\tau}) \leq -\gamma \leq 0$, and the buyer will not exercise. This is intuitive because the buyer is better off giving up the step-up option and double his protection by entering a separate CDS at the lower market spread $p(X_{\tau}; \alpha)$ at time τ .

	<i>Protection Buyer's Position</i>	<i>Protection Seller's Position</i>
<i>American Callable Step-Up CDS</i>	long a vanilla CDS & long an American payer default swaption	short a vanilla CDS & short an American payer default swaption
<i>American Callable Step-Down CDS</i>	long a vanilla CDS & long an American receiver default swaption	short a vanilla CDS & short an American receiver default swaption

TABLE 1. Four positions of American callable step-up/down CDS and their decompositions.

The seller's position is the opposite of the buyer's.

2.3. The American Puttable Step-Up/Down CDS. Applying the ideas from the previous subsection, we formulate the pricing problem for the perpetual *American puttable step-up/down CDS*. These CDSs allow the protection seller (and not the buyer) to change the protection premium and default payment for a fee anytime prior to default. Let p and α be the initial premium and default payment. The seller may select a time τ to switch to a new premium \hat{p} and default payment $\hat{\alpha}$ for a switching fee $\gamma \geq 0$. The seller's maximal expected cash flow is

$$\begin{aligned}
 (2.12) \quad U(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) &:= \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[\int_0^\tau e^{-rt} p \, dt + 1_{\{\tau < \infty\}} \left(\int_\tau^\theta e^{-rt} \hat{p} \, dt - e^{-r\tau} \gamma 1_{\{\tau < \theta\}} - e^{-r\theta} (\hat{\alpha} 1_{\{\tau < \theta\}} + \alpha 1_{\{\tau = \theta\}}) \right) \right] \\
 &= \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[1_{\{\tau < \infty\}} \left(- \int_\tau^\theta e^{-rt} \tilde{p} \, dt - e^{-r\tau} \gamma 1_{\{\tau < \theta\}} + e^{-r\theta} \tilde{\alpha} 1_{\{\tau < \theta\}} \right) \right] \\
 &\quad + \mathbb{E}^x \left[\int_0^\theta e^{-rt} p \, dt \right] - \alpha \mathbb{E}^x \left[e^{-r\theta} \right].
 \end{aligned}$$

In particular, we will study the American puttable CDS with a step-up option (i.e. $p < \hat{p}$ and $\alpha < \hat{\alpha}$) or step-down option (i.e. $p > \hat{p}$ and $\alpha > \hat{\alpha}$). Again, the credit spread p^* is chosen so that the seller's value function is zero, i.e. $U(x; p^*, \hat{p}, \alpha, \hat{\alpha}, \gamma) = 0$.

Following the procedure in the proof of Proposition 2.1 or by a no-arbitrage argument, we can simplify the seller's value U as follows:

Proposition 2.2. *The perpetual American puttable step-up/down CDS can be decomposed into a short vanilla CDS and a long perpetual American receiver/payer default swaption. Precisely, we have*

$$(2.13) \quad U(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) = -C(x; p, \alpha) + u(x; -\tilde{p}, -\tilde{\alpha}, \gamma),$$

where $C(\cdot)$ and $u(\cdot)$ are given in (2.2) and (2.6), respectively.

	<i>Protection Buyer's Position</i>	<i>Protection Seller's Position</i>
<i>American Puttable Step-Up CDS</i>	long a vanilla CDS & short an American receiver default swaption	short a vanilla CDS & long an American receiver default swaption
<i>American Puttable Step-Down CDS</i>	long a vanilla CDS & short an American payer default swaption	short a vanilla CDS & long an American payer default swaption

TABLE 2. Four positions of American puttable step-up/down CDS and their decompositions. The seller's position is the opposite of the buyer's.

We summarize the buyer's and seller's positions in the American puttable step-up/down CDS in Table 2. To gain intuition on the seller's exercise decision, let us look at the step-down case where $\hat{p} = 0.5p$ and $\hat{\alpha} = 0.5\alpha$. Recall that $C(x; p, \alpha)$ is decreasing in p and $C(X_\tau, p(X_\tau; \alpha), \alpha) = 0$ for any stopping time τ . If the market prevailing CDS spread $p(X_\tau; \alpha) \leq p$ at some τ , then the seller's default swaption payoff is $-C(X_\tau; \tilde{p}, \tilde{\alpha}) - \gamma \leq -\gamma \leq 0$. The seller will not exercise at τ since the protection of 0.5α can be purchased from a separate CDS at the lower prevailing spread $0.5p(X_\tau; \alpha) \leq 0.5p$.

2.4. Symmetry Between Callable and Puttable CDS. By Propositions 2.1 and 2.2, along with (2.7), we observe the following “put-call parity” and symmetry identities:

$$V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) - U(x; p, 2p - \hat{p}, \alpha, 2\alpha - \hat{\alpha}, \gamma) = 2C(x; p, \alpha),$$

$$V(x; p, \hat{p}, \alpha, \hat{\alpha}, \gamma) + U(x; p, 2p - \hat{p}, \alpha, 2\alpha - \hat{\alpha}, \gamma) = 2v(x; \hat{p} - p, \hat{\alpha} - \alpha, \gamma).$$

The first equality means a long position in an American callable step-up (step-down) CDS and a short position in an American puttable step-down (step-up) CDS result in a double long position in a vanilla CDS. From the second equality, a long position in both an American callable step-up (step-down) CDS and an American puttable step-down (step-up) CDS yields a double long position in an American payer (receiver) default swaption.

Furthermore, according to (2.9) and (2.13), the optimal exercise times for $V(x)$ and $U(x)$ are determined from $v(x)$ and $u(x)$ which depend on the triplet $(\tilde{p}, \tilde{\alpha}, \gamma)$ but not directly on p and α . Consequently, by (2.7), the same optimal exercising strategy applies for both

- (1) the protection buyer of an American callable CDS with a step-up (step-down) option with $(-\tilde{p}, -\tilde{\alpha}, \gamma)$,
and
- (2) the protection seller of an American puttable CDS with a step-down (step-up) option with $(\tilde{p}, \tilde{\alpha}, \gamma)$.

This observation means that it suffices to solve for two cases instead of four. Specifically, we shall solve for (i) the buyer's callable step-down case in (2.9) and (ii) the seller's puttable step-down case in (2.13), both with $\tilde{p} > 0$ and

$\tilde{\alpha} > 0$. In view of (2.10) and the proof of Proposition 2.1, this amounts to solving the following optimal stopping problems:

$$(2.14) \quad v(x) := v(x; -\tilde{p}, -\tilde{\alpha}, \gamma) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} h(X_\tau) 1_{\{\tau < \infty\}} \right],$$

$$(2.15) \quad u(x) := u(x; -\tilde{p}, -\tilde{\alpha}, \gamma) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} g(X_\tau) 1_{\{\tau < \infty\}} \right],$$

where $\tilde{p}, \tilde{\alpha} > 0$ and

$$(2.16) \quad h(x) := \left(\left(\frac{\tilde{p}}{r} - \gamma \right) - \left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) \zeta(x) \right) 1_{\{x > 0\}},$$

$$(2.17) \quad g(x) := \left(\left(-\frac{\tilde{p}}{r} - \gamma \right) + \left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) \zeta(x) \right) 1_{\{x > 0\}},$$

for $x \in \mathbb{R}$. Here, $h(x)$ and $g(x)$ are computed using formula (2.2).

By inspecting (2.14), it follows from (2.16) that $h(x) \leq 0 \forall x \in \mathbb{R}$ if $\gamma \geq \tilde{p}/r$. Financially, this means that the fee γ to be paid exceeds the maximum benefit of stepping down, i.e. perpetual annuity with premium $p - \hat{p} > 0$. It is clear that choosing $\tau = \theta$ is optimal and the protection buyer will never exercise the step-down option. Hence, we only need to study the non-trivial case with the condition

$$(2.18) \quad 0 \leq \gamma < \frac{\tilde{p}}{r}.$$

For (2.15), we have $g(x) \leq 0 \forall x \in \mathbb{R}$ if $g(0+) \leq 0$ because g is decreasing in x on $(0, \infty)$. Again, this means that θ is automatically optimal for the protection seller. Therefore, we shall focus on the case with $g(0+) > 0$ which also implies

$$(2.19) \quad 0 \leq \gamma < \tilde{\alpha}.$$

The intuition behind this is that the fee should not exceed the reduction in liability.

2.5. Solution Methods via Continuous and Smooth Fit. We conclude this section by describing our solution procedure for the optimal stopping problems under a general Lévy model. In the next section, we shall focus on the spectrally negative Lévy model and derive an analytical solution.

For our first problem (2.14), the protection buyer has an incentive to step-down when default is less likely, or equivalently when X is sufficiently high. Following this intuition, we denote the threshold strategy

$$(2.20) \quad \tau_B^+ := \inf \{t \geq 0 : X_t \notin (0, B)\}, \quad B \geq 0.$$

Clearly, $\tau_B^+ \in \mathcal{S}$. The corresponding expected payoff is given by

$$(2.21) \quad v_B(x) := \mathbb{E}^x \left[e^{-r\tau_B^+} h(X_{\tau_B^+}) 1_{\{\tau_B^+ < \infty\}} \right], \quad x \in \mathbb{R}.$$

Note that $v_B(x) = h(x) = 0$ for $x \leq 0$. Sometimes it is more intuitive to consider the difference

$$\Delta_B(x) := v_B(x) - h(x), \quad x \in \mathbb{R}.$$

One common solution approach for many optimal stopping problems is *continuous and smooth fit* (see [36, 38, 39, 40]). Applying to our problem, it involves two main steps:

- (a) obtain B^* that satisfies the continuous or smooth fit condition: $\Delta_{B^*}(B^* -) = 0$ or $\Delta'_{B^*}(B^* -) = 0$, and
- (b) verify the optimality of $\tau_{B^*}^+$ by showing (i) $v_{B^*}(x) \geq h(x)$ for $x \in \mathbb{R}$ and (ii) the process $M_t := e^{-r(t \wedge \theta)} v_{B^*}(X_{t \wedge \theta})$, $t \geq 0$, is a supermartingale.

To this end, an analytical expression for v_B or Δ_B would be useful.

Lemma 2.1. *Fix $B > 0$. The function Δ_B is given by*

$$(2.22) \quad \Delta_B(x) = \begin{cases} \left(\frac{\tilde{p}}{r} - \gamma \right) \Lambda_1(x; B) + \left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) \Lambda_2(x; B) + \gamma - \frac{\tilde{p}}{r}, & x \in (0, B), \\ 0, & x \notin (0, B), \end{cases}$$

where $\Lambda_1(x; B) := \mathbb{E}^x \left[e^{-r\tau_B^+} 1_{\{\tau_B^+ < \theta, \tau_B^+ < \infty\}} \right]$ and $\Lambda_2(x; B) := \mathbb{E}^x \left[e^{-r\tau_B^+} 1_{\{\tau_B^+ = \theta, \tau_B^+ < \infty\}} \right]$.

As we shall see in Section 3, the functions $\Lambda_1(\cdot; B)$ and $\Lambda_2(\cdot; B)$ can be computed via the scale functions for a spectrally negative Lévy model; see (3.3) below.

In our second problem (2.15), the protection seller tends to exercise the step-down option when default is likely, or equivalently when X is sufficiently small. Suppose the seller exercises at the first time X reaches or goes below some fixed threshold $A \geq 0$; namely,

$$\tau_A^- := \inf\{t \geq 0 : X_t \leq A\}.$$

Then, the corresponding expected payoff is given by

$$u_A(x) := \mathbb{E}^x \left[e^{-r\tau_A^-} g(X_{\tau_A^-}) 1_{\{\tau_A^- < \infty\}} \right], \quad x \in \mathbb{R}.$$

Again, we denote the difference between continuation and exercise by $\Delta_A(x) := u_A(x) - g(x)$ for $x \in \mathbb{R}$.

For this problem, the continuous and smooth fit solution approach is to

- (a) obtain A^* that satisfies the smooth fit condition: $\Delta_{A^*}(A^* +) = 0$ or $\Delta'_{A^*}(A^* +) = 0$, and
- (b) verify the optimality of $\tau_{A^*}^-$ by showing (i) $u_{A^*}(x) \geq g(x)$ for $x \in \mathbb{R}$ and (ii) the process $\widetilde{M}_t := e^{-r(t \wedge \theta)} u_{A^*}(X_{t \wedge \theta})$, $t \geq 0$, is a supermartingale.

This method requires some expression for Δ_A , which is summarized as follows:

Lemma 2.2. Fix $A > 0$. The function Δ_A is given by

$$(2.23) \quad \Delta_A(x) = \begin{cases} \left(\gamma + \frac{\tilde{p}}{r}\right) (1 - \zeta(x - A)) - (\tilde{\alpha} - \gamma) \Gamma(x; A), & x > A, \\ 0, & x \leq A, \end{cases}$$

where

$$(2.24) \quad \Gamma(x; A) := \mathbb{E}^x \left[e^{-r\tau_A^-} 1_{\{X_{\tau_A^-} < 0, \tau_A^- < \infty\}} \right].$$

The function $\Gamma(\cdot; A)$ and Laplace transform $\zeta(\cdot)$ can be also expressed in terms of the scale function for a spectrally negative Lévy model; see (3.4) and Lemma 3.10 below.

3. SOLUTION METHODS UNDER THE SPECTRALLY NEGATIVE LÉVY MODEL

We proceed to solve the optimal stopping problems (2.14) and (2.15) for spectrally negative Lévy processes. With the analytical solutions (see Theorems 3.1 and 3.2 below), the American callable/putable step-up/down CDS can be immediately priced in view of Propositions 2.1 and 2.2.

3.1. The Spectrally Negative Lévy Process and Scale Function. Let X be a spectrally negative Lévy process with the *Laplace exponent*

$$(3.1) \quad \psi(s) := \log \mathbb{E}^0 [e^{sX_1}] = cs + \frac{1}{2}\sigma^2 s^2 + \int_{(0,\infty)} (e^{-sx} - 1 + sx1_{\{0 < x < 1\}}) \Pi(dx), \quad s \in \mathbb{C},$$

where $c \in \mathbb{R}$, $\sigma \geq 0$ is called the Gaussian coefficient, and Π is a Lévy measure on $(0, \infty)$ such that $\int_{(0,\infty)} (1 \wedge x^2) \Pi(dx) < \infty$. See [28], p.212. The risk neutral condition requires that $\psi(1) = r$ so that the discounted stock price is a \mathbb{P} -martingale. In particular, when

$$(3.2) \quad \int_{(0,\infty)} (1 \wedge x) \Pi(dx) < \infty,$$

we can rewrite

$$\psi(s) = \mu s + \frac{1}{2}\sigma^2 s^2 + \int_{(0,\infty)} (e^{-sx} - 1) \Pi(dx), \quad s \in \mathbb{C}$$

where

$$\mu := c + \int_{(0,1)} x \Pi(dx).$$

Recall that the process has paths of bounded variation if and only if $\sigma = 0$ and (3.2) holds. A special example is a compound Poisson process where $\Pi(0, \infty) < \infty$. We ignore the case X is a negative subordinator (decreasing a.s.). This means that we require μ to be strictly positive if $\sigma = 0$. See also Remark 3.1 .

For any spectrally negative Lévy process, there exists an (r -)scale function

$$W^{(r)} : \mathbb{R} \mapsto \mathbb{R}, \quad r \geq 0,$$

whose Laplace transform is

$$\int_0^\infty e^{-sx} W^{(r)}(x) dx = \frac{1}{\psi(s) - r}, \quad s > \Phi(r),$$

where Φ is the right inverse of ψ , defined by

$$\Phi(r) := \sup\{\lambda \geq 0 : \psi(\lambda) = r\}, \quad r \geq 0.$$

Here, $W^{(r)}(x) = 0$ on $(-\infty, 0)$. Fix $B \geq x > 0$. By the property of the scale function (see, for example, Theorem 8.1 of [28]), we obtain the formulas:

$$(3.3) \quad \begin{aligned} \Lambda_1(x; B) &= \mathbb{E}^x \left[e^{-r\tau_B^+} \mathbf{1}_{\{\tau_B^+ < \theta, \tau_B^+ < \infty\}} \right] = \frac{W^{(r)}(x)}{W^{(r)}(B)}, \\ \Lambda_2(x; B) &= \mathbb{E}^x \left[e^{-r\tau_B^+} \mathbf{1}_{\{\tau_B^+ = \theta, \tau_B^+ < \infty\}} \right] = Z^{(r)}(x) - Z^{(r)}(B) \frac{W^{(r)}(x)}{W^{(r)}(B)}, \end{aligned}$$

where

$$Z^{(r)}(x) := 1 + r \int_0^x W^{(r)}(y) dy, \quad x \in \mathbb{R}.$$

Notice that $Z^{(r)}(x) = 1$ for every $x \in (-\infty, 0]$. The Laplace transform of θ in (2.3) becomes

$$(3.4) \quad \zeta(x) = Z^{(r)}(x) - \frac{r}{\Phi(r)} W^{(r)}(x), \quad x \in \mathbb{R}.$$

By Proposition 1 of [41], the scale function $W^{(r)}$ increases exponentially in x on \mathbb{R}_+ , and we have

$$(3.5) \quad W^{(r)}(x) \sim \frac{e^{\Phi(r)x}}{\psi'(\Phi(r))} \quad \text{as } x \rightarrow \infty.$$

There also exists a version of the scale function $W_{\Phi(r)} = \{W_{\Phi(r)}(x); x \in \mathbb{R}\}$ that satisfies

$$(3.6) \quad W^{(r)}(x) = e^{\Phi(r)x} W_{\Phi(r)}(x), \quad x \in \mathbb{R},$$

and

$$\int_0^\infty e^{-sx} W_{\Phi(r)}(x) dx = \frac{1}{\psi(s + \Phi(r)) - r}, \quad s > 0.$$

The function $W_{\Phi(r)}(x)$ is increasing and

$$(3.7) \quad W_{\Phi(r)}(x) \sim \frac{1}{\psi'(\Phi(r))} \quad \text{as } x \rightarrow \infty.$$

From Lemmas 4.3 and 4.4 of [30], we also summarize the behavior in the neighborhood of zero.

Lemma 3.1. For every $r \geq 0$, we have

$$W^{(r)}(0) = \left\{ \begin{array}{ll} 0, & \text{unbounded variation} \\ \frac{1}{\mu}, & \text{bounded variation} \end{array} \right\} \quad \text{and} \quad W^{(r)'}(0+) = \left\{ \begin{array}{ll} \frac{2}{\sigma^2}, & \sigma > 0 \\ \infty, & \sigma = 0 \text{ and } \Pi(0, \infty) = \infty \\ \frac{r + \Pi(0, \infty)}{\mu^2}, & \text{compound Poisson} \end{array} \right\}.$$

Remark 3.1. When X is a negative subordinator, the scale function is not defined, but we can remedy this by changing the measure via

$$\frac{d\mathbb{P}_{\Phi(r)}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = e^{\Phi(r)X_t - rt}, \quad t \geq 0.$$

Indeed, under $\mathbb{P}_{\Phi(r)}$, X_t converges to ∞ a.s. as $t \rightarrow \infty$ and the corresponding scale function becomes $W_{\Phi(r)}$. See p.213 of [28]. Although the resulting problem becomes an undiscounted problem, it can be solved easily by extending the results in this paper.

3.2. Callable Step-Down CDS. We now solve (2.14) for the callable step-down CDS. First, we write down the expected payoff function $v_B(x)$ in (2.21) with threshold B . Precisely, we have

$$(3.8) \quad v_B(x) = \begin{cases} h(x), & x \in [B, \infty), \\ h(x) + \Delta_B(x), & x \in (0, B), \\ 0, & x \in (-\infty, 0], \end{cases}$$

for $0 \leq B < \infty$. If $B = \infty$, then $v_B(x) = 0$, $x \in \mathbb{R}$. If $B = 0$, then $v_B(x) = h(x)$, $x \in \mathbb{R}$. Here, the stopping value $h(x)$ is computed using the scale function and (3.4), and is given by

$$(3.9) \quad h(x) = \left[\tilde{p} \left(\frac{1}{r} (1 - Z^{(r)}(x)) + \frac{1}{\Phi(r)} W^{(r)}(x) \right) - \tilde{\alpha} \left(Z^{(r)}(x) - \frac{r}{\Phi(r)} W^{(r)}(x) \right) - \gamma \right] 1_{\{x > 0\}}, \quad x \in \mathbb{R}.$$

By Lemma 2.1 and (3.3), the difference function $\Delta_B(x)$ is given by

$$(3.10) \quad \Delta_B(x) = \frac{\tilde{p}}{r} (Z^{(r)}(x) - 1) + \tilde{\alpha} Z^{(r)}(x) - \frac{W^{(r)}(x)}{W^{(r)}(B)} G^{(r)}(B) + \gamma, \quad x \in (0, B),$$

where

$$G^{(r)}(B) := \frac{\tilde{p}}{r} (Z^{(r)}(B) - 1) + \tilde{\alpha} Z^{(r)}(B) + \gamma, \quad B \geq 0.$$

Remark 3.2. Whenever $x = B > 0$, we have $\tau_B^+ = 0 < \theta$ a.s. and hence $\Delta_B(B) = 0$ or $v_B(B) = h(B)$. In view of (3.10), it is clear that $\Delta_B(B-) = 0$. In other words, continuous fit $v_B(B-) = v_B(B)$ holds for all values of $B > 0$.

To obtain the candidate optimal threshold, we consider the smooth fit condition. To this end, we compute the derivative

$$\Delta'_B(x) = (\tilde{p} + \tilde{\alpha}r) W^{(r)}(x) - \frac{W^{(r)'}(x)}{W^{(r)}(B)} G^{(r)}(B), \quad 0 < x < B.$$

Then, we define the function

$$\varrho(B) := \Delta'_B(B-) = (\tilde{p} + \tilde{\alpha}r) W^{(r)}(B) - \frac{W^{(r)'}(B)}{W^{(r)}(B)} G^{(r)}(B), \quad B > 0.$$

Its derivative is

$$\varrho'(B) = - \left(\frac{\partial}{\partial B} \frac{W^{(r)'}(B)}{W^{(r)}(B)} \right) G^{(r)}(B), \quad B > 0.$$

Here, notice that $G^{(r)}(B) \geq \tilde{\alpha} + \gamma > 0$ for every $B \geq 0$. Therefore, the function $\varrho(B)$ is increasing in B if and only if

$$(3.11) \quad \frac{\partial}{\partial B} \frac{W^{(r)'}(B)}{W^{(r)}(B)} \leq 0, \quad B > 0.$$

Assumption 3.1. We assume that (3.11) holds for (2.14).

Remark 3.3. Assumption 3.1 is rather natural. To see this, we apply $W^{(r)'}(B) = \Phi(r)e^{\Phi(r)B}W_{\Phi(r)}(B) + e^{\Phi(r)B}W'_{\Phi(r)}(B)$ to write

$$\frac{\partial}{\partial B} \frac{W^{(r)'}(B)}{W^{(r)}(B)} = \frac{\partial}{\partial B} \left(\frac{W'_{\Phi(r)}(B)}{W_{\Phi(r)}(B)} \right) = \frac{1}{(W_{\Phi(r)}(B))^2} \left(W''_{\Phi(r)}(B)W_{\Phi(r)}(B) - (W'_{\Phi(r)}(B))^2 \right).$$

In view of this, (3.11) is guaranteed to hold when $W''_{\Phi(r)}(x) \leq 0$ for every $x \geq 0$ as $W_{\Phi(r)}(x)$ is a non-negative increasing function. The concavity of $W_{\Phi(r)}$ is a reasonable assumption as confirmed by the numerical analysis by Surya [41]. Furthermore, if the Lévy measure Π has a completely monotone density π , namely

$$(-1)^n \pi^{(n)}(x) \geq 0, \quad x \geq 0,$$

then $W'_{\Phi(r)}$ is completely monotone according to Loeffen [32] and therefore $W_{\Phi(r)}$ is concave. Finally, we remark that Assumption 3.1 is not needed for the problem $u(x)$ in (2.15).

Under Assumption 3.1, there exists at most one $B \in (0, \infty)$, denoted by B^* , that satisfies the **smooth fit condition**:

$$(3.12) \quad \varrho(B^*) = 0.$$

If it exists, then this is our candidate optimal threshold, and $v_{B^*}(x)$ is the candidate value function for (2.14).

The smooth fit condition fails if (a) $\varrho(B) \geq 0 \forall B > 0$, or (b) $\varrho(B) < 0 \forall B > 0$. Under each of these scenarios, we need another way to deduce the candidate optimal threshold. To this end, let us consider the derivative of $v_B(x)$ with respect to B . For $0 < x < B$,

$$(3.13) \quad \frac{\partial}{\partial B} v_B(x) = \frac{\partial}{\partial B} \Delta_B(x) = -\frac{W^{(r)}(x)}{W^{(r)}(B)} \left[(r\tilde{\alpha} + \tilde{p}) W^{(r)}(B) - \frac{W^{(r)'}(B)}{W^{(r)}(B)} G^{(r)}(B) \right] = -\frac{W^{(r)}(x)}{W^{(r)}(B)} \varrho(B).$$

Under scenario (a), $\varrho(B) \geq 0$ in (3.13) implies that $v_B(x)$ is decreasing in B for any $x < B$, so we choose $B^* = 0$ as our candidate optimal threshold. In this case, the buyer will stop immediately ($\tau_0^+ = 0$), and the corresponding expected payoff $v_{B^*} = h(x)$ (see (2.21)). As we show next, $B^* = 0$ is possible only when X is of bounded variation.

Lemma 3.2. *We have $B^* = 0$ if and only if*

$$(1) \sigma = 0, \quad (2) \Pi(0, \infty) < \infty, \quad \text{and} \quad (3) \tilde{p} - r\gamma - (\tilde{\alpha} + \gamma)\Pi(0, \infty) \geq 0.$$

As for scenario (b), it follows from (3.13) that $v_B(x)$ is increasing in B for any $x < B$. Therefore, we set $B^* = \infty$, meaning that the buyer will never exercise ($\tau_\infty^+ = \theta$), and the corresponding expected payoff is $v_\infty(x) = 0$ (see (2.21)). In fact, this corresponds to the case where the payoff $h(x) \leq 0 \forall x \in \mathbb{R}$. To see this, we deduce from (3.13) and continuous fit in Remark 3.2 that, for any arbitrarily fixed $x > 0$, $\lim_{B \rightarrow \infty} v_B(x) \geq v_x(x) = h(x)$. Then, applying dominated convergence theorem (as h is bounded) and $\tau_B^+ \xrightarrow{B \uparrow \infty} \theta$ a.s., we obtain $\lim_{B \rightarrow \infty} v_B(x) = \mathbb{E}^x \left[\lim_{B \rightarrow \infty} e^{-r\tau_B^+} h(B) 1_{\{\tau_B^+ \neq \theta\}} \right] = 0$, and thus $0 \geq h(x)$. Therefore, under our earlier assumption (2.18), we have ruled out scenario (b).

To summarize, working under the assumption (2.18), we propose the candidate value function $v_{B^*}(x)$ corresponding to the threshold B^* easily obtained from (3.12) for $0 < B^* < \infty$, or $B^* = 0$ otherwise. By direct computation using (3.8)-(3.10) we get

$$(3.14) \quad v_{B^*}(x) = \begin{cases} W^{(r)}(x) \left(\frac{\tilde{p} + \tilde{\alpha}r}{\Phi(r)} - \frac{G^{(r)}(B^*)}{W^{(r)}(B^*)} \right) 1_{\{x \neq 0\}}, & -\infty < x < B^*, \\ h(x), & x \geq B^*. \end{cases}$$

Here, on $(-\infty, 0)$, $v_{B^*}(x) = 0$ because $W^{(r)}(x) = 0$.

Our next task is to show that $v_{B^*}(x)$ is indeed optimal.

Theorem 3.1. *The candidate function $v_{B^*}(x)$ is optimal for (2.14). Precisely,*

$$v_{B^*}(x) = v(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} h(X_\tau) 1_{\{\tau < \infty\}} \right],$$

and $\tau_{B^*}^+$ is the optimal stopping time.

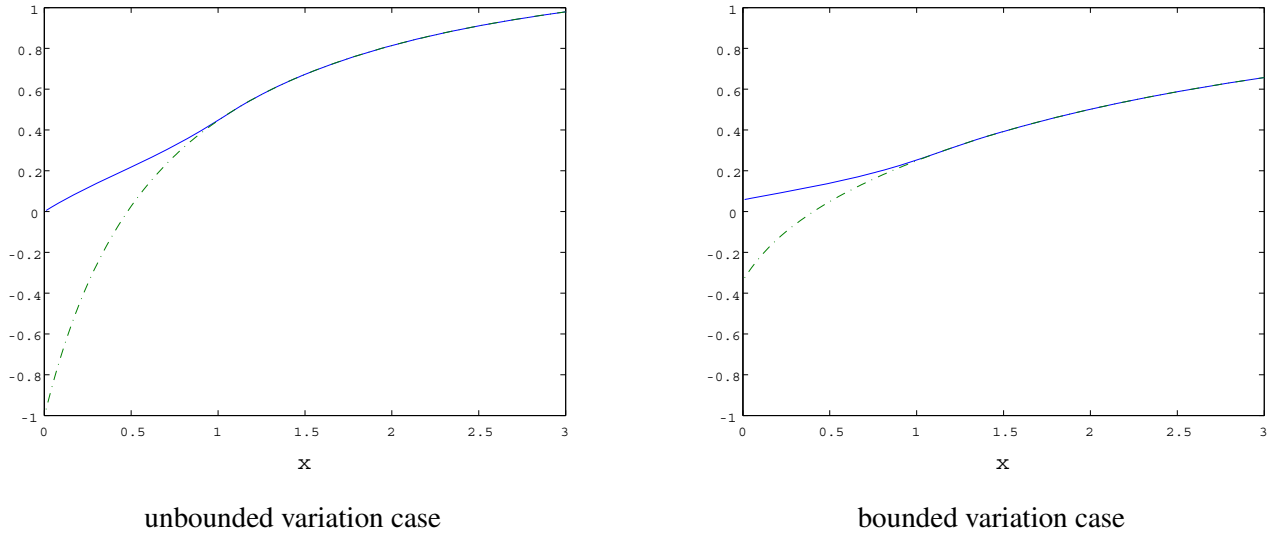


FIGURE 1. Illustrating the continuous and smooth fits of the value function $v_{B^*}(\cdot)$ (solid curve) and stopping value $h(\cdot)$ (dashed curve). Here $v_{B^*}(0+) = 0$ for the unbounded variation case while $v_{B^*}(0+) > 0$ for the bounded variation case.

In order to verify the optimality of v_{B^*} , we shall show that (i) v_{B^*} dominates h and (ii) the stochastic process

$$(3.15) \quad M_t := e^{-r(t \wedge \theta)} v_{B^*}(X_{t \wedge \theta}), \quad t \geq 0$$

is a supermartingale.

We address the first part as follows. When $B^* = 0$, it is clear that $v_{B^*}(x) = h(x)$ from (2.21) or the arguments above. On the other hand, when the smooth fit condition is met, i.e. $B^* \in (0, \infty)$, $\varrho(B)$ is monotonically increasing and attains 0 at B^* . Therefore, when $B^* > x$, $v_B(x)$ is increasing in B for $B \in [x, B^*]$ by (3.13). Then, by continuous fit in Remark 3.2, for any arbitrarily fixed x , taking $B = x$, we have $v_{B^*}(x) \geq v_x(x) = h(x)$. Moreover, $v_{B^*}(x) = h(x)$ on $x \leq B^*$ by definition. Consequently, (i) holds.

Lemma 3.3. *We have $v_{B^*}(x) \geq h(x)$ for every $x \in \mathbb{R}$.*

We now pursue the supermartingale property of the process M in (3.15). Define generator \mathcal{L} by

$$\mathcal{L}f(x) = cf'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_0^\infty [f(x-z) - f(x) + f'(x)z1_{\{0 < z < 1\}}] \Pi(dz)$$

for the unbounded variation case and

$$\mathcal{L}f(x) = \mu f'(x) + \int_0^\infty [f(x-z) - f(x)] \Pi(dz)$$

for the bounded variation case. We shall show that $(\mathcal{L} - r)v_{B^*}(x) \leq 0$ for every $x \in (0, \infty) \setminus \{B^*\}$, again by way of the scale functions.

Lemma 3.4. *Fix $0 < x \leq B < \infty$. The stochastic processes*

$$\left\{ e^{-r(t \wedge \tau_B^+)} W^{(r)}(X_{t \wedge \tau_B^+}); t \geq 0 \right\} \quad \text{and} \quad \left\{ e^{-r(t \wedge \tau_B^+)} Z^{(r)}(X_{t \wedge \tau_B^+}); t \geq 0 \right\}$$

are \mathbb{P}^x -martingales.

The following is immediate by the previous lemma and the definition of v_{B^*} in (3.14).

Lemma 3.5. *If $0 < B^* < \infty$, then $(\mathcal{L} - r)v_{B^*}(x) = 0$ for every $x \in (0, B^*)$.*

Now we establish via the following lemmas that $(\mathcal{L} - r)v_{B^*}(x) \leq 0$ for every $x > B^*$.

Lemma 3.6. *If $B^* = 0$, then we have $(\mathcal{L} - r)v_{B^*}(0+) = (\mathcal{L} - r)h(0+) \leq 0$.*

Lemma 3.7. *If $0 < B^* < \infty$, then we have $\Delta''(B^*-) \geq 0$.*

Suppose $0 < B^* < \infty$. Since $v_{B^*}(\cdot)$ and $v'_{B^*}(\cdot)$ are continuous at B^* by the continuous and smooth fit conditions, it follows from Lemma 3.7 that $(\mathcal{L} - r)v_{B^*}(B^*-) \geq (\mathcal{L} - r)v_{B^*}(B^*+)$, and then by Lemma 3.5 that $(\mathcal{L} - r)v_{B^*}(B^*+) \leq 0$. Suppose $B^* = 0$, we have $(\mathcal{L} - r)v_{B^*}(B^*+) \leq 0$ by Lemma 3.6. Now, in order to show $(\mathcal{L} - r)v_{B^*}(x) \leq 0$ on (B^*, ∞) , it is sufficient to prove that $(\mathcal{L} - r)v_{B^*}(x)$ is decreasing on (B^*, ∞) . Rewrite v_{B^*} as in the following:

$$v_{B^*}(x) = \left(\tilde{h}(x) + \tilde{\Delta}(x) \right) 1_{\{x \neq 0\}}, \quad x \in \mathbb{R}$$

where

$$(3.16) \quad \begin{aligned} \tilde{h}(x) &= - \left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) \left(Z^{(r)}(x) - \frac{r}{\Phi(r)} W^{(r)}(x) \right), \quad x \in \mathbb{R}, \\ \tilde{\Delta}(x) &= \begin{cases} \left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) Z^{(r)}(x) - \frac{W^{(r)}(x)}{W^{(r)}(B^*)} G^{(r)}(B^*), & x \in (-\infty, B^*), \\ \frac{\tilde{p}}{r} - \gamma, & x \in [B^*, \infty). \end{cases} \end{aligned}$$

Lemma 3.4 implies $(\mathcal{L} - r)\tilde{h}(x) = 0$ for every $x > 0$. Furthermore, $\tilde{\Delta}'(x) = \tilde{\Delta}''(x) = 0$ on $x > B^*$, and hence

$$(\mathcal{L} - r)v_{B^*}(x) = (\mathcal{L} - r)\tilde{\Delta}(x) = \int_0^\infty \left(\tilde{\Delta}(x - z) - \tilde{\Delta}(x) \right) \Pi(dz) - (\tilde{p} - r\gamma), \quad x > B^*.$$

In order to show that this is decreasing in x on (B^*, ∞) , it is sufficient to show that the integrand in the right-hand side is decreasing in x or equivalently $\tilde{\Delta}(x - z)$ is decreasing in x for every fixed z by noting that $\tilde{\Delta}(x)$ is a constant on (B^*, ∞) .

Lemma 3.8. *The function $\tilde{\Delta}(\cdot)$ is decreasing on \mathbb{R} and is uniformly bounded below by $\tilde{p}/r - \gamma > 0$.*

Therefore, we have $(\mathcal{L} - r)v_{B^*}(x) \leq 0$ for every $x > B^*$, with $0 \leq B^* < \infty$. Along with Lemma 3.5, we conclude that

Lemma 3.9. *The function $v_{B^*}(\cdot)$ satisfies*

$$(\mathcal{L} - r)v_{B^*}(x) \leq 0, \quad x \in (0, \infty) \setminus \{B^*\}.$$

Lemmas 3.3 and 3.9 prove the optimality. Due to the potential discontinuity of the value function at zero, care must be taken in the proof. The rest of the proof is given in the appendix.

3.3. Putable Step-Down CDS. We now turn our attention to the putable step-down CDS, which amounts to solving $u(x)$ in (2.15) with $\tilde{p} > 0$ and $\tilde{\alpha} > 0$. Under the spectrally negative Lévy model, the function $\Gamma(\cdot; A)$ in (2.24) can be expressed explicitly by scale functions as shown in Lemma 3.10 below. This together with (3.4) expresses explicitly $\Delta_A(\cdot)$ in (2.23), which is needed to apply continuous and smooth fit principle.

Define

$$(3.17) \quad \rho(A) := \int_A^\infty \Pi(du) \left(1 - e^{-\Phi(r)(u-A)}\right), \quad A > 0.$$

Here $\rho(A)$ decreases monotonically in A because

$$\rho'(A) = - \int_A^\infty \Pi(du) \Phi(r) e^{-\Phi(r)(u-A)} < 0.$$

Next, we express $\Gamma(\cdot; A)$ using $\rho(A)$ and the scale function.

Lemma 3.10. *Fix $A > 0$, we have*

$$\Gamma(x; A) = \frac{1}{\Phi(r)} W^{(r)}(x - A) \rho(A) - \frac{1}{r} \int_A^\infty \Pi(du) \left(Z^{(r)}(x - A) - Z^{(r)}(x - u) \right), \quad \text{for } A \leq x,$$

and $\Gamma(x; A) = 0$ for $x < A$.

We proceed to consider the continuous fit condition. First, it follows from (2.23) that, for every $A > 0$,

$$(3.18) \quad \begin{aligned} \Delta_A(A+) &= \left(\gamma + \frac{\tilde{p}}{r} \right) \left(\frac{r}{\Phi(r)} W^{(r)}(0) \right) - (\tilde{\alpha} - \gamma) \left(\frac{1}{\Phi(r)} W^{(r)}(0) \rho(A) \right) \\ &= \frac{W^{(r)}(0)}{\Phi(r)} \left((r\gamma + \tilde{p}) - (\tilde{\alpha} - \gamma) \rho(A) \right). \end{aligned}$$

If X is of unbounded variation, then $W^{(r)}(0) = 0$ by Lemma 3.1, and therefore continuous fit holds for every $A > 0$. Nevertheless, for the bounded variation case, we can apply the **continuous fit condition**: $\Delta_A(A+) = 0$, which is equivalent to

$$(3.19) \quad (\tilde{\alpha} - \gamma)\rho(A) = (\gamma r + \tilde{p}).$$

For the unbounded variation case, we apply the smooth fit condition. By differentiation, we have

$$\frac{\partial}{\partial x} \zeta(x - A) \Big|_{x=A+} = -\frac{r}{\Phi(r)} W^{(r)'}(0+) \quad \text{and} \quad \frac{\partial}{\partial x} \Gamma(x; A) \Big|_{x=A+} = \frac{\rho(A)}{\Phi(r)} W^{(r)'}(0+),$$

where $W^{(r)'}(0+) > 0$ (in particular, $W^{(r)'}(0+) = \infty$ if $\sigma = 0$ and $\Pi(0, \infty) = \infty$ by Lemma 3.1). Therefore,

$$\Delta'_A(A+) = \frac{W^{(r)'}(0+)}{\Phi(r)} ((\gamma r + \tilde{p}) - (\tilde{\alpha} - \gamma)\rho(A)).$$

Consequently, the **smooth fit condition**, $\Delta'_A(A+) = 0$, is equivalent to (3.19).

In summary, we look for the solution to (3.19), denoted by A^* , which will be our candidate optimal threshold. Since $\rho(A)$ is monotonically decreasing, there exists *at most one* A^* that satisfies (3.19). If it does not exist, we set the threshold $A^* = 0$.

When X has paths of bounded variation, we must have $A^* > 0$ under assumption (2.19). Indeed, if $A^* = 0$, then it follows from (3.18) that $\Delta_A(A+) > 0$ for every $A > 0$. This implies that there exists $\varepsilon > 0$ such that $u_\varepsilon(\varepsilon+) > g(0+)$. However, since $g(0+)$ attains the global maximum (because $g(0+) > 0$ by (2.19)), this is a contradiction. Hence $A^* = 0$ is impossible when X is of unbounded variation.

In the case with $A^* > 0$, we take A^* to be our candidate optimal threshold, and the corresponding stopping time is $\tau_{A^*}^-$. The candidate value function is given by

$$u_{A^*}(x) = \begin{cases} g(x) + \Delta_{A^*}(x), & x > A^*, \\ g(x), & 0 < x \leq A^*, \\ 0, & x \leq 0. \end{cases}$$

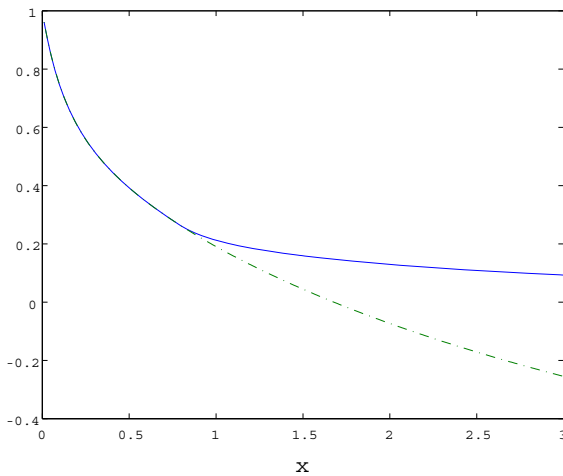
For $x > 0$, we can apply (3.19) to express it as

$$(3.20) \quad u_{A^*}(x) = (\tilde{\alpha} - \gamma) \left(\frac{1}{r} \int_{A^*}^{\infty} \Pi(du) \left[Z^{(r)}(x - A^*) - Z^{(r)}(x - u) \right] \right) - \left(\gamma + \frac{\tilde{p}}{r} \right) Z^{(r)}(x - A^*) + \left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) \zeta(x).$$

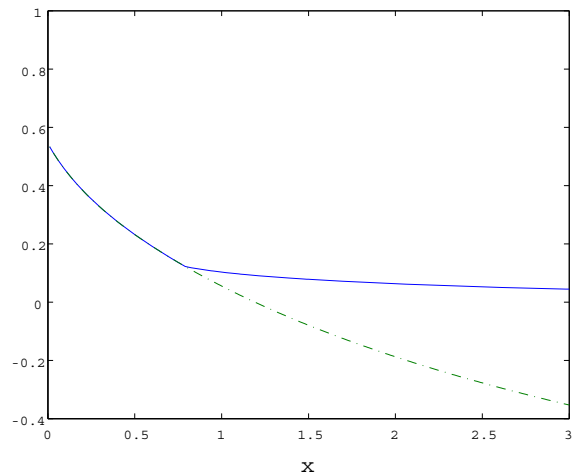
When $A^* = 0$, we consider the candidate value function defined by

$$\begin{aligned}
 (3.21) \quad u_{A^*}(x) &:= \lim_{A \downarrow 0} u_A(x) \\
 &= -(\tilde{\alpha} - \gamma)\Gamma(x; A^*) - \left(\gamma + \frac{\tilde{p}}{r}\right)\zeta(x - A^*) + \left(\frac{\tilde{p}}{r} + \tilde{\alpha}\right)\zeta(x) \\
 &= (\tilde{\alpha} - \gamma)(\zeta(x) - \Gamma(x; 0))
 \end{aligned}$$

for all $x > 0$ and $u_{A^*}(x) = 0$ for every $x \leq 0$. This means that the seller will delay the exercise until X is arbitrarily close to zero, by exercising at a sufficiently small level $\varepsilon > 0$. This can be realized by monitoring X as it *creeps downward* through zero (see Section 5.3 of [28]). The seller may lose the opportunity to exercise prior to default if X suddenly jumps across (below) zero.



unbounded variation case



bounded variation case

FIGURE 2. Illustrating the continuous and smooth fits of the value function $u_{A^*}(\cdot)$ (solid curve) and stopping value $g(\cdot)$ (dashed curve). Here $u_{A^*}(\cdot)$ is C^1 at A^* for the unbounded variation case while it is C^0 for the bounded variation case.

Theorem 3.2. *The candidate function $u_{A^*}(x)$ is indeed optimal. That is,*

$$u_{A^*}(x) = u(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-r\tau} g(X_\tau) 1_{\{\tau < \infty\}} \right].$$

Similarly to the previous subsection, we shall show (i) $u_{A^*} \geq g$ and (ii) the supermartingale property of the value function. We shall first show the former.

Lemma 3.11. *For every $0 < A < x$, we have*

$$\frac{\partial}{\partial A} u_A(x) = \frac{\partial}{\partial A} \Delta_A(x) < 0 \iff (\tilde{\alpha} - \gamma)\rho(A) - (\gamma r + \tilde{p}) < 0 \iff A > A^*.$$

Suppose $A^* > 0$. For the unbounded variation case, we can apply Lemma 3.11 for any fixed $x \geq A^*$, along with the continuous fit (which holds at any A), to get the inequality

$$u_{A^*}(x) \geq u_x(x) = g(x), \quad x \geq A^*.$$

When X is of bounded variation, we have, by (3.18), $\Delta_x(x+) \geq 0$ for every $x \geq A^*$ and hence

$$(3.22) \quad u_{A^*}(x) \geq \lim_{A \uparrow x} u_A(x) = g(x) + \lim_{A \uparrow x} \Delta_A(x) \geq g(x), \quad x \geq A^*.$$

This also holds for the case $A^* = 0$ by (3.21). Therefore, because $u_{A^*}(x) = g(x)$ on $(-\infty, A^*)$ by definition, we have the following.

Lemma 3.12. *We have $u_{A^*}(x) \geq g(x)$ for every $x \in \mathbb{R}$.*

By taking advantage of Lemma 3.4 and how A^* is chosen, we can show the following.

Lemma 3.13. *We have $(\mathcal{L} - r)u_{A^*}(x) \leq 0$ for every $x \in (0, \infty) \setminus \{A^*\}$.*

Finally, Lemmas 3.12 and 3.13 show Theorem 3.2. The proof that $u_{A^*} \geq u$ can be carried out as in the proof of Theorem 3.1. We also have $u_{A^*} \leq u$. Indeed, u_{A^*} is attained by $\tau_{A^*}^-$ when $A^* > 0$ and can be approximated arbitrarily closely by u_ε (attained by τ_ε^-) for sufficiently small ε when $A^* = 0$.

4. NUMERICAL EXAMPLES

In this section, we illustrate the investor's optimal exercise strategy and the credit spread behaviors through a series of numerical examples, where the underlying spectrally negative Lévy process is assumed to have *hyperexponential jumps*. The class of hyperexponential distributions is dense in the class of all positive-valued distributions with completely monotone densities (see, e.g., [17, 19]). In a related work, Asmussen et al. [3] approximate the Lévy density of the CGMY process by a hyperexponential distribution.

4.1. Spectrally negative Lévy processes with hyperexponential jumps. Let X be a spectrally negative Lévy process of the form

$$(4.1) \quad X_t - X_0 = \mu t + \sigma B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \leq t < \infty.$$

Here $B = \{B_t; t \geq 0\}$ is a standard Brownian motion, $N = \{N_t; t \geq 0\}$ is a Poisson process with arrival rate λ , and $Z = \{Z_n; n = 1, 2, \dots\}$ is an i.i.d. sequence of hyperexponential distributed random variables with density function

$$f(z) = \sum_{i=1}^m \alpha_i \eta_i e^{-\eta_i z}, \quad z > 0,$$

for some $0 < \eta_1 < \dots < \eta_m < \infty$. Its Laplace exponent (3.1) is then

$$(4.2) \quad \psi(s) = \mu s + \frac{1}{2} \sigma^2 s^2 - \lambda \sum_{i=1}^m \alpha_i \frac{s}{\eta_i + s}.$$

Notice in this case that $\{-\eta_i; i = 1, \dots, m\}$ are the poles of the Laplace exponent ψ . Here we assume $\sigma > 0$; for the case with $\sigma = 0$, see [17].

In this case, there are $m + 1$ *negative* solutions to the equation $\psi(s) = r$ and their absolute values $\{\xi_{i,r}; i = 1, \dots, m + 1\}$ satisfy the interlacing condition:

$$0 < \xi_{1,r} < \eta_1 < \xi_{2,r} < \dots < \eta_m < \xi_{m+1,r} < \infty.$$

The scale function becomes

$$(4.3) \quad \begin{aligned} W^{(r)}(x) &= \frac{2}{\sigma^2 \sum_{i=1}^{m+1} A_{i,r} \xi_{i,r}} \sum_{i=1}^{m+1} A_{i,r} \left(\frac{\xi_{i,r}}{\Phi(r) + \xi_{i,r}} \right) \left[e^{\Phi(r)x} - e^{-\xi_{i,r}x} \right], \\ Z^{(r)}(x) &= 1 + \frac{2r}{\sigma^2 \sum_{i=1}^{m+1} A_{i,r} \xi_{i,r}} \sum_{i=1}^{m+1} A_{i,r} \left(\frac{\xi_{i,r}}{\Phi(r) + \xi_{i,r}} \right) \left[\frac{1}{\Phi(r)} \left(e^{\Phi(r)x} - 1 \right) + \frac{1}{\xi_{i,r}} \left(e^{-\xi_{i,r}x} - 1 \right) \right] \end{aligned}$$

for every $x \geq 0$ where

$$A_{k,r} := \frac{\prod_{j \in \{1, \dots, m\}} \left(1 - \frac{\xi_{k,r}}{\eta_j} \right)}{\prod_{i \in \{1, \dots, m\} \setminus \{k\}} \left(1 - \frac{\xi_{k,r}}{\xi_{i,r}} \right)}, \quad 1 \leq k \leq m + 1.$$

Note in this case that

$$W_{\Phi(r)}(x) = \frac{2}{\sigma^2 \sum_{i=1}^{m+1} A_{i,r} \xi_{i,r}} \sum_{i=1}^{m+1} A_{i,r} \left(\frac{\xi_{i,r}}{\Phi(r) + \xi_{i,r}} \right) \left[1 - e^{-(\Phi(r) + \xi_{i,r})x} \right], \quad x \geq 0,$$

which is concave in x . Hence, Assumption 3.1 is satisfied. Furthermore, (3.17) becomes

$$\rho(A) = \lambda \sum_{i=1}^m \alpha_i \frac{\Phi(r)}{\eta_i + \Phi(r)} e^{-\eta_i A}, \quad A > 0.$$

4.2. Hyperexponential Fitting. If a density function $f(\cdot)$ is completely monotone, then it can be approximated arbitrarily closely by those of hyperexponential distributions. Consequently, as discussed in [17], the scale function of any spectrally negative Lévy process with a completely monotone Lévy measure can be approximated in terms of the scale functions defined in (4.3). Here we give an example using the results obtained by Feldmann and Whitt [19] for the case Z in (4.1) is Pareto distributed.

The Pareto distribution with positive parameters a and b is given by

$$F(t) = 1 - (1 + bt)^{-a}, \quad t \geq 0.$$

These have long-tails, namely $e^{\delta t}(1 - F(t)) \rightarrow \infty$ as $t \rightarrow \infty$ for any $\delta > 0$; see [24] for more details. Feldmann and Whitt [19] introduced a recursive algorithm to fit hyperexponential distributions and specifically computed for Pareto and Weibull distributions. We adopt their procedure for the process in the form (4.1) with Z replaced by Pareto random variables with $a = 1.2$ and $b = 5$. Table 3 shows the fitted data. The corresponding scale functions can be approximated by those in the form (4.3). For the convergence results, we refer to the author's previous work [17].

i	α_i	η_i	i	α_i	η_i
1	8.37E-11	8.3E-09	8	0.000147	0.0020
2	7.18E-10	6.8E-08	9	0.001122	0.0100
3	5.56E-09	3.9E-07	10	0.008462	0.0570
4	4.27E-08	2.2E-06	11	0.059768	0.3060
5	3.27E-07	1.2E-05	12	0.307218	1.5460
6	2.50E-06	6.5E-05	13	0.533823	6.5160
7	1.92E-05	3.5E-04	14	0.089437	23.304

TABLE 3. Parameters of the hyperexponential distributions fitted to a Pareto distribution with $a = 1.2$ and $b = 5$ (taken from Table 9 of Feldmann and Whitt [19]).

4.3. Numerical Results. Let us denote the step-up/down ratio by $q := \hat{p}/p = \hat{\alpha}/\alpha$. For both callable and putable CDSs, we consider the following three cases:

- (C) Cancellable CDS with $q = 0$,
- (D) Step-Down CDS with $q = 0.5$,
- (U) Step-Up CDS with $q = 1.5$.

Hence, there are in total 6 cases. The model parameters are $r = 0.03$ and $\sigma = 0.2$, $\alpha = 1$, $x = 1.5$ and $\gamma = 50\text{bps}$, unless specified otherwise. We shall adjust the values of λ and μ so that the risk-neutral condition $\psi(1) = r$ is satisfied.

By the symmetry argument discussed in Section 2.4, the optimal stopping problems for callable-(D) CDS and putable-(U) CDS are equivalent. The callable-(U) and putable-(D) CDSs are also equivalent. Figure 3 shows the optimal thresholds B^* for callable-(C), callable-(D) and putable-(U) CDSs, and also the optimal threshold A^* for callable-(U), putable-(C) and putable-(D) CDSs. Both B^* and A^* are decreasing in the premium p . This is intuitive: for callable-(C/D) CDS, the higher premium (cost to the protection buyer) induces the buyer to exercise earlier at a lower level B^* to cancel or step-down the position; in contrast, for the putable-(C/D) CDS, receiving a higher premium motivates the protection seller to delay exercising to cancel or step-down by choosing a lower exercise threshold A^* . Also, both A^* and B^* increase as λ increases because it raises the chance of down-crossing the level zero. In particular, A^* becomes zero when $\lambda = 0$.

In Figure 4, we plot the CDS values V and U with respect to p (recall (2.8) and (2.12) for definitions). In all cases, V is decreasing while U is increasing in p . For case (C), the values of V and U go to $-\gamma$ when p is sufficiently high and when it is sufficient low, respectively, because immediate stopping is optimal. For cases (D) and (U), since the premium remains positive even after exercise, the value functions go to $-\infty$ (for callable) and ∞ (for putable) as p increases. We also observe that, as λ increases, V increases and U decreases. Using the functions V and U , the credit spread (or fair premiums) p^* is determined where $V(p^*) = 0$ for the callable case, and $U(p^*) = 0$ for the putable case.

Figures 5 and 6 show the credit spread p^* as a function of the distance-to-default x for the callable and putable CDSs, respectively, along with those for the vanilla CDSs (2.4). By the monotonicity of V and U in p , the premium value p^* must be unique, and it is computed by the bisection method here. As seen in the figures, as x is farther away from default level 0, meaning a lower chance of default, the credit spread p^* decreases. For the callable CDSs in Figure 5, the credit spreads are naturally higher than the vanilla case due to the embedded step-up/down option. In contrast, in the putable CDSs in Figure 6, the credit spreads are lower than the vanilla case because the buyer is subject to the step-up/down exercise by the seller.

Although the cases (C), (D) and (U) are very different in terms of contractual features and exercise strategies, we observe that the cancellation option (C) appears to have the strongest impact on the credit spread (relative to the vanilla case) among all cases. One intuition is that the cancellation option has the most significant one-time change to the premium and default payment. If multiple step-up/down options are given, however, then other interesting situations may arise. We will discuss future research in the next section.

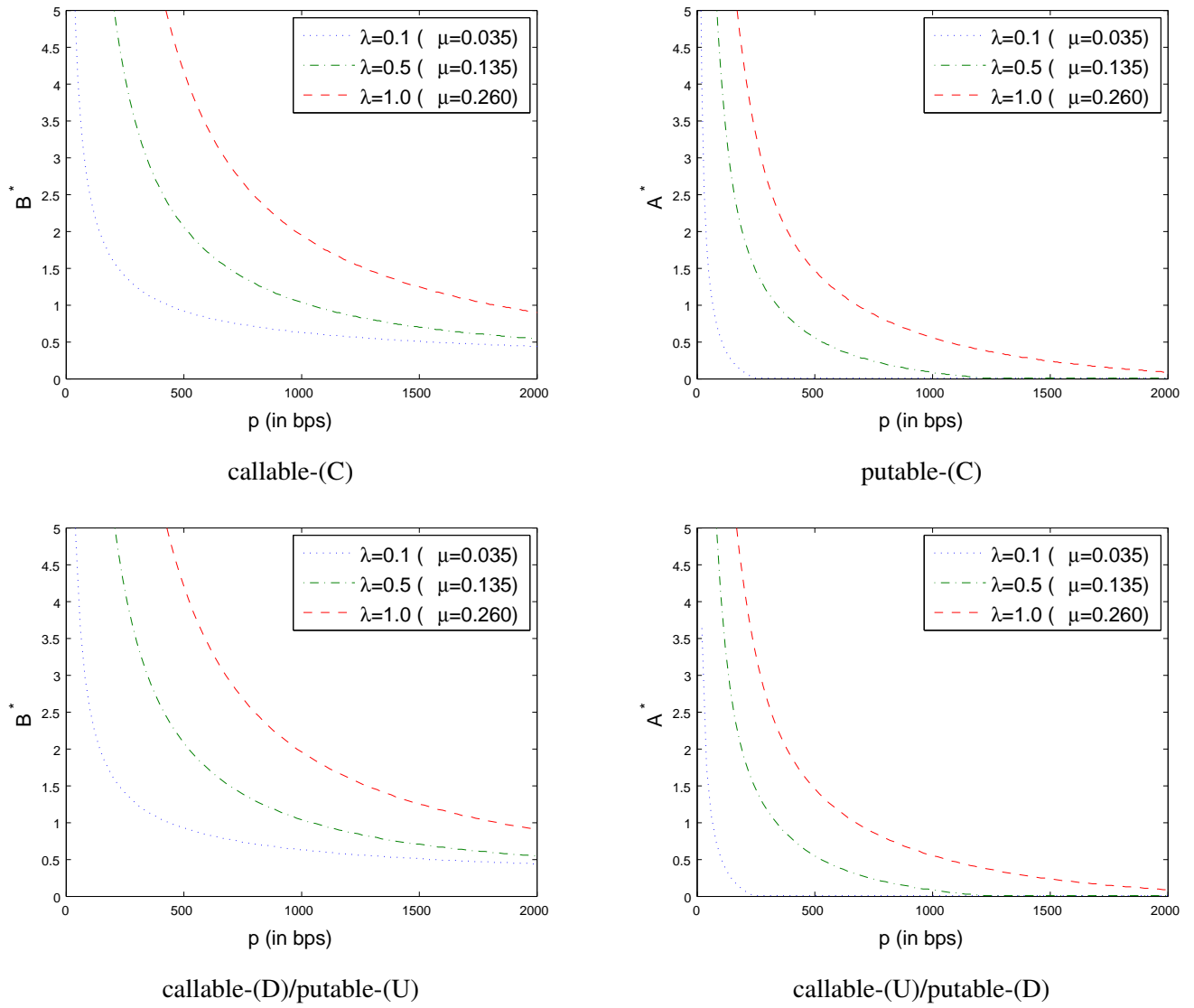


FIGURE 3. The optimal thresholds for all six CDS specifications.

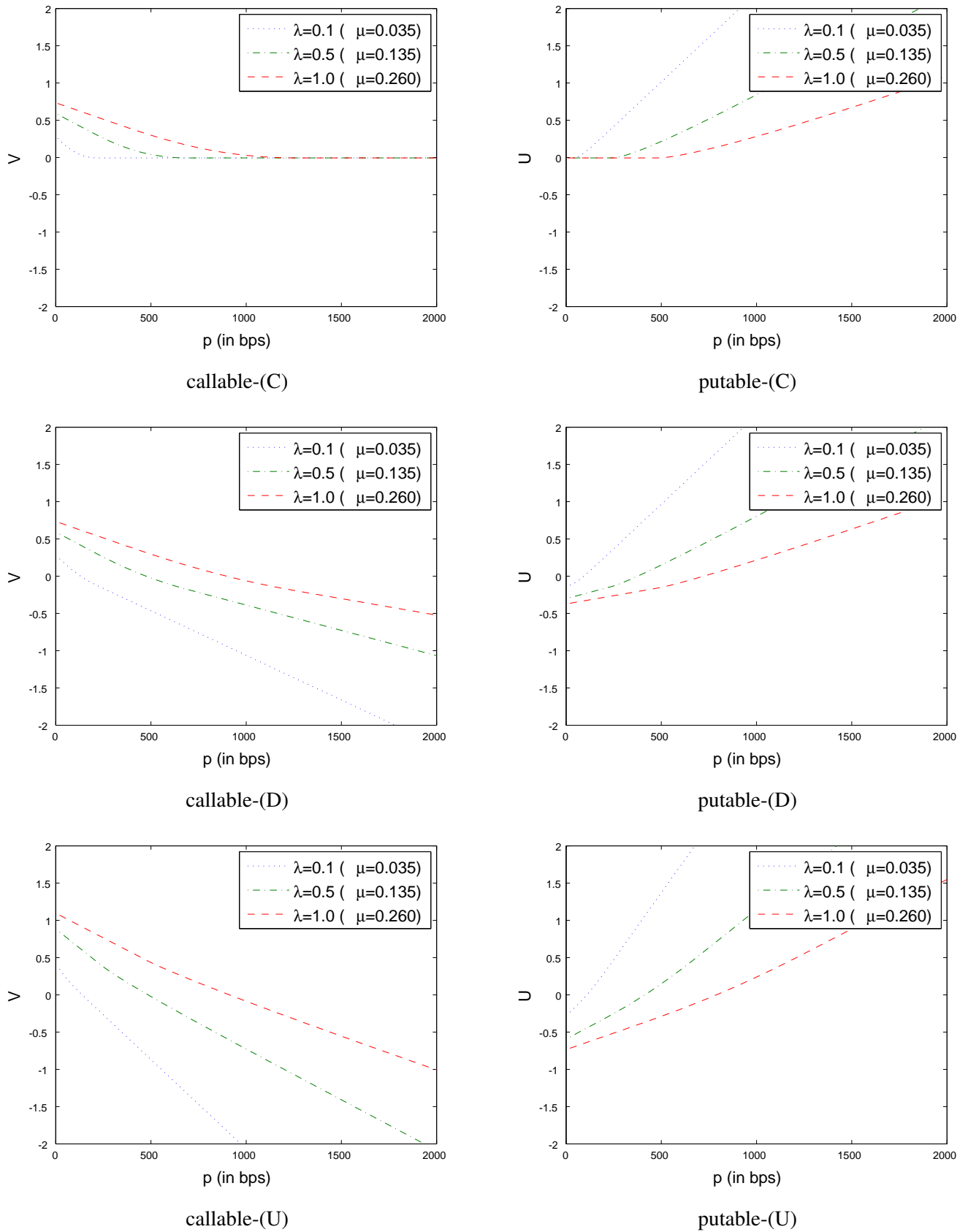
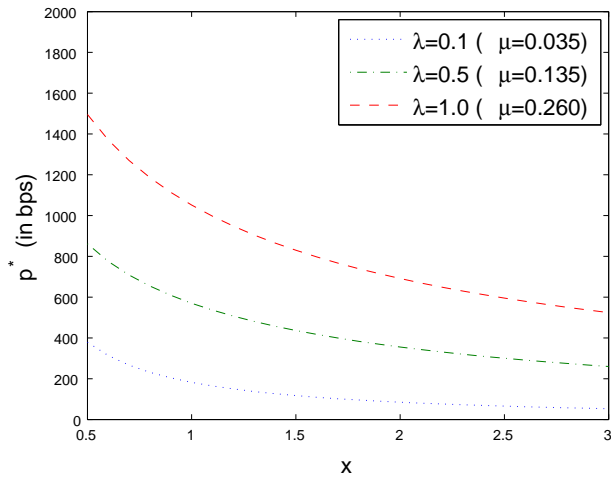
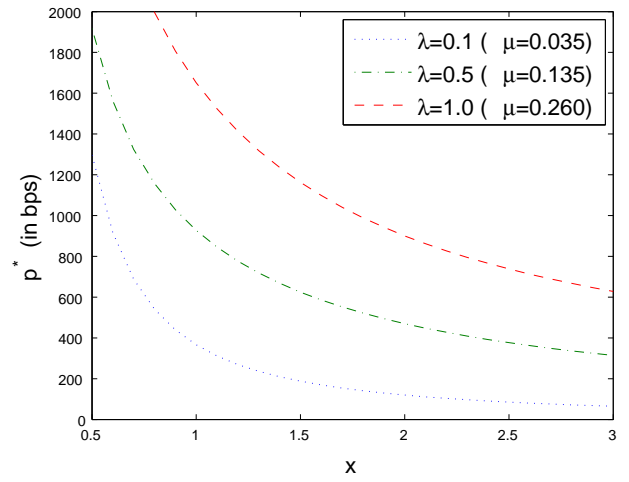


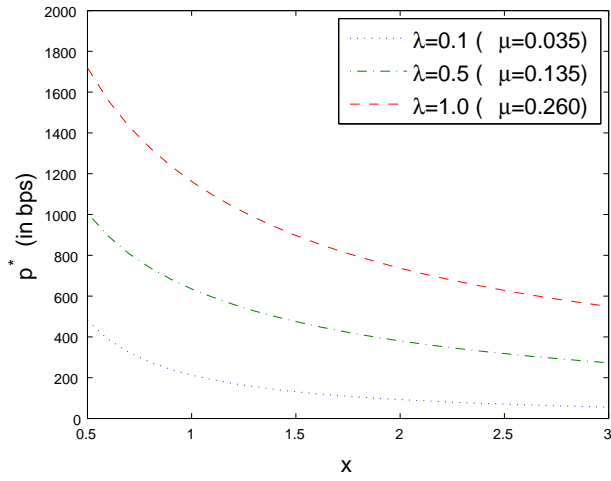
FIGURE 4. The CDS value as a function of p for all six CDS specifications.



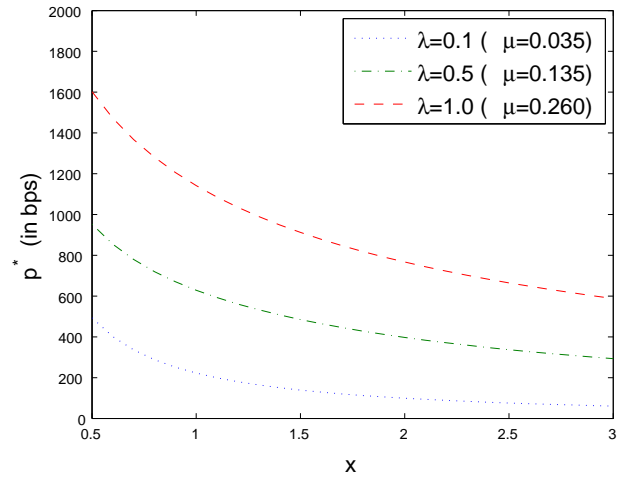
vanilla



callable-(C)

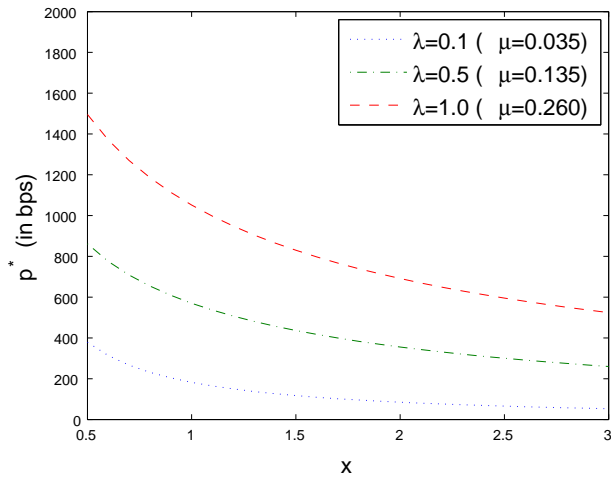


callable-(D)

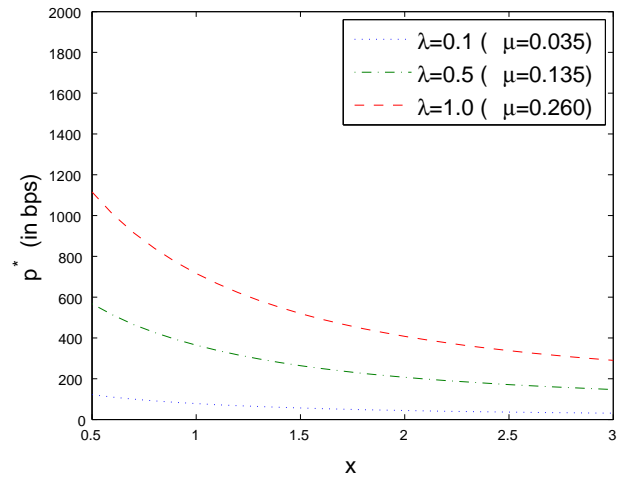


callable-(U)

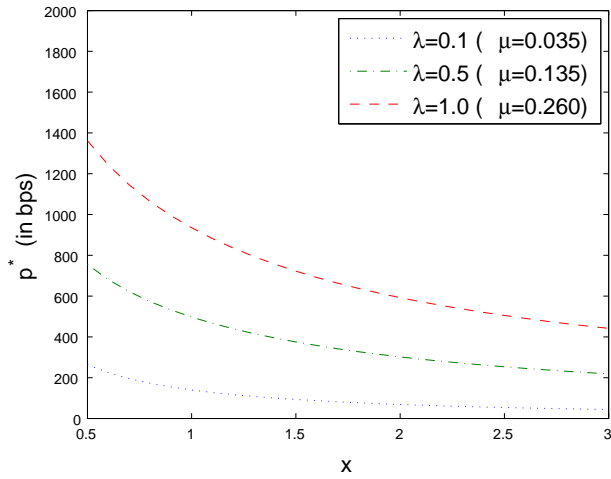
FIGURE 5. The credit spreads as a function of distance-to-default for the vanilla CDS (top-left), buyer cancellable (top-right), callable step-down (bottom-left) and callable step-up (bottom-right) CDSs.



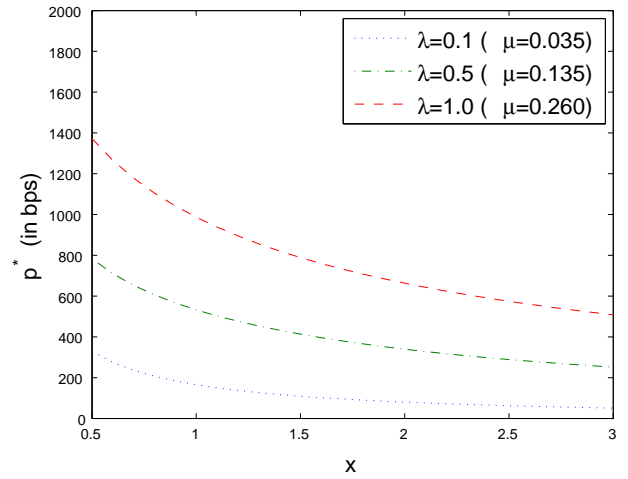
vanilla



puttable-(C)



puttable-(D)



puttable-(U)

FIGURE 6. The credit spreads as a function of distance-to-default for the vanilla CDS (top-left), seller cancellable (top-right), puttable step-down (bottom-left) and puttable step-up (bottom-right) CDSs.

5. CONCLUDING REMARKS

In summary, the incorporation of American step-up and step-down options give CDS investors the additional flexibility to manage and trade credit risks. The valuation of these contracts requires solving for the optimal timing to step-up/down for the protection buyer/seller. The perpetual nature of the contract allows us to compute analytically the investor's optimal exercise threshold under quite general Lévy credit risk models. Using the symmetry properties between step-up and step-down contracts, we gain better intuition on various contract specifications, and drastically simplify the procedure to determine the credit spreads.

There are a number of avenues for future research. For instance, it would be interesting to value a CDS where both the protection buyer and seller can terminate the contract early. Then, the valuation problem can be formulated as a modified game option as introduced by Kifer [26]. In this case, we conjecture that threshold strategies will again be optimal for both parties and constitute Nash or Stackelberg equilibrium [18, 37]. Another direction for future research is to consider derivatives with multiple early exercisable step-up/down options. This is related to some optimal multiple stopping problems arising in other financial applications, such as swing options [13] and employee stock options [31].

APPENDIX A. PROOFS

A.1. Proofs for Section 2.

Proof of Lemma 2.1. Applying the definitions of $\Delta_B(x)$ and $h(x)$ (see (2.11)) and noting that $\theta = \infty$ whenever $\tau_B^+ = \infty$, we obtain, for every $x \in (0, B)$,

$$\begin{aligned}
& \Delta_B(x) \\
&= \mathbb{E}^x \left[1_{\{\tau_B^+ < \infty\}} \left(\int_{\tau_B^+}^{\theta} e^{-rt} \tilde{p} dt - e^{-r\tau_B^+} \gamma 1_{\{\tau_B^+ < \theta\}} - e^{-r\theta} \tilde{\alpha} 1_{\{\tau_B^+ < \theta\}} \right) \right] - \mathbb{E}^x \left[\int_0^{\theta} e^{-rt} \tilde{p} dt - e^{-r\theta} \tilde{\alpha} \right] + \gamma \\
&= \mathbb{E}^x \left[1_{\{\tau_B^+ < \infty\}} \left(- \int_0^{\tau_B^+} e^{-rt} \tilde{p} dt - e^{-r\tau_B^+} \gamma 1_{\{\tau_B^+ < \theta\}} + e^{-r\theta} \tilde{\alpha} 1_{\{\tau_B^+ = \theta\}} \right) - 1_{\{\tau_B^+ = \infty\}} \left(\int_0^{\theta} e^{-rt} \tilde{p} dt - e^{-r\theta} \tilde{\alpha} \right) \right] + \gamma \\
&= \mathbb{E}^x \left[1_{\{\tau_B^+ < \infty\}} e^{-r\tau_B^+} \left(\tilde{\alpha} 1_{\{\tau_B^+ = \theta\}} - \gamma 1_{\{\tau_B^+ < \theta\}} \right) - \left(\int_0^{\tau_B^+} e^{-rt} \tilde{p} dt \right) \right] + \gamma \\
&= \mathbb{E}^x \left[1_{\{\tau_B^+ < \infty\}} e^{-r\tau_B^+} \left(\left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) 1_{\{\tau_B^+ = \theta\}} + \left(\frac{\tilde{p}}{r} - \gamma \right) 1_{\{\tau_B^+ < \theta\}} \right) \right] + \gamma - \frac{\tilde{p}}{r}.
\end{aligned}$$

□

Proof of Lemma 2.2. Using the same argument as in the proof of Lemma 2.1, we can write

$$\Delta_A(x) = \left(\frac{\tilde{p}}{r} + \gamma \right) - \mathbb{E}^x \left[e^{-r\tau_A^-} \left(\left(\tilde{\alpha} + \frac{\tilde{p}}{r} \right) 1_{\{\tau_A^- = \theta, \tau_A^- < \infty\}} + \left(\gamma + \frac{\tilde{p}}{r} \right) 1_{\{\tau_A^- < \theta, \tau_A^- < \infty\}} \right) \right].$$

Therefore, we have

$$\begin{aligned} \Delta_A(x) &= \left(\frac{\tilde{p}}{r} + \gamma \right) - \mathbb{E}^x \left[e^{-r\tau_A^-} \left(\left(\gamma + \frac{\tilde{p}}{r} \right) 1_{\{\tau_A^- < \infty\}} + (\tilde{\alpha} - \gamma) 1_{\{\tau_A^- = \theta, \tau_A^- < \infty\}} \right) \right] \\ &= \left(\gamma + \frac{\tilde{p}}{r} \right) (1 - \zeta(x - A)) - (\tilde{\alpha} - \gamma) \Gamma(x; A). \end{aligned}$$

□

A.2. Proofs for Subsection 3.2.

Proof of Lemma 3.2. When $\sigma > 0$ or $\Pi(0, \infty) = \infty$, we have $\varrho(0+) = -\infty$ because $W^{(r)'}(0+) > 0$ and $W^{(r)}(0) = 0$ by Lemma 3.1. This implies that both (1) and (2) are necessary conditions for $B^* = 0$. Now suppose (1) and (2). Again by Lemma 3.1 we have after some algebra

$$\varrho(0+) = \frac{1}{\mu} (\tilde{p} - r\gamma - (\tilde{\alpha} + \gamma)\Pi(0, \infty)).$$

Therefore, condition (3) implies that $\varrho(\cdot) \geq 0$ (i.e. scenario (a)) since $\varrho(\cdot)$ is increasing. The proof is complete. □

Proof of Lemma 3.4. By (3.3) and strong Markov property, we write

$$\mathbb{E}^x \left[e^{-r\tau_B^+} 1_{\{\tau_B^+ < \theta\}} \middle| \mathcal{F}_{t \wedge \tau_B^+} \right] = e^{-r(t \wedge \tau_B^+)} \frac{W^{(r)}(X_{t \wedge \tau_B^+})}{W^{(r)}(B)}, \quad t \geq 0.$$

Therefore, by taking expectation on both sides, we obtain

$$\mathbb{E}^x \left[e^{-r(t \wedge \tau_B^+)} \frac{W^{(r)}(X_{t \wedge \tau_B^+})}{W^{(r)}(B)} \right] = \mathbb{E}^x \left[e^{-r\tau_B^+} 1_{\{\tau_B^+ < \theta\}} \right] = \frac{W^{(r)}(x)}{W^{(r)}(B)}, \quad t \geq 0.$$

Hence

$$\mathbb{E}^x \left[e^{-r(t \wedge \tau_B^+)} W^{(r)}(X_{t \wedge \tau_B^+}) \right] = W^{(r)}(x), \quad t \geq 0.$$

Furthermore, we have the bound $0 \leq W^{(r)}(X_{t \wedge \tau_B^+}) \leq W^{(r)}(B)$. This shows the first claim.

Similarly, (3.3) and strong Markov property yield that

$$\mathbb{E}^x \left[e^{-r\tau_B^+} 1_{\{\tau_B^+ = \theta\}} \middle| \mathcal{F}_{t \wedge \tau_B^+} \right] = e^{-r(t \wedge \tau_B^+)} \left(Z^{(r)}(X_{t \wedge \tau_B^+}) - \frac{r}{\Phi(r)} W^{(r)}(X_{t \wedge \tau_B^+}) \right), \quad t \geq 0.$$

By taking expectation on both sides, we get

$$\mathbb{E}^x \left[e^{-r(t \wedge \tau_B^+)} \left(Z^{(r)}(X_{t \wedge \tau_B^+}) - \frac{r}{\Phi(r)} W^{(r)}(X_{t \wedge \tau_B^+}) \right) \right] = \mathbb{E}^x \left[e^{-r\tau_B^+} 1_{\{\tau_B^+ = \theta\}} \right] = Z^{(r)}(x) - \frac{r}{\Phi(r)} W^{(r)}(x).$$

With this and the bound $1 \leq Z^{(r)}(X_{t \wedge \tau_B^+}) \leq Z^{(r)}(B)$, we conclude the second claim. □

Proof of Lemma 3.6. By using the martingale property of Lemma 3.4, we have

$$(\mathcal{L} - r)h(0+) = (\mathcal{L} - r)L(0+),$$

where

$$L(x) := \left(\frac{\tilde{p}}{r} - \gamma \right) + (\tilde{\alpha} + \gamma) 1_{\{x \leq 0\}}, \quad x \in \mathbb{R}.$$

Now, with $L'(0+) = 0$ and $\sigma = 0$ and $\Pi(0, \infty) < \infty$ by Lemma 3.2, we obtain

$$(\mathcal{L} - r)L(0+) = \int_0^\infty (L(0 - z) - L(0)) \Pi(dz) - r \left(\frac{\tilde{p}}{r} - \gamma \right) = (\tilde{\alpha} + \gamma)\Pi(0, \infty) - (\tilde{p} - \gamma r).$$

Hence, $(\mathcal{L} - r)h(0+) \leq 0$ by Lemma 3.2. \square

Proof of Lemma 3.7. Notice that

$$\Delta''(B^* -) = (r\tilde{\alpha} + \tilde{p}) W^{(r)'}(B^*) - \frac{W^{(r)''}(B^*)}{W^{(r)}(B^*)} G^{(r)}(B^*).$$

By (3.12), we have

$$(A.1) \quad G^{(r)}(B^*) = \frac{(W^{(r)}(B^*))^2}{W^{(r)'}(B^*)} (\tilde{p} + \tilde{\alpha}r).$$

Direction substitution yields that

$$\begin{aligned} \Delta''(B^* -) &= (r\tilde{\alpha} + \tilde{p}) \frac{1}{W^{(r)'}(B^*)} \left[\left(W^{(r)'}(B^*) \right)^2 - W^{(r)''}(B^*) W^{(r)}(B^*) \right] \\ &= - (r\tilde{\alpha} + \tilde{p}) \frac{(W^{(r)}(B^*))^2}{W^{(r)'}(B^*)} \frac{\partial}{\partial B} \frac{W^{(r)'}(B)}{W^{(r)}(B)}, \end{aligned}$$

which is non-negative by Assumption 3.1, as desired. \square

Proof of Lemma 3.8. It is clear that $\tilde{\Delta}(\cdot)$ in (3.16) is monotonically decreasing when $B^* = 0$ because $\tilde{\Delta}(0-) = \frac{\tilde{p}}{r} + \tilde{\alpha} > \frac{\tilde{p}}{r} - \gamma = \tilde{\Delta}(0+)$. Suppose $0 < B^* < \infty$. By differentiating $\tilde{\Delta}(\cdot)$, we get

$$\begin{aligned} \tilde{\Delta}'(x) &= (\tilde{p} + \tilde{\alpha}r) W^{(r)}(x) - \frac{W^{(r)'}(x)}{W^{(r)}(B^*)} G^{(r)}(B^*) \\ &= W^{(r)'}(x) \left\{ (r\tilde{\alpha} + \tilde{p}) \left[\frac{W^{(r)}(x)}{W^{(r)'}(x)} - \frac{W^{(r)}(B^*)}{W^{(r)'}(B^*)} \right] \right\}, \quad 0 < x < B^*, \end{aligned}$$

where the last equation follows from (A.1).

Now, $\tilde{\Delta}'(\cdot)$ is non-positive because Assumption 3.1 implies that

$$\frac{W^{(r)'}(x)}{W^{(r)}(x)} \geq \frac{W^{(r)'}(B^*)}{W^{(r)}(B^*)} \iff \frac{W^{(r)}(x)}{W^{(r)'}(x)} \leq \frac{W^{(r)}(B^*)}{W^{(r)'}(B^*)}, \quad x \leq B^*.$$

Furthermore, we note that

$$\tilde{\Delta}(0+) - \tilde{\Delta}(0-) = -\frac{W^{(r)}(0)}{W^{(r)}(B^*)}G^{(r)}(B^*) \leq 0.$$

Finally, it follows from continuous fit that $\tilde{\Delta}(B^*-) = \tilde{\Delta}(B^*+) = \frac{\tilde{p}}{r} - \gamma > 0$ (see (2.18)). This completes the proof because $\tilde{\Delta}$ is constant on $(-\infty, 0) \cup (B^*, \infty)$. \square

Proof of Theorem 3.1. Due to the potential discontinuity of the value function at zero, we need to proceed carefully. Before showing the main result, we first prove that $0 = v_{B^*}(0) \leq v_{B^*}(0+)$ as illustrated in Figure 1.

Suppose $0 < B^* < \infty$. Both $h(x)$ and $v_{B^*}(x)$ are both increasing in x (see (2.16) and (3.14)). Since $W^{(r)}$ is increasing and non-negative, we must have

$$\frac{\tilde{p} + \tilde{\alpha}r}{\Phi(r)} - \frac{G^{(r)}(B^*)}{W^{(r)}(B^*)} \geq 0.$$

Therefore,

$$v_{B^*}(0+) = W^{(r)}(0) \left(\frac{\tilde{p} + \tilde{\alpha}r}{\Phi(r)} - \frac{G^{(r)}(B^*)}{W^{(r)}(B^*)} \right) \geq 0,$$

and this is strictly positive if and only if X has paths of bounded variation by Lemma 3.1. On the other hand, if $B^* = 0$, we also have $v_{B^*}(0+) \geq 0$ because (3.13) implies that, for any $\varepsilon > 0$, $\partial v_B(\varepsilon)/\partial B \leq 0$ for every $B > \varepsilon$ and $\lim_{B \rightarrow \infty} v_B(\varepsilon) = 0$.

We now prove the optimality. We first construct a sequence of functions $v_n(\cdot)$ such that (1) it is C^2 everywhere except at B^* , (2) $v_n(x) = v_{B^*}(x)$ on $x \in (0, \infty)$ and (3) $v_n(x) \downarrow v_{B^*}(x)$ pointwise for every fixed $x \in (-\infty, 0)$. This implies, by noting that $v'_{B^*}(x) = v'_n(x)$ and $v''_{B^*}(x) = v''_n(x)$ on $(0, \infty) \setminus \{B^*\}$, that $(\mathcal{L} - r)(v_n - v_{B^*})(x)$ decreases monotonically in n to zero for every fixed $x \in (0, \infty) \setminus \{B^*\}$ by monotone convergence theorem. Notice that $v_{B^*}(\cdot)$ is uniformly bounded because $h(\cdot)$ is. Hence, we can choose so that v_n is also uniformly bounded for every fixed $n \geq 1$.

Notice that

$$(A.2) \quad \mathbb{E}^x \left[\int_0^\nu e^{-rs} ((\mathcal{L} - r)v_n(X_s)) ds \right] \leq \mathbb{E}^x \left[\int_0^\nu e^{-rs} ((\mathcal{L} - r)(v_n - v_{B^*})(X_s)) ds \right] < \infty, \quad \nu \in \mathcal{S}.$$

To see this, we have by Lemma 3.9

$$\begin{aligned} \mathbb{E}^x \left[\int_0^\nu e^{-rs} ((\mathcal{L} - r)v_n(X_s)) ds \right] &= \mathbb{E}^x \left[\int_0^\nu e^{-rs} ((\mathcal{L} - r)v_{B^*}(X_s)) ds \right] + \mathbb{E}^x \left[\int_0^\nu e^{-rs} ((\mathcal{L} - r)(v_n - v_{B^*})(X_s)) ds \right] \\ &\leq \mathbb{E}^x \left[\int_0^\nu e^{-rs} ((\mathcal{L} - r)(v_n - v_{B^*})(X_s)) ds \right] \leq K \mathbb{E}^x \left[\int_0^\theta e^{-rs} \Pi(X_s, \infty) ds \right] \end{aligned}$$

where $K < \infty$ is the maximum difference between v_{B^*} and v_n . Using N as the Poisson random measure for $-X$, we have by compensation formula (see, e.g., Theorem 4.4 in [28])

$$\begin{aligned} \mathbb{E}^x \left[\int_0^\theta e^{-rs} \Pi(X_s, \infty) ds \right] &= \mathbb{E}^x \left[\int_0^\infty \int_0^\infty e^{-rs} 1_{\{\theta \geq s, u > X_{s-}\}} \Pi(du) ds \right] \\ &= \mathbb{E}^x \left[\int_0^\infty \int_0^\infty e^{-rs} 1_{\{\theta \geq s, u > X_{s-}\}} N(du \times ds) \right] = \mathbb{E}^x \left[e^{-r\theta} \right] < \infty. \end{aligned}$$

Notice that, although v_n is not C^2 at B^* , the Lebesgue measure of v_n at which $X = B^*$ is zero and hence $v_n''(B^*)$ can be chosen arbitrarily for the unbounded variation case. By applying Ito's formula to $\{e^{-r(t \wedge \theta)} v_n(X_{t \wedge \theta}); t \geq 0\}$, we see that

$$(A.3) \quad \left\{ e^{-r(t \wedge \theta)} v_n(X_{t \wedge \theta}) - \int_0^{t \wedge \theta} e^{-rs} ((\mathcal{L} - r)v_n(X_s)) ds; \quad t \geq 0 \right\}$$

is a local martingale. Suppose $\{\sigma_k; k \geq 1\}$ is the corresponding localizing sequence, namely,

$$\mathbb{E}^x \left[e^{-r(t \wedge \theta \wedge \sigma_k)} v_n(X_{t \wedge \theta \wedge \sigma_k}) \right] = v_n(x) + \mathbb{E}^x \left[\int_0^{t \wedge \theta \wedge \sigma_k} e^{-rs} ((\mathcal{L} - r)v_n(X_s)) ds \right], \quad k \geq 1.$$

Now by applying dominated convergence theorem on the left-hand side and Fatou's lemma on the right-hand side via (A.2), we obtain

$$\mathbb{E}^x \left[e^{-r(t \wedge \theta)} v_n(X_{t \wedge \theta}) \right] \leq v_n(x) + \mathbb{E}^x \left[\int_0^{t \wedge \theta} e^{-rs} ((\mathcal{L} - r)v_n(X_s)) ds \right].$$

Hence, (A.3) is in fact a supermartingale.

Now fix $\nu \in \mathcal{S}$. By optional sampling theorem, we have, for any $M \geq 0$, that

$$\begin{aligned} \mathbb{E}^x \left[e^{-r(\nu \wedge M)} v_n(X_{\nu \wedge M}) \right] &\leq v_n(x) + \mathbb{E}^x \left[\int_0^{\nu \wedge M} e^{-rs} ((\mathcal{L} - r)v_n(X_s)) ds \right] \\ &= v_n(x) + \mathbb{E}^x \left[\int_0^{\nu \wedge M} e^{-rs} ((\mathcal{L} - r)v_{B^*}(X_s)) ds \right] + \mathbb{E}^x \left[\int_0^{\nu \wedge M} e^{-rs} ((\mathcal{L} - r)(v_n - v_{B^*})(X_s)) ds \right] \end{aligned}$$

where the last equality holds because the expectation can be split by (A.2). Applying dominated convergence theorem on the left-hand side and monotone convergence theorem on the right-hand side (here the integrands in the two expectations are negative and positive, respectively), along with Lemma 3.9, we obtain

$$(A.4) \quad \mathbb{E}^x \left[e^{-r\nu} v_n(X_\nu) \right] \leq v_n(x) + \mathbb{E}^x \left[\int_0^\nu e^{-rs} ((\mathcal{L} - r)(v_n - v_{B^*})(X_s)) ds \right].$$

Furthermore, monotone convergence theorem yields the followings:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}^x \left[e^{-r\nu} v_n(X_\nu) \right] &= \mathbb{E}^x \left[e^{-r\nu} v_{B^*}(X_\nu) \right], \\ \lim_{n \rightarrow \infty} \mathbb{E}^x \left[\int_0^\nu e^{-rs} ((\mathcal{L} - r)(v_n - v_{B^*})(X_s)) ds \right] &= 0. \end{aligned}$$

Therefore, by taking $n \rightarrow \infty$ on both sides of (A.4) (note $v_{B^*}(x) = v_n(x)$), we have

$$v_{B^*}(x) \geq \mathbb{E}^x [e^{-r\nu} v_{B^*}(X_\nu)] \geq \mathbb{E}^x [e^{-r\nu} h(X_\nu)], \quad \nu \in \mathcal{S},$$

where the last inequality follows from Lemma 3.3. This together with the fact that the stopping time $\tau_{B^*}^+$ corresponds to the value function v_{B^*} completes the proof of Theorem 3.1. \square

A.3. Proofs for Subsection 3.3.

Proof of Lemma 3.10. Let N be the Poisson random measure for $-X$. By the compensation formula, we get

$$\begin{aligned} \Gamma(x; A) &= \mathbb{E}^x \left[\int_0^\infty \int_0^\infty N(dt \times du) e^{-rt} 1_{\{\tau_A^- \geq t, X_{t-} - u < 0\}} \right] \\ &= \mathbb{E}^x \left[\int_0^\infty dt e^{-rt} \int_0^\infty \Pi(du) 1_{\{\tau_A^- \geq t, X_{t-} - u < 0\}} \right] = \int_0^\infty \Pi(du) \int_0^\infty dt [e^{-rt} \mathbb{P}^x \{X_{t-} < u, \tau_A^- \geq t\}]. \end{aligned}$$

Recall, for example from p.225 of [28], the fact

$$\int_0^\infty dt [e^{-rt} \mathbb{P}^x \{X_t \in dy, \tau_0^- > t\}] = dy [e^{-\Phi(r)y} W^{(r)}(x) - W^{(r)}(x - y)], \quad y \geq 0.$$

Applying this yields

$$\begin{aligned} \int_0^\infty dt [e^{-rt} \mathbb{P}^x \{X_t \in dy, \tau_A^- > t\}] &= \int_0^\infty dt [e^{-rt} \mathbb{P}^{x-A} \{X_t \in d(y-A), \tau_0^- > t\}] \\ &= dy [e^{-\Phi(r)(y-A)} W^{(r)}(x-A) - W^{(r)}(x-y)], \end{aligned}$$

when $y \geq A$, and it is 0 otherwise. Therefore, for $u \geq A$, we have

$$\begin{aligned} \int_0^\infty dt [e^{-rt} \mathbb{P}^x \{X_{t-} < u, \tau_A^- \geq t\}] &= \int_A^u dy [e^{-\Phi(r)(y-A)} W^{(r)}(x-A) - W^{(r)}(x-y)] \\ &= \int_0^{u-A} dz [e^{-\Phi(r)z} W^{(r)}(x-A) - W^{(r)}(x-z-A)] \\ &= \frac{W^{(r)}(x-A)}{\Phi(r)} [1 - e^{-\Phi(r)(u-A)}] - \int_0^{u-A} dz W^{(r)}(x-z-A), \end{aligned}$$

and it is 0 on $0 \leq u \leq A$. Substituting this back into $\Gamma(x; A)$ above, we obtain

$$\Gamma(x; A) = \int_A^\infty \Pi(du) \left\{ \frac{1}{\Phi(r)} W^{(r)}(x-A) (1 - e^{-\Phi(r)(u-A)}) - \int_0^{u-A} dz W^{(r)}(x-A-z) \right\}.$$

Next, we apply (3.17) and that $W^{(r)}(x) = 0$ and $Z^{(r)}(x) = 1$ on $(-\infty, 0)$ to get

$$\begin{aligned}
 & \Gamma(x; A) \\
 &= \frac{1}{\Phi(r)} W^{(r)}(x - A) \rho(A) - \int_A^x \Pi(du) \int_0^{u-A} dz W^{(r)}(x - A - z) - \int_x^\infty \Pi(du) \int_0^{x-A} dz W^{(r)}(x - A - z) \\
 &= \frac{1}{\Phi(r)} W^{(r)}(x - A) \rho(A) - \frac{1}{r} \int_A^x \Pi(du) \left(Z^{(r)}(x - A) - Z^{(r)}(x - u) \right) - \frac{1}{r} \int_x^\infty \Pi(du) \left(Z^{(r)}(x - A) - 1 \right) \\
 &= \frac{1}{\Phi(r)} W^{(r)}(x - A) \rho(A) - \frac{1}{r} \int_A^\infty \Pi(du) \left(Z^{(r)}(x - A) - Z^{(r)}(x - u) \right),
 \end{aligned}$$

as desired. \square

Proof of Lemma 3.11. Fix $A > 0$. We differentiate to get

$$\begin{aligned}
 \frac{\partial}{\partial A} \zeta(x - A) &= -r W^{(r)}(x - A) + \frac{r}{\Phi(r)} W^{(r)'}(x - A), \\
 \frac{\partial}{\partial A} \Gamma(x; A) &= -\frac{\rho(A)}{\Phi(r)} W^{(r)'}(x - A) + W^{(r)}(x - A) \int_A^\infty \Pi(du) \left(1 - e^{-\Phi(r)(u-A)} \right) \\
 &= \left(W^{(r)}(x - A) - \frac{1}{\Phi(r)} W^{(r)'}(x - A) \right) \rho(A).
 \end{aligned}$$

Therefore, it follows from (2.23) that

$$\frac{\partial}{\partial A} \Delta_A(x) = \left(W^{(r)}(x - A) - \frac{1}{\Phi(r)} W^{(r)'}(x - A) \right) ((\gamma r + \tilde{p}) - (\tilde{\alpha} - \gamma) \rho(A)).$$

Applying (3.6), we have the expressions

$$\begin{aligned}
 W^{(r)}(x - A) &= e^{\Phi(r)(x-A)} W_{\Phi(r)}(x - A), \\
 W^{(r)'}(x - A) &= \Phi(r) W^{(r)}(x - A) + e^{\Phi(r)(x-A)} W'_{\Phi(r)}(x - A),
 \end{aligned}$$

which yield

$$\frac{\partial}{\partial A} \Delta_A(x) = \frac{e^{\Phi(r)(x-A)}}{\Phi(r)} W'_{\Phi(r)}(x - A) ((\tilde{\alpha} - \gamma) \rho(A) - (\gamma r + \tilde{p})).$$

This completes the proof because it is known that $W'_{\Phi(r)}(x - A)$ is positive and $\rho(A)$ is decreasing. \square

Proof of Lemma 3.13. (1) We first show $(\mathcal{L} - r)u_{A^*}(x) = 0$ for every $x > A^*$. Let \tilde{u} be defined by (3.20) and (3.21) on the whole real line (while $u_{A^*}(x) = 0$ on $x \in (-\infty, 0]$). Suppose $A^* > 0$. Using (3.20) and recalling

that $(\mathcal{L} - r)Z^{(r)}(y) = 0$ and $(\mathcal{L} - r)W^{(r)}(y) = 0$ for every $y > 0$, we have

$$\begin{aligned} (\mathcal{L} - r)\tilde{u}(x) &= (\tilde{\alpha} - \gamma) \left[\frac{1}{r} \int_{A^*}^{\infty} \Pi(du) \left[(\mathcal{L} - r)Z^{(r)}(x - A^*) - (\mathcal{L} - r)Z^{(r)}(x - u) \right] \right] \\ &\quad - \left(\gamma + \frac{\tilde{p}}{r} \right) (\mathcal{L} - r)Z^{(r)}(x - A^*) + \left(\frac{\tilde{p}}{r} + \tilde{\alpha} \right) \left[(\mathcal{L} - r)Z^{(r)}(x) - \frac{q}{\Phi(r)} (\mathcal{L} - r)W^{(r)}(x) \right] \\ &= -(\tilde{\alpha} - \gamma) \left[\frac{1}{r} \int_x^{\infty} \Pi(du) \left[(\mathcal{L} - r)Z^{(r)}(x - u) \right] \right] \\ &= (\tilde{\alpha} - \gamma)\Pi(x, \infty). \end{aligned}$$

Suppose $A^* = 0$. By (3.21), we obtain $(\mathcal{L} - r)\tilde{u}(x) = (\tilde{\alpha} - \gamma)\Pi(x, \infty)$ similarly to the case above. Furthermore, it is easy to show that

$$(\mathcal{L} - r)(u_{A^*}(x) - \tilde{u}(x)) = -\Pi(x, \infty)(\tilde{\alpha} - \gamma),$$

and hence the proof is complete.

(2) We now show for the case $A^* > 0$ that we have $(\mathcal{L} - r)u_{A^*}(x) > 0$ for every $0 < x < A^*$. Notice that $u_{A^*}(x) = g(x)$ on $0 < x < A^*$. We have

$$(\mathcal{L} - r)u_{A^*}(x) = (\mathcal{L} - r)L(x) \quad \text{where} \quad L(x) := - \left(\gamma + \frac{\tilde{p}}{r} \right) - (\tilde{\alpha} - \gamma)1_{\{x \leq 0\}}.$$

Therefore, we have

$$\begin{aligned} (\mathcal{L} - r)u_{A^*}(x) &= (\mathcal{L} - r)L(x) = \int_0^{\infty} (L(x - u) - L(x)) \Pi(du) - rL(x) \\ &= -(\tilde{\alpha} - \gamma) \Pi(x, \infty) + (r\gamma + \tilde{p}) \leq -(\tilde{\alpha} - \gamma) \rho(A^*) + (r\gamma + \tilde{p}) = 0. \end{aligned}$$

Here the inequality holds by (3.17) and because $x < A^*$. The last equality holds by (3.19). \square

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