On the HJB equation arising in the consumption-investment problem with transaction costs

Yuri Kabanov

Laboratoire de Mathématiques, Université de Franche-Comté

August, 27, 2009

History

- Merton. JET, 1971.
- Magill, Constantinides. JET, 1976.
- Davis, Norman. Math. Oper. Res., 1990.
- Shreve, Soner. AAP, 1994.
- K., Klüppelberg. FS, 2004.
- Benth, Karlsen, Reikvam. 2000.

Model

Let $Y = (Y_t)$ be an $\mathbf{R}^d$-valued Lévy process modelling **relative** price movements (i.e. $dY_t^i = dS_t^i / S_{t-}^i$ or $S_t^i = S_0^i \mathcal{E}_t(Y^i)$) :

$$dY_t = \mu t + \Xi dw_t + \int z(p(dz, dt) - \Pi(dz)dt)$$

w is a Wiener process and $p(dt, dx)$ is a Poisson random measure with the compensator $\Pi(dz)dt$ where $\Pi(dz)$ is concentrated on $]-1, \infty]^d$. The matrix Ξ is such that $A = \Xi\Xi^*$ is non-degenerated,

$$\int (|z|^2 \wedge |z|)\Pi(dz) < \infty.$$

Let K and $\mathcal{C}$ be **proper** cones in $\mathbf{R}^d$ such that $\mathcal{C} \subseteq \operatorname{int} K \neq \emptyset$. The set $\mathcal{A}_a$ of controls $\pi = (B, C)$ is the set of **predictable** càdlàg processes of bounded variation such that $dC_t = c_t dt$ and

$$\dot{B} \in -K, \qquad c \in \mathcal{C}.$$

Dynamics

The process $V = V^{x,\pi}$ is the solution of the linear system

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i - dC_t^i, \quad V_{0-}^i = x^i, \quad i = 1, ..., d.$$

This solution can be expressed explicitly using the Doléans-Dade exponentials $S_t^i = \mathcal{E}_t(Y^i)$ (we assume that $S_0 = \mathbf{1}$) :

$$V_t^i = S_t^i x^i + S_t^i \int_{[0,t]} \frac{1}{S_{s-}^i}(dB_s^i - dC_s^i), \quad i = 1, ..., d.$$

We introduce the stopping time

$$\theta = \theta^{x,\pi} := \inf\{t : \; V_t^{x,\pi} \notin \mathrm{int}\, K\}.$$

For $x \in \mathrm{int}\, K$ we consider the subset $\mathcal{A}_a^x$ of "admissible" controls for which $\pi = I_{[0,\theta^{x,\pi}]}\pi$, i.e. the process $V^{x,\pi}$ stops at the moment of ruin : no more consumption.

Goal Functional

Let $U : \mathcal{C} \to \mathbf{R}_+$ be a concave function such that $U(0) = 0$ and $U(x)/|x| \to 0$ as $|x| \to \infty$. For $\pi = (B, C) \in \mathcal{A}_a^x$ we put

$$J_t^\pi := \int_0^t e^{-\beta s} U(c_s) ds$$

and consider the infinite horizon maximization problem with the *goal functional* EJ_∞^π. The *Bellman function*

$$W(x) := \sup_{\pi \in \mathcal{A}_a^x} EJ_\infty^\pi \,, \quad x \in \operatorname{int} K \,,$$

is **increasing** with respect to the partial ordering $\geq_K$.
The process $V^{\lambda x_1 + (1-\lambda) x_2, \lambda \pi_1 + (1-\lambda)\pi_2}$ is the convex combination of V^{x_i, π_i} with the same coefficients. For **continuous** Y the ruin time is the maximum of θ^{x_i, π_i} and the concavity of u implies the concavity of W. But if Y has **jumps**, the ruin times are not related in this way and we cannot guarantee (at least, by the above argument) that the Bellman function is concave.

The Hamilton–Jacobi–Bellman Equation, I

Let $G := (-K) \cap \partial\mathcal{O}_1(0)$ where $\mathcal{O}_r(y) := \{x \in \mathbf{R}^d : \ |x-y| < r\}$
Then $-K = \mathrm{cone}\, G$. We denote by Σ_G the *support function* of G, i.e. $\Sigma_G(p) = \sup_{x \in G} px$. Put

$$F(X, p, \mathcal{I}(f,x), W, x) = \max\{F_0(X, p, \mathcal{I}(f,x), W, x) + U^*(p), \Sigma_G(p)\},$$

where $X \in \mathcal{S}_d$, the set of $d \times d$ symmetric matrices, $p, x \in \mathbf{R}^d$, $W \in \mathbf{R}$, $f \in C_1(K) \cap C^2(x)$ and the function F_0 is given by

$$\begin{aligned} F_0(X, p, \mathcal{I}(f,x), W, x) &= \frac{1}{2}\mathrm{tr}\, A(x)X + \mu(x)p + \mathcal{I}(f,x) - \beta W(x) \\ &= \frac{1}{2}\sum_{i,j} a^{ij}x^i x^j X^{ij} + \sum_i \mu^i x^i p^i + \mathcal{I}(f,x) - \beta W(x) \end{aligned}$$

where $A(x)$ is the matrix with $A^{ij}(x) = a^{ij}x^i x^j$, $\mu^i(x) = \mu^i x^i$,

$$\mathcal{I}(f,x) = \int (f(x+\mathrm{diag}\, xz) - f(x) - \mathrm{diag}\, xzf'(x))I(z,x)\Pi(dz), \quad x \in \mathrm{int}\, K,$$

$$I(z,x) = I_{\{z:\ x+\mathrm{diag}\, xz \in K\}} = I_K(x + \mathrm{diag}\, xz).$$

The Hamilton–Jacobi–Bellman Equation, II

If ϕ is a smooth function, we put

$$\mathcal{L}\phi(x) := F(\phi''(x), \phi'(x), \mathcal{I}(\phi, x), \phi(x), x).$$

In a similar way, $\mathcal{L}_0$ corresponds to the function F_0.
We show, under mild hypotheses, that W is the unique viscosity solution of the Dirichlet problem for the HJB equation

$$\begin{aligned} F(W''(x), W'(x), \mathcal{I}(W, x), W(x), x) &= 0, \qquad x \in \operatorname{int} K, \\ W(x) &= 0, \qquad x \in \partial K. \end{aligned}$$

In general, W has no derivatives at some points $x \in \operatorname{int} K$ and the notation above needs to be interpreted. The idea of viscosity solutions is to substitute W in F by suitable test functions.

Viscosity Solutions

- A function $v \in C(K)$ is called *viscosity supersolution* (of HJB) if for every $x \in \operatorname{int} K$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \geq f$ the inequality $\mathcal{L}f(x) \leq 0$ holds.
- A function $v \in C(K)$ is called *viscosity subsolution* of if for every $x \in \operatorname{int} K$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \leq f$ the inequality $\mathcal{L}f(x) \geq 0$ holds.
- $v \in C(K)$ is *viscosity solution* of if v is simultaneously a viscosity super- and subsolution.
- $v \in C_1(K) \cap C^2(\operatorname{int} K)$ is *classical supersolution* of HJB if $\mathcal{L}v \leq 0$ on $\operatorname{int} K$. We add the adjective *strict* when $\mathcal{L}v < 0$ on the set $\operatorname{int} K$.

Lemma

Suppose that the function v is a viscosity solution. If v is twice differentiable at $x_0 \in \operatorname{int} K$, then it satisfies HJB at this point in the classical sense.

Jets

For $p \in \mathbf{R}^d$ and $X \in \mathcal{S}_d$ we put $Q_{p,X}(z) = pz + (1/2)\langle Xz, z\rangle$ and define the *super-* and *subjets* of a function v at the point x :

$$\begin{aligned} J^+v(x) &= \{(p,X):\ v(x+h) \le v(x) + Q_{p,X}(h) + o(|h|^2)\}, \\ J^-v(x) &= \{(p,X):\ v(x+h) \ge v(x) + Q_{p,X}(h) + o(|h|^2)\}. \end{aligned}$$

I.e. $J^+v(x)$ (resp. $J^-v(x)$) is the family of coefficients of quadratic functions $v(x) + Q_{p,X}(y - .)$ dominating $v(.)$ (resp., dominated by $v(.)$) near x up to the 2nd order and coinciding with $v(.)$ at x.
For integro-differential operators viscosity solution does not admit an equivalent formulation in terms of jets.

Lemma

Let v be a viscosity supersolution, $x \in \operatorname{int} K$, and $(p,X) \in J^-v(x)$. Then there is a function $f \in C_1(K) \cap C^2(x)$ such that $f'(x) = p$, $f''(x) = X$, $f(x) = v(x)$, $f \ge v$ on K and, hence,

$$F(X, p, \mathcal{I}(f,x), W(x), x) \le 0.$$

Supermaringales and Majorants of W

Put $\tilde{V} = V^{\theta-} = VI_{[0,\theta[} + V_{\sigma-}I_{[\theta,\infty[}$ where θ is the ruin time. Let Φ be the set of continuous functions $f : K \to \mathbf{R}_+$ increasing with respect to $\geq_K$ and such that for each $x \in \operatorname{int} K$, $\pi \in \mathcal{A}_a^x$

$$X^f = X^{f,x,\pi} = e^{-\beta t} f(\tilde{V}) + J^\pi,$$

is a supermartingale. This set is convex and stable under the operation $\wedge$. Any continuous function which is a monotone limit of functions from Φ also belongs to Φ.

Lemma

(*a*) *If $f \in \Phi$, then $W \leq f$;*
(*b*) *if a point $y \in \partial K$ is such that there is $f \in \Phi$ with $f(y) = 0$, then W is continuous at y.*

Proof. Indeed : $EJ_t^\pi \leq EX_t^f \leq f(\tilde{V}_0) = f(V_0) \leq f(V_{0-}) = f(x)$.

Supermaringales and Supersolutions of HJB, I

Lemma

Let $f : K \to \mathbf{R}_+$ be a function in $C_1(K) \cap C^2(\mathrm{int}\, K)$. If f is a classical supersolution of HJB, then f is a monotone function and X^f is a supermartingale, i.e. $f \in \Phi$.

Proof. A classical supersolution is increasing with respect to $\geq_K$. Indeed, for any $x, h \in \mathrm{int}\, K$ there is $\vartheta \in [0,1]$ such that

$$f(x+h) - f(x) = f'(x+\vartheta h)h \geq 0$$

because for the supersolution $\Sigma_G(f'(y)) \leq 0$ when $y \in \mathrm{int}\, K$, or, equivalently, $f'(y)h \geq 0$ for every $h \in K$. By continuity, $f(x+h) - f(x) \geq 0$ for every $x, h \in K$.

Supermaringales and Supersolutions of HJB, II

Using the Itô formula we have :

$$X_t^f = f(x) + \int_0^{t\wedge\theta} e^{-\beta s}[\mathcal{L}_0 f(\tilde{V}_s) - c_s f'(\tilde{V}_s) + U(c_s)]ds + R_t + m_t,$$

where the integral is a decreasing process (since $[\ldots] \le \mathcal{L}f(\tilde{V}_s)$),

$$R_t = \int_0^{t\wedge\theta} e^{-\beta s} f'(V_{s-})dB_s^c + \sum_{s\le t} e^{-\beta s}[f(\tilde{V}_{s-} + \Delta B_s) - f(\tilde{V}_{s-})]$$

is also decreasing and m is the local martingale with

$$\begin{aligned} m_t &= \int_0^{t\wedge\theta} e^{-\beta s} f'(\tilde{V}_{s-}) \mathrm{diag}\, \tilde{V}_s \Xi dw_s \\ &\quad + \int_0^t \int e^{-\beta s}[f(\tilde{V}_{s-} + \mathrm{diag}\, \tilde{V}_{s-} z) - f(\tilde{V}_{s-})] I(\tilde{V}_{s-}, z)\tilde{p}(dz, ds). \end{aligned}$$

$\tilde{p}(dz, ds) = p(dz, ds) - \Pi(dz)ds.$

Strict Local Supersolutions

We fix a ball $\bar{\mathcal{O}}_r(x) \subseteq \operatorname{int} K$ and define τ^π as the exit time of $V^{\pi,x}$ from $\mathcal{O}_r(x)$, i.e.

$$\tau^\pi = \inf\{t \geq 0 : \ |V_t^{\pi,x} - x| \geq r\}.$$

Lemma

Let $f \in C_1(K) \cap C^2(\bar{\mathcal{O}}_r(x))$ be such that $\mathcal{L}f \leq -\varepsilon < 0$ on $\bar{\mathcal{O}}_r(x)$. Then there exist a constant $\eta > 0$ and an interval $]0, t_0]$ such that

$$\sup_{\pi \in \mathcal{A}_a^x} E X_{t \wedge \tau^\pi}^{f,x,\pi} \leq f(x) - \eta t \qquad \forall\, t \in]0, t_0].$$

Dynamic Programming Principle

For the following two assertions we need to assume that Ω is a path space.

Lemma

Let $\mathcal{T}_f$ and $\mathcal{T}_b$ be, respectively, the sets of all finite and bounded stopping times. Then

$$W(x) \le \sup_{\pi \in \mathcal{A}_a^x} \inf_{\tau \in \mathcal{T}_f} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi})\right).$$

Lemma

Assume that $W(x)$ is continuous on $\operatorname{int} K$. Then for any $\tau \in \mathcal{T}_f$

$$W(x) \ge \sup_{\pi \in \mathcal{A}_a^x} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi})\right).$$

Bellman Function and HJB

Theorem

Assume that the Bellman function W is in $C(K)$. Then W is a viscosity solution of the HJB equation).

Proof.

Uniqueness Theorem for HJB

Definition. We say that a positive function $\ell \in C(K) \cap C^2(\operatorname{int} K)$ is the *Lyapunov function* if the following properties are satisfied :
1) $\ell'(x) \in \operatorname{int} K^*$ and $\mathcal{L}_0 \ell(x) \le 0$ for all $x \in \operatorname{int} K$,
2) $\ell(x) \to \infty$ as $|x| \to \infty$.

Theorem

Assume that the jump measure Π does not charge $(d-1)$-dimensional surfaces. Suppose that there exists a Lyapunov function ℓ. Then the Dirichlet problem for the HJB equation has at most one viscosity solution in the class of continuous functions satisfying the growth condition

$$W(x)/\ell(x) \to 0, \quad |x| \to \infty.$$

Uniqueness Theorem for HJB. Idea of the proof, I

Let W and $\tilde{W}$ be two viscosity solutions of HJB coinciding on ∂K. Suppose that $W(z) > \tilde{W}(z)$ for some $z \in K$. Take $\varepsilon > 0$ such that

$$W(z) - \tilde{W}(z) - 2\varepsilon\ell(z) > 0.$$

Define continuous functions $\Delta_n : K \times K \to \mathbf{R}$

$$\Delta_n(x,y) := W(x) - \tilde{W}(y) - \frac{1}{2}n|x-y|^2 - \varepsilon[\ell(x)+\ell(y)], \qquad n \geq 0.$$

Note that $\Delta_n(x,x) = \Delta_0(x,x)$ for all $x \in K$ and $\Delta_0(x,x) \leq 0$ when $x \in \partial K$. Since ℓ has a higher growth rate than W we deduce that $\Delta_n(x,y) \to -\infty$ as $|x|+|y| \to \infty$. The sets $\{\Delta_n \geq a\}$ are compacts and Δ_n attains its maximum. I.e., there is $(x_n, y_n) \in K \times K$ such that

$$\Delta_n(x_n,y_n) = \bar{\Delta}_n := \sup_{(x,y)\in K\times K} \Delta_n(x,y) \geq \bar{\Delta} := \sup_{x\in K} \Delta_0(x,x) > 0.$$

All (x_n, y_n) belong to the compact $\{(x,y) : \ \Delta_0(x,y) \geq 0\}$. Thus, the sequence $n|x_n - y_n|^2$ is bounded. We assume wlg that (x_n, y_n) converge to $(\widehat{x},\widehat{x})$. Also, $n|x_n - y_n|^2 \to 0$ (otherwise we $\Delta_0(\widehat{x},\widehat{x}) > \bar{\Delta}$). Clearly, $\bar{\Delta}_n \to \Delta_0(\widehat{x},\widehat{x}) = \bar{\Delta}$. Thus, $\widehat{x}$ is in the interior of K and so are x_n and y_n.

Lemma

Let v and $\tilde v$ be two continuous functions on an open subset $\mathcal{O} \subseteq \mathbf{R}^d$. Consider the function $\Delta(x,y) = v(x) - \tilde v(y) - \frac{1}{2}n|x-y|^2$ with $n > 0$. Suppose that Δ attains a local maximum at $(\widehat{x}, \widehat{y})$. Then there are symmetric matrices X and Y such that

$$(n(\widehat{x}-\widehat{y}), X) \in \bar{J}^+ v(\widehat{x}), \qquad (n(\widehat{x}-\widehat{y}), Y) \in \bar{J}^- \tilde v(\widehat{y}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Here $\bar{J}^+ v(x)$ and $\bar{J}^- v(x)$ are values of the set-valued mappings whose graphs are closures of graphs of $J^+ v$ and $J^- v$.

The matrix inequality implies the bound

$$\operatorname{tr}(A(x)X - A(y)Y) \le 3n|A|^{1/2}|x-y|^2.$$

Uniqueness Theorem for HJB. Idea of the proof, II

By the Ishii lemma applied to $v = W - \varepsilon\ell$ and $\tilde{v} = \tilde{W} + \varepsilon\ell$ at the point (x_n, y_n) there exist matrices X^n and Y^n such that

$$(n(x_n - y_n), X^n) \in \bar{J}^+ v(x_n), \qquad (n(x_n - y_n), Y^n) \in \bar{J}^- \tilde{v}(y_n).$$

Using the notations $p_n = n(x_n - y_n) + \varepsilon\ell'(x_n)$, $q_n = n(x_n - y_n) - \varepsilon\ell'(y_n)$, $X_n = X^n + \varepsilon\ell''(x_n)$, $Y_n = Y^n - \varepsilon\ell''(y_n)$, we may rewrite the last relations in the following equivalent form :

$$(p_n, X_n) \in \bar{J}^+ W(x_n), \qquad (q_n, Y_n) \in \bar{J}^- \tilde{W}(y_n).$$

Since W and $\tilde{W}$ are viscosity sub- and supersolutions, one can find, the functions $f_n \in C_1(K) \cap C^2(x_n)$ and $\tilde{f}_n \in C_1(K) \cap C^2(y_n)$ such that $f_n'(x_n) = p_n$, $f_n''(x_n) = X_n$, $f_n(x_n) = W(x_n)$, $f_n \le W$ on K, and $\tilde{f}_n'(y_n) = q_n$, $\tilde{f}_n''(y_n) = Y_n$, $\tilde{f}_n(y_n) = \tilde{W}(y_n)$, $\tilde{f}_n \ge \tilde{W}$ on K,

$$F(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) \ge 0 \ge F(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n).$$

Uniqueness Theorem for HJB. Idea of the proof, III

The second inequality implies that $mq_n \le 0$ for each $m \in G = (-K) \cap \partial\mathcal{O}_1(0)$. But for the Lyapunov function $\ell'(x) \in \operatorname{int} K^*$ when $x \in \operatorname{int} K$ and, therefore,

$$mp_n = mq_n + \varepsilon m(\ell'(x_n) + \ell'(y_n)) < 0.$$

Since G is a compact, $\Sigma_G(p_n) < 0$. It follows that

$$\begin{aligned} F_0(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) + U^*(p_n) &\ge 0, \\ F_0(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n) + U^*(q_n) &\le 0. \end{aligned}$$

Recall that U^* is decreasing with respect to the partial ordering generated by $\mathcal{C}^*$ hence also by K^*. Thus, $U^*(p_n) \le U^*(q_n)$ and we obtain the inequality

$$b_n = F_0(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) - F_0(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n) \ge 0$$

Uniqueness Theorem for HJB. Idea of the proof, IV

Clearly,

$$
\begin{aligned}
b_n \;&=\; \frac{1}{2}\sum_{i,j=1}^{d}(a^{ij}x_n^i x_n^j X_{ij}^n - a^{ij}y_n^i y_n^j Y_{ij}^n) + n\sum_{i=1}^{d}\mu^i(x_n^i - y_n^i)^2 \\
&\quad -\frac{1}{2}\beta n|x_n - y_n|^2 - \beta\Delta_n(x_n, y_n) + \mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n) \\
&\quad +\varepsilon(\mathcal{L}_0\ell(x_n) + \mathcal{L}_0\ell(y_n)).
\end{aligned}
$$

The first term in the rhs is dominated by a constant multiplied by $n|x_n - y_n|^2$; a similar bound for the second sum is obvious ; the last term is negative according to the definition of the Lyapunov function. To complete the proof, it remains to show that

$$\limsup_n \left(\mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n)\right) \le 0.$$

Indeed, with this we have that $\limsup b_n \le -\beta\bar{\Delta} < 0$.

Uniqueness Theorem for HJB. Idea of the proof, V

Let

$$\begin{aligned} F_n(z) &= \big[(f_n-\varepsilon\ell)(x_n+\operatorname{diag} x_n z)-(f_n-\varepsilon\ell)(x_n) \\ &\quad -\operatorname{diag} x_n z(f_n'-\varepsilon\ell')(x_n)\big] I(z,x_n), \\ \tilde F_n(z) &= \big[(\tilde f_n+\varepsilon\ell)(y_n+\operatorname{diag} y_n z)-(\tilde f_n+\varepsilon\ell)(y_n) \\ &\quad -\operatorname{diag} y_n z(\tilde f_n'+\varepsilon\ell')(y_n)\big] I(z,y_n). \end{aligned}$$

and $H_n(z)=F_n(z)-\tilde F_n(z)$ With this notation

$$\mathcal{I}(f_n-\varepsilon\ell,x_n)-\mathcal{I}(\tilde f_n+\varepsilon\ell,y_n)=\int H_n(z)\Pi(dz)$$

and the needed inequality will follow from the Fatou lemma if we show that there is a constant C such that for all sufficiently large n

$$H_n(z)\le C(|z|\wedge|z|^2)\quad \text{for all } z\in K \tag{1}$$

and

$$\limsup_n H_n(z)\le 0 \quad \Pi\text{-a.s.} \tag{2}$$

Uniqueness Theorem for HJB. Idea of the proof, VI

Using the properties of f_n we get the bound :

$$F_n(z) \leq \Big[(W-\varepsilon\ell)(x_n+\operatorname{diag} x_n z)-(W-\varepsilon\ell)(x_n) \\ -\operatorname{diag} x_n z n(x_n-y_n)\Big] I(z,x_n)$$

Since the continuous function W and I are of sublinear growth and the sequences x_n and $n(x_n-y_n)$ are converging (hence bounded), absolute value of the function in the right-hand side of this inequality is dominated by a function $c(1+|z|)$. The arguments for $-\tilde{F}_n(z)$ are similar. So, the function H_n is of sublinear growth.
We have the following identity :

$$\begin{aligned} H_n(z) &= (\Delta_n(x_n+\operatorname{diag} x_n z, y_n+\operatorname{diag} y_n z)-\Delta_n(x_n,y_n) \\ &\quad +(1/2)n|\operatorname{diag}(x_n-y_n)z|^2)I(z,x_n)I(z,y_n) \\ &\quad +(f_n(x_n+\operatorname{diag} x_n z)-W(x_n+\operatorname{diag} x_n z))I(z,x_n)I(z,y_n) \\ &\quad -(\tilde{f}_n(y_n+\operatorname{diag} y_n z)-\tilde{W}(y_n+\operatorname{diag} y_n z))I(z,x_n)I(z,y_n) \\ &\quad +F_n(z)(1-I(z,y_n))-\tilde{F}_n(z)(1-I(z,x_n)). \end{aligned}$$

Uniqueness Theorem for HJB. Idea of the proof, VII

The function $\Delta(x,y)$ attains its maximum at (x_n, y_n) and $f_n \le W$, $\tilde f_n \ge \tilde W$. It follows that

$$H_n(z) \le (1/2)n|x_n - y_n|^2|z|^2 + F_n(z)(1 - I(z, y_n)) - \tilde F_n(z)(1 - I(z, x_n)).$$

Let $\delta > 0$ be the distance between $\widehat{x}$ from and ∂K. Then $x_n, y_n \in \mathcal{O}_{\delta/2}(\widehat{x})$ for large n and, hence, the second and the third terms in the rhs above are functions vanishing on $\mathcal{O}_1(0)$. So, for such n the function H_n is dominated from above on $\mathcal{O}_1(0)$ by $c_n|z|^2$ where $c_n := (1/2)n|x_n - y_n|^2 \to 0$ as $n \to \infty$. Therefore, (1) holds. The relation (1) also holds because the second and the first terms tends to zero (stationarily) for all z except the set $\{z : \ \widehat{x} + \mathrm{diag}\,\widehat{x} z \in \partial K\}$. The coordinates of points of $\partial K \setminus \{0\}$ are non-zero. So this set is empty if $\widehat{x}$ has a zero coordinate. If all components $\widehat{x}$ are nonzero, the operator $\widehat{x}$ is non-degenerated and the set in question is of zero measure Π in virtue of our assumption.

Lyapunov Functions, I

Let $u \in C(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus \{0\})$ be increasing strictly concave, $u(0) = 0$, $u(\infty) = \infty$. Define $R = -u'^2/(u''u)$ and assume that $\bar{R} = \sup_{z>0} R(z) < \infty$.

For $p \in K^* \setminus \{0\}$ define the function $f(x) = f_p(x) := u(px)$. If $y \in K$ and $x \neq 0$, then $yf'(x) = (py)u'(px) \leq 0$. The inequality is strict when $p \in \operatorname{int} K^*$.

Recall that $A^{ij}(x) = A^{ij}x^ix^j$ and $\mu^i(x) = \mu^i x^i$. Suppose that $\langle A(x)p, p\rangle \neq 0$. Isolating the full square we get that $\mathcal{L}_0 f(x)$ is equal to

$$\frac{1}{2}\left[\langle A(x)p,p\rangle u''(px) + 2\langle \mu(x),p\rangle u'(px) + \frac{\langle \mu(x),p\rangle^2}{\langle A(x)p,p\rangle}\frac{u'^2(px)}{u''(px)}\right]$$
$$+\frac{1}{2}\frac{\langle \mu(x),p\rangle^2}{\langle A(x)p,p\rangle}R(px)u(px) + \mathcal{I}(f,x) - \beta u(px).$$

Note that

$$(f(x+\operatorname{diag} xz) - f(x) - \operatorname{diag} xzf'(x)) = (1/2)u''(\ldots)(px)^2 \leq 0.$$

Lyapunov Functions, II

It follows that $\mathcal{L}_0 f(x) \leq 0$ if $\beta \geq \eta(p)\bar{R}$ where

$$\eta(p) := \frac{1}{2} \sup_{x \in K} \frac{\langle \mu(x), p \rangle^2}{\langle A(x)p, p \rangle}.$$

If $\langle A(x)p, p \rangle = 0$ and $\langle \mu(x), p \rangle = 0$, then $\mathcal{L}_0 f(x) = -\beta u(px) \leq 0$ for any $\beta \geq 0$.

Proposition

Let $p \in \operatorname{int} K^*$. Suppose that $\langle \mu(x), p \rangle$ vanishes on the set $\{x \in \operatorname{int} K : \ \langle A(x)p, p \rangle = 0\}$. If $\beta \geq \eta(p)\bar{R}$, then f_p is a Lyapunov function.

Existence of Classical Supersolutions

The same ideas are useful also in the search of supersolutions. Since $\mathcal{L}f = \mathcal{L}_0 f + U^*(f')$, it is natural to choose u related to U. For the case where $\mathcal{C} = \mathbf{R}^d_+$ and $U(c) = u(e_1 c)$, with u satisfying the postulated properties and assuming, moreover, the inequality

$$u^*(au'(z)) \le g(a)u(z)$$

we get, using the homogeneity of $\mathcal{L}_0$, the following result.

Proposition

Assume $\langle A(x)p, p\rangle \neq 0$ for all $x \in \operatorname{int} K$ and $p \in K^* \setminus \{0\}$. Suppose that $g(a) = o(a)$ as $a \to \infty$. If $\beta > \bar{\eta}\bar{R}$, then there is a_0 such that for every $a \ge a_0$ the function af_p is a classical supersolution of HJB, whatever is $p \in K^*$ with $p^1 \neq 0$. Moreover, if $p \in \operatorname{int} K^*$, then af_p is a strict supersolution on any compact subset of $\operatorname{int} K$.

Power Utility Function

For the power utility function $u(z) = z^\gamma/\gamma$, $\gamma \in]0,1[$, we have :

$$R(z) = \gamma/(1-\gamma) = \bar{R},$$

$$u^*(au'(z)) = (1-\gamma)a^{\gamma/(\gamma-1)}u(z) = g(a)u(z).$$

If $A = \operatorname{diag}\sigma$, $\sigma^1 = 0$, $\mu^1 = 0$ (the first asset is the *numéraire*) and $\sigma^i \neq 0$ for $i \neq 1$, then, by the Cauchy–Schwarz inequality,

$$\eta(p) \leq \frac{1}{2}\sum_{i=2}^{d}\left(\frac{\mu^i}{\sigma^i}\right)^2.$$

The inequality

$$\beta > \frac{\gamma}{1-\gamma}\frac{1}{2}\sum_{i=2}^{d}\left(\frac{\mu^i}{\sigma^i}\right)^2$$

(implying the bound $\beta > \bar{\eta}\bar{R}$) ensures the existence of a classical supersolution.