On High Frequency Estimation of the Frictionless Price: The Use of Observed Liquidity Variables

Selma Chaker*

Observed high-frequency prices are always contaminated with liquidity costs or market microstructure noise. Inspired by the market microstructure literature, I explicitly model this noise and remove it from observed prices to obtain an estimate of the frictionless price. I then formally test whether the prices adjusted for the estimated liquidity costs are either totally or partially free from noise. If the liquidity costs are only partially removed, the residual noise is smaller and closer to an exogenous white noise than the original noise is. To illustrate my approach, I use the adjusted prices to improve volatility estimation in the presence of noise. If the noise is totally absorbed, I show that the sum of squared returns – which would be inconsistent for return variance when based on observed returns – becomes consistent when based on adjusted returns. This novel estimator achieves the maximum possible rate of convergence. If the noise is partially absorbed, however, I show that the two time scales volatility estimator – which would be inconsistent for return variance when based on observed returns – becomes consistent when based on adjusted returns even if the original noise is endogenous, heteroskedastic and autocorrelated.

KEY WORDS: Stochastic volatility; Hidden semimartingale model; Infill regression ; Endogenous noise; Semiparametric volatility estimation.

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1. INTRODUCTION

The frictionless price – also referred to as the true price, the efficient price or the equilibrium price – is the expectation of the asset's final value conditional on all publicly available information. However, the frictionless price is latent as observed prices are contaminated with market microstructure noise.

Why then is it important to estimate the frictionless price? From a high frequency financial econometrics perspective, specifically nonparametric methods, the frictionless price is either treated as observable or suffering from a measurement error. In this literature, the object of interest varies from integrated volatility - pioneered by Andersen et al. (2003) - to spot volatility, leverage effects, integrated betas and jumps. The financial applications range from risk management to options hedging, execution of transactions, portfolio optimization, and forecasting. Therefore, the frictions in the observed price will impact the estimation of the object of interest as well as the application. In this paper, I show that estimating the frictionless price before using it to measure the integrated volatility not only improves the accuracy but also relaxes the assumptions underlying the traditional robust-to-noise volatility estimators. In the empirical illustration, I find that for more than half of the 2010 business days, I can fully recover the frictionless price for Alcoa Aluminum.

From a market microstructure perspective, several papers provide estimators of the liquidity or transaction costs which may induce a measure of the frictionless price. However, the underlying assumptions for the frictionless price usually restrict its volatility to be constant; key models are surveyed in Hasbrouck (2007). This paper allows the volatility of the frictionless price to be time-varying as a result of adopting the standard model in financial econometrics, the hidden semimartingale model. The latter model assumes that the observed price is the sum of a semimartingale, which usually has stochastic volatility, and a noise term.

I introduce the liquidity costs in the context of a model that is consistent with both the standard additive price model of high-frequency financial econometrics and several transaction-cost models from the market microstructure literature. The standard model is given by

$$p_t = p_t^* + \varepsilon_t, \quad t \in [0,1], \tag{1}$$

where p_t is the observed log price, p_t^* is the log of the frictionless price and ε_t is a measurement error term summarizing the market microstructure noise generated by the trading process[1]. The fixed interval $[0,1]$ is a day, for example. In this context, the observed price is the sum of two unobservable components, the frictionless price and the noise.

Regarding the noise, within the market microstructure literature, Stoll (2000) studies various sources of trading frictions. The presence of a bid-ask spread and the corresponding bounces is one such source modeled by Roll (1984). Glosten and Harris (1988) extend Roll's model by adding a trading volume component to capture size-varying costs of providing liquidity service. This model is nested in (1) and is given by

$$p_t = p_t^* + \beta_1 q_t + \beta_2 q_t v_t, \tag{2}$$

where q_t is the trade-direction indicator, which takes the value +1 if the trade is buyer-initiated and -1 if the trade is seller-initiated, and v_t is the trading volume. β_1 and β_2 are two parameters to be estimated.

If an estimator of the noise is denoted by $\widehat{\varepsilon_t}$, then

$$\widehat{p_t^*} = p_t - \widehat{\varepsilon_t} \tag{3}$$

[1] In a similar framework, Aït-Sahalia and Yu (2009) relate statistical measures of the noise ε to financial measures of the stock liquidity.

is an estimator of the frictionless price. For example, in the context of model (2), $\widehat{\varepsilon}_t$ would be written as $\widehat{\beta}_1 q_t + \widehat{\beta}_2 q_t v_t$ where $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are consistent estimators of β_1 and β_2, respectively.

The rest of this paper is organized as follows. Section 2 describes the model for market microstructure noise based on liquidity costs. In Section 3, I discuss the estimation of this model and describe a test for the performance of the frictionless-price measure. In Section 4, I study volatility estimation based on adjusting prices for the liquidity measure introduced in Section 2. Section 5 is an empirical application to assess with data the performance of the price model of Section 2. In Section 6, I offer several conclusions.

2. PRICE MODEL AND SETUP

A generalized model of Glosten and Harris (1988) introduced in (2) is given by

$$p_t = p_t^* + \mathbf{F}_t^{'}\boldsymbol{\beta}, \tag{4}$$

where $\mathbf{F}$ is an M-vector of liquidity-cost variables and $\boldsymbol{\beta}$ is a parameter to be estimated from the data. In addition to q_t and $q_t v_t$, $\mathbf{F}$ could contain the bid-ask spread - a natural measure of frictions - and the quoted depths[2]. Indeed, in Kavajecz (1999), the depths are used to capture inventory-control costs as well as asymmetric-information costs.

The linear form $\mathbf{F}_t^{'}\boldsymbol{\beta}$ in (4) could be misspecified in the sense that it does not capture the entire noise ε_t. The model of this paper accounts for this misspecification in the following way:

$$p_t = p_t^* + \mathbf{F}_t^{'}\boldsymbol{\beta} + \xi_t. \tag{5}$$

The residual noise ξ_t captures all the trading frictions that are misspecified by the $\mathbf{F}_t^{'}\boldsymbol{\beta}$ form. The magnitude of ξ_t could also be seen as a measure of the performance of

[2]The ask (bid) depth specifies the maximum quantity for which the ask (bid) price applies.

the liquidity costs $\mathbf{F}_t^{'}\boldsymbol{\beta}$. If ξ_t is small, then $\mathbf{F}_t^{'}\boldsymbol{\beta}$ is a good measure of liquidity costs.

To present the model in discrete time, I introduce the following notation. I consider $n+1$ equidistant price observations at $i = 0, 1, .., n$ over [0,1]. To simplify notation under the infill sampling design, an intraday variable Y_i stands for $Y_{i/n}$. Then, I denote r_i and r_i^* the intraday observed and latent returns $p_i - p_{i-1}$ and $p_i^* - p_{i-1}^*$, respectively. Finally, the first differences or variations of the regressors and the noises are denoted by $\mathbf{X}_i = \mathbf{F}_i - \mathbf{F}_{i-1}$, $\Delta\varepsilon_i = \varepsilon_i - \varepsilon_{i-1}$ and $\Delta\xi_i = \xi_i - \xi_{i-1}$, respectively. Using the model (5), the high-frequency returns are written as

$$r_i = r_i^* + \mathbf{X}_i^{'}\boldsymbol{\beta} + \Delta\xi_i, \ i = 1..n. \tag{6}$$

Next, I turn to the assumptions underlying the frictionless price and liquidity costs. I make the arbitrage-free semimartingale assumption for the frictionless price as in the standard hidden semimartingale model. This one-dimensional price process, which evolves in continuous time over the fixed interval [0,1], is defined on a complete probability space $(\mho, \mathscr{F}, \mathbf{P})$. I consider an information filtration, the increasing family of σ-fields $(\mathscr{F}_t)_{t\in[0,1]} \subseteq \mathscr{F}$, which satisfies the usual conditions of $\mathbf{P}$-completeness and right continuity. The prices and noise explanatory variables are included in the information set $\mathscr{F}_t$.

Assumption 1

The frictionless price p^* follows the dynamics

$$dp_t^* = \mu_t dt + \sigma_t dW_t, \tag{7}$$

where W_t is a standard Brownian motion, σ_t is a *càdlàg* volatility function, independent from the frictionless price (no leverage), and μ_t is the drift coefficient.

I make the following set of assumptions for the different components of the noise.

Assumption 2

(i) $\mathbf{F}_t$ and p_t^* are not independent.

(ii) The increments of $\mathbf{F}_t$ are $O_P(1)$ and $E[\mathbf{F}_t] = \mathbf{0}$.

Essentially, Assumption 2 allows the first component of the noise to be endogenous with the frictionless return, autocorrelated and heteroskedastic[3]. In fact, the return-noise endogeneity is empirically evidenced (see, e.g., Hansen and Lunde 2006) and theoretically modeled (see, e.g., Diebold and Strasser 2013).

Assumption 3

(i) ξ_t is independent from p_t^* and $\mathbf{F}_t$.

(ii) ξ_t is normally identically distributed and $E[\xi_t] = 0$.

In Assumption 3, the residual noise is an exogenous white noise. I explain in Section 3.3 how the normality assumption could be relaxed.

3. FRICTIONLESS-PRICE ESTIMATION

In this section, I estimate the liquidity costs which yield an estimate of the frictionless price, as in (3). To check whether the proposed liquidity-cost model is misspecified, I derive a formal econometric test to distinguish between models (4) and (5).

The idea of the estimation is to write the price-impact regression in (6) such that all latent variables, including the frictionless return, are in the regression's residual:

$$r_i = \mathbf{X}_i'\boldsymbol{\beta} + (r_i^* + \Delta\xi_i), \ i = 1, ..., n. \tag{8}$$

In matrix notation, the regression (8) is written as $\mathbf{r} = \mathbf{X}\boldsymbol{\beta} + \mathbf{r}^* + \Delta\boldsymbol{\xi}$ where $\mathbf{r} = (r_1, .., r_n)'$, $\mathbf{X} = (\mathbf{X}^{(1)'}, .., \mathbf{X}^{(M)'})$, $\mathbf{X}^{(m)} = (X_1^{(m)}, .., X_n^{(m)})$ for $m = 1..M$, and $\Delta\boldsymbol{\xi} =$

[3]For example, the trading volume is highly persistent because of the clustering of small-size trades, heteroskedastic as a result of its U-shaped intraday pattern, and endogenous with the frictionless price as modeled in Glosten and Harris (1988).

$(\Delta\xi_1, .., \Delta\xi_n)'$. Using a similar notation, I also write the price model in matrix form: $\mathbf{p} = \mathbf{p}^* + \boldsymbol{\varepsilon} = \mathbf{p}^* + \mathbf{F}\boldsymbol{\beta} + \boldsymbol{\xi}$.

3.1. *Asymptotic theory*

In this subsection, I show the consistency and the asymptotic normality of the estimator of $\boldsymbol{\beta}$. Let $\widehat{\boldsymbol{\beta}}$ be the ordinary least squares (OLS) estimator of $\boldsymbol{\beta}$, defined by

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{r}. \tag{9}$$

I make the following assumptions.

Assumption A $\frac{\mathbf{X}'\mathbf{X}}{n} \xrightarrow{P} \boldsymbol{\Omega}$, a matrix of rank M.

Assumption B $\mathbf{X}'Var[\mathbf{r}^*|\mathbf{X}]\mathbf{X} \xrightarrow{P} \boldsymbol{\Omega}^*$, a positive definite matrix.

Assumption C $\frac{\mathbf{X}'Var[\Delta\boldsymbol{\xi}]\mathbf{X}}{n} \xrightarrow{P} \mathbf{S}$, a positive definite matrix.

Assumption A concerns the regressors in (8), whereas Assumptions B and C are related to the residual of the price-impact regression.

I next derive the asymptotic theory for the estimator of the liquidity-cost parameters. All proofs are given in the Appendix. Convergence in probability is denoted by $\xrightarrow{P}$, whereas convergence in law is denoted by $\xrightarrow{L}$.

Theorem 1 *Suppose Assumptions 1 and 2 hold.*

(i) If $Var[\xi_t] = 0$*:*

Under Assumptions A and B, $n(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{L} \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}^*\boldsymbol{\Omega}^{-1}\right)$.

(ii) If $Var[\xi_t] \neq 0$*:*

Under Assumptions 3, A and C, $\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{L} \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{\Omega}^{-1}\boldsymbol{S}\boldsymbol{\Omega}^{-1}\right)$.

In Theorem 1 (i), consistency is obtained with a faster rate of convergence than the usual $\sqrt{n}$. In that case, the residual is the frictionless return, which is very small at high frequencies. On the other hand, the noise $\mathbf{X}\boldsymbol{\beta}$ is relatively big. Therefore,

the regression performs well and $\widehat{\boldsymbol{\beta}}$ is supra-convergent. In Stock (1987), the supra-convergence rate is obtained in a similar setting.
For the case where $Var[\xi_t] \neq 0$ in Theorem 1 (ii), I obtain the usual $\sqrt{n}$ rate of convergence because the regression residual $r^* + \Delta\xi$ is $O_P(1)$. The frictionless-return moments do not appear in the asymptotic variance of $\widehat{\boldsymbol{\beta}}$. Indeed, the stochastic magnitude of the frictionless return is negligible compared to $\Delta\xi$.

Once $\boldsymbol{\beta}$ is consistently estimated, $\widehat{\boldsymbol{\varepsilon}} = \mathbf{F}'\widehat{\boldsymbol{\beta}}$ is the liquidity-costs measure proposed in this paper. By subtracting the liquidity-costs measure from the observed prices, I decontaminate the latter from noise and obtain a proxy for the frictionless price. Let the adjusted price $\widehat{p}^*$ and the adjusted return $\widehat{r}^*$ be defined, respectively, as

$$\begin{aligned} \widehat{p}_i^* &= p_i - \mathbf{F}_i'\widehat{\boldsymbol{\beta}}, \\ \widehat{r}_i^* &= r_i - \mathbf{X}_i'\widehat{\boldsymbol{\beta}}. \end{aligned} \tag{10}$$

3.2. *Testing misspecification*

In this subsection, I formally test whether the adjusted returns still have a noise component. The null hypothesis H_0 and the alternative hypothesis H_1 are, respectively,

$$\begin{aligned} &H_0 : Var[\xi_t] = 0, \\ &H_1 : Var[\xi_t] \neq 0. \end{aligned} \tag{11}$$

The idea of the test is that the presence of noise usually causes negative serial correlation in high-frequency returns. The next proposition formally presents this idea.

Proposition 1 *Suppose Assumptions 1 and 2 hold.*
(i) If $Var[\xi_t] = 0$*:*

Under Assumptions A and B, $Cov[\widehat{r}_i^*, \widehat{r}_{i-1}^*] \xrightarrow{P} 0, \ i = 1..n.$

(ii) If $Var[\xi_t] \neq 0$*:*

Under Assumptions 3, A and C, $Cov[\widehat{r}_i^*, \widehat{r}_{i-1}^*] \xrightarrow{P} -Var[\xi_t] < 0, \ i = 1..n.$

According to Proposition 1, if $Var[\xi_t] = 0$, the covariance between successive adjusted returns is asymptotically zero. By contrast, if $Var[\xi_t] \neq 0$, the covariance between successive adjusted returns is asymptotically negative.

I denote by $RC(\widehat{p}^*) = \sum_{i=1}^{n} \widehat{r}_i^* \widehat{r}_{i-1}^*$ the realized autocovariance of order one for the adjusted returns. The test statistic S_n is defined as $S_n = \frac{\sqrt{n} RC(\widehat{p}^*)}{\sqrt{\frac{n}{3} \sum_{i=1}^{n} \widehat{r}_i^{*4}}}$. In the next theorem, I provide the asymptotic distribution of S_n.

Theorem 2

(i) Suppose Assumptions 1, 2, A and B hold.

Under H_0, $S_n \xrightarrow{L} \mathcal{N}(0, 1)$.

(ii) Suppose Assumptions 1-3, A and C hold.

Under H_1, $S_n \xrightarrow{P} \infty$.

According to Theorem 2, I reject H_0 at the confidence level α when $|S_n| > c_{1-\frac{\alpha}{2}}$, where $c_{1-\frac{\alpha}{2}}$ denotes the $1 - \frac{\alpha}{2}$-quantile of the $\mathcal{N}(0, 1)$ distribution.

From a market microstructure perspective, the misspecification test may be interpreted as a test for the quality of the trading-costs measure $\mathbf{F}'\widehat{\boldsymbol{\beta}}$. If this is a good measure of the noise ε, then the residual noise ξ should go to zero. Otherwise, the trading-costs measure does not capture all the frictions and the term ξ does not vanish.

3.3. *Relaxing the normality assumption for* ξ

The normality assumption for the residual noise ξ is critical to deriving the asymptotic distribution in Theorem 1 (ii). Indeed, in the context of the infill asymptotic framework of this paper, Lahiri (1996, section 2.1) shows that for any regression with autocorrelated residuals as in (8), the OLS estimator $\widehat{\boldsymbol{\beta}}$ cannot be consistent for $\boldsymbol{\beta}$

and the asymptotic distribution of $\widehat{\boldsymbol{\beta}}$ is not necessarily Gaussian. While Lahiri does not assume normality for the regression residual, this paper does so in order to derive asymptotic normality under infill asymptotics. For a comparison of infill asymptotics to the more standard increasing-domain asymtotics, see Zhang and L. (2005) where the authors study covariance parameters estimation in Gaussian spatial models.

A natural way to relax the normality assumption for ξ is to apply Generalized Least Squares (GLS) instead of OLS, since the regression residual in (8) is an MA(1) process in the limit with a known covariance matrix, a function only of $Var[\xi_t]$. Thus transforming the regression residual from MA(1) to i.i.d., using infill asymptotics does not cause inconsistency and non-normality of the estimator of $\boldsymbol{\beta}$. However, the regression cannot be transformed before testing whether $Var[\xi_t] = 0$ using the misspecification test in Section 3.2. Once H_0 is rejected, GLS could be applied instead of OLS to estimate $\boldsymbol{\beta}$ consistently, more efficiently and without imposing normality for ξ.

3.4. *Endogeneity analysis*

The consistency results for $\widehat{\boldsymbol{\beta}}$ in Theorem 1 would not be achievable in a standard setting because of endogenous regression residual; instrumental variables would be needed to achieve consistency. However, in the setting of this paper, the endogeneity of the residual does not cause inconsistency because its source - r^* - is asymptotically negligible. In finite sample, to consistently estimate $\boldsymbol{\beta}$, it might be important to use instruments. The lag of the regressor $\mathbf{X}$ would be a valid instrument.

In Proposition 2, I derive some properties for the return-noise covariance and I show that $Cov[r_i, \mathbf{X}_i'\widehat{\boldsymbol{\beta}}] > Cov[\widehat{r}_i^*, \mathbf{X}_i'\widehat{\boldsymbol{\beta}}]$ asymptotically.

Proposition 2 *Suppose Assumptions 1 and 2 hold. For i=1..n,*

(i) If $Var[\xi_t] = 0$*:*

Under Assumptions A and B,

$Cov[r_i, \boldsymbol{X}_i'\widehat{\boldsymbol{\beta}}] = Var[\boldsymbol{X}_i'\boldsymbol{\beta}] + Cov[r_i^*, \Delta\varepsilon_i] + O_P(1/n),$

$Cov[\widehat{r}_i^*, \boldsymbol{X}_i'\widehat{\boldsymbol{\beta}}] = Cov[r_i^*, \Delta\varepsilon_i] + O_P(1/n).$

(ii) If $Var[\xi_t] \neq 0$*:*

Under Assumptions 3, A and C,

$Cov[r_i, \boldsymbol{X}_i'\widehat{\boldsymbol{\beta}}] = Var[\boldsymbol{X}_i'\boldsymbol{\beta}] + Cov[r_i^*, \Delta\varepsilon_i] + O_P(1/\sqrt{n}),$

$Cov[\widehat{r}_i^*, \boldsymbol{X}_i'\widehat{\boldsymbol{\beta}}] = Cov[r_i^*, \Delta\varepsilon_i] + O_P(1/\sqrt{n}).$

To give an economic intuition for the return-noise endogeneity, I argue that it is due to informational frictions. For instance, in the asymmetric-information models of Glosten and Harris (1988) and Hasbrouck (1991), the trading volume captures the adverse selection in the efficient (or the frictionless) price. Therefore, having the volume as part of both the frictionless price and the liquidity costs results in the endogeneity between these two components. Moreover, in Glosten and Harris (1988), the trade-direction indicator is present in the efficient price as well as the liquidity costs (see Huang and Stoll 1997). So, this trade indicator is also a source of endogeneity of the noise.

4. APPLICATION TO VOLATILITY ESTIMATION

The object of interest in this section is the integrated variance, defined as

$$IV = \int_0^1 \sigma_u^2 du. \tag{12}$$

I denote the realized variance by $RV(.)$ of a given process. If the frictionless return were observed, then the realized variance $RV(p^*) = \sum_{i=1}^n r_i^{*2}$ would be a consistent estimator of IV, as first shown in Meyer (1967). However, the realized variance based on observed prices, $RV(p) = \sum_{i=1}^n r_i^2$, is inconsistent for IV because of the market microstructure noise.

The first consistent IV estimator which is robust to noise is the two time scales estimator of Zhang et al. (2005). This estimator relies on standard noise assumptions rather than endogenous, autocorrelated and heteroskedastic noise, as in this paper. In the same line of nonparametric volatility estimators, Barndorff-Nielsen et al. (2008a) and Jacod et al. (2009) derive the kernel and the pre-averaging estimators, respectively. All the mentioned three estimators are inconsistent for IV if applied to the price model (5) because of the endogeneity in the noise[4]. However, the noise heteroskedasticy is allowed only for the pre-averaging estimator. As for the autocorrelation in the noise, it could be accommodated for in the two time scales estimator and the pre-averaging estimator; see the extensions in Aït-Sahalia et al. (2011) and Hautsch and Podolskij (2013), respectively.

In this section, I propose a novel volatility estimator - based on adjusted prices defined in (10) - which relaxes the underlying noise assumptions of the aforementioned nonparametric volatility estimators. Improved volatility estimation is due to the fact that the adjusted price $\widehat{p}^*$ is closer to the frictionless price p^*. And more importantly, $\widehat{p}^*$ fits the assumptions justifying the use of model-free volatility estimators better than p .

In Theorem 3, I show that if the liquidity costs are fully removed or H_0 is not rejected, the realized variance based on adjusted returns, $RV(\widehat{p}^*) = \sum_{i=1}^n \widehat{r}_i^{*2}$, is consistent for IV. For the case where the liquidity costs are only partially absorbed or H_0 is rejected, a robust-to-noise volatility estimator is needed. I show that, based on adjusted rather than observed prices, the two time scale estimator - denoted by $RV^{tts}(\widehat{p}^*)$ - becomes consistent for IV.

$RV^{tts}(\widehat{p}^*)$ is defined in the Appendix, in the proof of Theorem 3. For mixed normal-limit distributions, I denote the stable convergence[5] as $\xrightarrow{st}$.

Theorem 3 *Suppose Assumptions 1 and 2 hold.*

(i) Suppose $Var[\xi_t] = 0$*:*

Under Assumptions A and B, $\sqrt{n}(RV(\widehat{p}^*) - IV) \xrightarrow{st} \mathcal{N}(0, 2\ IQ)$,

[4]See Li and Mykland (2007) for a sensitivity analysis of a specific endogeneity form.

[5]The stable convergence concept is discussed in Aldous and Eagleson (1978).

where $IQ = \int_0^1 \sigma_u^4 du$.

(ii) Suppose $Var[\xi_t] \neq 0$*:*

Under Assumptions 3, A, C and if the increments of $\boldsymbol{F}_t$ *have bounded fourth moments,*
$n^{1/6}\left(RV^{tts}(\widehat{p}^*) - IV\right) \xrightarrow{st} \mathcal{N}(0, \Gamma_\xi)$,
where $\Gamma_\xi = \frac{8}{c^2}E[\xi^2]^2 + c\frac{4}{3}IQ$, $c = \left(\frac{1}{12E[\xi^2]^2}IQ\right)^{-1/3}$.

According to Theorem 3, the estimation error in $\hat{\boldsymbol{\beta}}$ impacts neither the consistency nor the asymptotic distribution of the estimator based on the adjusted returns. In case (i), the result is similar to that where p^* is observed. And for case (ii), the result is similar to that where $p^* + \xi$ is observed.

The new volatility estimator $RV(\widehat{p}^*)$ of Theorem 3 (i) has no tuning parameters as would be the case if a robust-to-noise volatility estimator had to be used. The result is obtained because the adjusted return could be written as $\widehat{r}^* = r^* + o_P(1/n)$, where the $o_P(1/n)$ term is a consequence of the rate of convergence of $\hat{\boldsymbol{\beta}}$ in Theorem 1 (i). Finally, the rate of convergence of $RV(\widehat{p}^*)$ is $\sqrt{n}$, which is not achievable using any robust-to-noise volatility estimator. Indeed, Gloter and Jacod (2001) show that the rate of convergence of any robust-to-noise integrated volatility estimator is bounded by $n^{-1/4}$, where n is the sample size.

In Theorem 3 (ii), the new volatility estimator $RV^{tts}(\widehat{p}^*)$ has a limit distribution as if the noise were ξ - exogenous, independent and identically distributed - instead of ε - endogenous[6], autocorrelated and heteroskedastic. The idea of the two time scales estimator is to combine a slow-scale sampling return frequency (to mitigate the noise effect) with a high-scale one (to correct the noise-induced bias). The same idea still holds if the adjusted returns are used. First, for the slow-scale sampling return, low-frequency adjusted returns contain an $o_P(1/\sqrt{n})$ term resulting from the estimation of $\hat{\boldsymbol{\beta}}$ in Theorem 1 (ii). This term is negligible compared to low-frequency frictionless returns, which are $o_P(1/\sqrt{\overline{n}})$, where $\overline{n}$ is the slow frequency verifying $\overline{n} < n$. Second,

[6]Within the statistical approach of the nonparametric estimation of volatility, Kalnina and Linton (2008) propose an alternative specification of the return-noise endogeneity.

for the high-scale frequency, high-frequency adjusted returns also contain an $o_P(1/\sqrt{n})$ term resulting from $\hat{\boldsymbol{\beta}}$ estimation in Theorem 1 (ii). This term is negligible compared to the residual noise component $\Delta\xi$, which is $O_P(1)$. Therefore, the bias correction is not impacted.

To summarize, the new semiparametric volatility estimator is given by

$$\begin{aligned} IV^{new}(p) &= RV(\widehat{p}^{*}) \quad if\ Var[\xi_t] = 0, \\ &= RV^{tts}(\widehat{p}^{*}) \quad if\ Var[\xi_t] \neq 0. \end{aligned} \tag{13}$$

Confidence intervals for $IV^{new}(p)$ are computed using Theorem 3. And, as in Gonçalves and Meddahi (2009), more accurate confidence intervals could be obtained using the bootstrap method.

5. EMPIRICAL ILLUSTRATION

In this section, I assess with data the performance of the model presented in Section 2. I use Alcoa Aluminum stock, listed on the NYSE, and the data cover the 2010 period. To clean this data, I apply the same procedure as in Barndorff-Nielsen et al. (2008b). I use five explanatory variables to capture the liquidity costs: the inferred trade-direction indicator[7], trading volume, bid-ask spread, bid depth and ask depth.

For the misspecification test of Theorem 2, I find that for 139 business days out of 252, the test is not rejected, implying that the liquidity-cost measure absorbs all the noise in more than half of the sample. Figure 1 shows the first-order realized autocovariance of the observed returns and adjusted returns. Consistently with Proposition 1, the stylized fact of the negativity of the first-order autocovariance of the high-

[7] I infer the binary series q_t from observed trade and quote prices using the Lee and Ready (1991) trade classification algorithm. Trade classification requires that the trade series be matched with the quote series because in the Trade and Quote (TAQ) database the two series are offered separately. I match trades and quotes by assuming a zero time lag.

frequency returns disappears, or at least becomes much less pronounced, by adjusting the returns for liquidity costs.

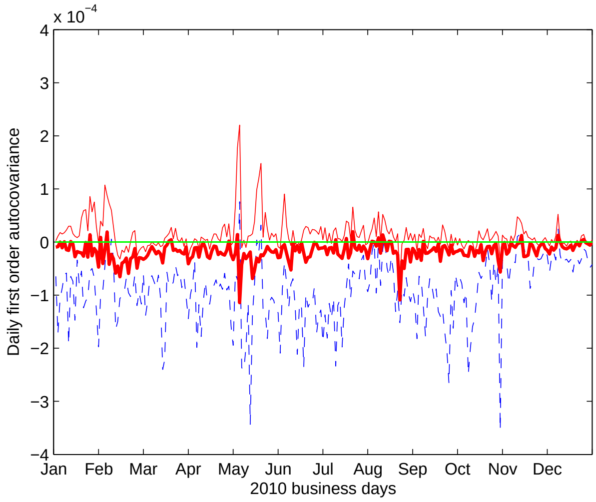

Figure 1: The daily first-order realized autocovariance for Alcoa, 2010. The autocovariance involving the following returns: original, $\widehat{\boldsymbol{\beta}}$-adjusted, $\widehat{\boldsymbol{\beta}}_{inst}$-adjusted, are plotted in the dashed lines, solid thin line, solid thick line, respectively.

Figure 1 also shows the first-order realized autocovariance of the adjusted returns using the lag of $\mathbf{X}$ as instrumental variable to estimate $\boldsymbol{\beta}$. Indeed, as mentioned in Section 3.4, using instrumental variables might be important in finite sample. The adjusted return expression becomes $\widehat{r}_i^* = r_i - \mathbf{X}_i^{'}\widehat{\boldsymbol{\beta}}_{inst}$ where $\widehat{\boldsymbol{\beta}}_{inst} = (\mathbf{Z}^{'}\mathbf{X})^{-1}\mathbf{Z}^{'}\mathbf{r}$ and $\mathbf{Z}$ denotes the lag of $\mathbf{X}$. $\mathbf{Z}$ is a valid instrument since $\mathbf{Z}^{'}\mathbf{X}$ is nonsingular - as the liquidity-cost variables are persistent - and it is uncorrelated with the regression residual at high frequencies - as a consequence of the p^* semimartingale property. For Alcoa 2010, the first-order realized autocovariance using adjusted returns with instrumental variables is mostly negative and close to zero. This is an evidence that any residual noise would be an exogenous white noise. However, OLS-based $RC(\widehat{p}^*)$ could be positive in contradiction with a residual noise that is an exogenous white noise.

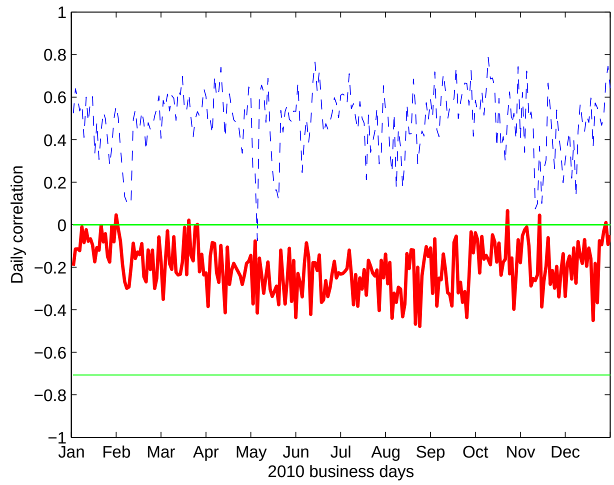

Figure 2: The daily realized correlation of the estimated noise and the returns for Alcoa, 2010. The correlation involving the original (adjusted) return is plotted in the dashed lines (solid line). The horizontal lines are the $-1/\sqrt{2}$ and 0 bounds of Diebold and Strasser (2013).

Finally, Figure 2 plots the correlation of the returns and the fitted noise $\mathbf{X}'\widehat{\boldsymbol{\beta}}_{inst}$, using observed returns and $\widehat{\boldsymbol{\beta}}_{inst}$-based adjusted returns. I also plot the return-noise correlation bounds derived by proposition 4 of Diebold and Strasser (2013). The authors find that the return-noise correlation is between $-1/\sqrt{2}$ and 0 for a one-period model of market making. In Figure 2, the return-noise correlation computed using observed returns is positive, whereas the return-noise correlation based on adjusted returns is mostly in the interval $[-1/\sqrt{2}, 0]$, consistent with the theoretical result of Diebold and Strasser (2013). I also find that, for all 2010's business days, $\sum_{i=1}^{n} r_i \mathbf{X}_i'\widehat{\boldsymbol{\beta}} > \sum_{i=1}^{n} \widehat{r}_i^* \mathbf{X}_i'\widehat{\boldsymbol{\beta}}$ as stated in Proposition 2.

6. CONCLUSION

In light of the market microstructure literature that provides economic drivers for trading frictions or noise, I propose a semiparametric price model. Thus, by exploiting a much bigger set of available trade and quote data, I estimate the frictionless price. I derive a new volatility measure using the estimated frictionless price. Compared to traditional robust-to-noise volatility estimators, this new volatility estimator does not rely on the absence of endogeneity for the noise, and allows by construction for heteroskedastic and autocorrelated noise. Moreover, if the noise is completely removed by the liquidity-cost variables considered, then the new volatility estimator is as accurate as if the frictionless price were observed.

There are many possible extensions to this work. Potentially, a nonlinear or index model of liquidity costs would capture more noise than a linear one. Indeed, nonlinearities are well documented in market microstructure theory. Another extension would be to add jumps to the frictionless-price dynamics. There is evidence of jumps in the data, so accounting for discontinuities should be explored. In this paper, I focus on integrated volatility estimation, but the approach could improve the measurement of intraday quantities such as spot volatility, powers of volatility, the leverage effect and integrated betas in a multivariate setting. These extensions would broaden the applicability of my approach to portfolio allocation, risk management and asset evaluation.

APPENDIX: PROOFS OF RESULTS

Proof of Theorem 1

By substituting the return expression into the definition of $\widehat{\boldsymbol{\beta}}$ given in (9), I obtain

$$\begin{aligned}\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{r} - \boldsymbol{\beta} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{r}^* + \mathbf{X}\boldsymbol{\beta} + \Delta\boldsymbol{\xi}) - \boldsymbol{\beta} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{r}^* + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Delta\boldsymbol{\xi} \\ &= \left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1}\frac{\mathbf{X}'\mathbf{r}^*}{n} + \left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1}\frac{\mathbf{X}'\Delta\boldsymbol{\xi}}{n}.\end{aligned} \tag{A.1}$$

- **Consistency**:

(i) If $Var[\xi_t] = 0$, then equation (A.1) becomes

$$\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} = \left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1} \frac{\mathbf{X}'\mathbf{r}^*}{n} + o_P(1). \tag{A.2}$$

The non negligible term of (A.2) is the product of $\frac{\mathbf{X}'\mathbf{X}}{n}$ - which has a finite limit, Ω, according to Assumption A - and the vector $\mathbf{X}'\mathbf{r}^*$, given by

$$\mathbf{X}'\mathbf{r}^* = \sum_{i=1}^{n} \mathbf{X}_i r_i^* = \begin{pmatrix} \sum_{i=1}^n X_i^{(1)} r_i^* \\ \vdots \\ \sum_{i=1}^n X_i^{(M)} r_i^* \end{pmatrix}. \tag{A.3}$$

I apply the Cauchy-Schwartz inequality for each element of the vector $\mathbf{X}'\mathbf{r}^*$, for m=1..M,

$$\left(\frac{\sum_{i=1}^n X_i^{(m)} r_i^*}{n}\right)^2 \leq \frac{1}{n}\left(\frac{\sum_{i=1}^n X_i^{(m)2}}{n}\right)\left(\sum_{i=1}^{n} r_i^{*2}\right). \tag{A.4}$$

Using Assumption A, $\frac{\sum_{i=1}^n X_i^{(m)2}}{N} \to \boldsymbol{\Omega}(m,m)$, where $\boldsymbol{\Omega}(m,m)$ is the m^{th} diagonal element of the matrix $\boldsymbol{\Omega}$. On the other hand, the realized variance $\sum_{i=1}^n r_i^{*2} \xrightarrow{P} IV$. Therefore $\frac{\mathbf{X}'\mathbf{r}^*}{n} \xrightarrow{P} 0$, which implies along with (A.2) and Assumption A that $\widehat{\boldsymbol{\beta}} \xrightarrow{P} \boldsymbol{\beta}$.

(ii) If $Var[\xi_t] \neq 0$, I show that both terms in (A.1) converge to zero. For the first term, I use the consistency result (i) demonstrated above; $\left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1} \frac{\mathbf{X}'\mathbf{r}^*}{n} = o_p(1)$. For the second term - $\left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1} \frac{\mathbf{X}'\Delta\boldsymbol{\xi}}{n}$ - I need to show that $\frac{\mathbf{X}'\Delta\boldsymbol{\xi}}{N} \xrightarrow{P} 0$ to obtain the consistency result since $\frac{\mathbf{X}'\mathbf{X}}{n} \xrightarrow{P} \boldsymbol{\Omega}$ according to Assumption A. By applying the Law of Large numbers to each element $\frac{\sum_{i=1}^n X_i^m \Delta\xi_i}{n}$ of the sample mean, I obtain the outcome $\frac{\mathbf{X}'\Delta\boldsymbol{\xi}}{n} \xrightarrow{P} E[\mathbf{X}_t \Delta\xi_t]$. The limit is zero since $E[\mathbf{X}_t\Delta\xi_t] = E[\mathbf{X}_t]E[\Delta\xi_t] = 0$ using Assumption 3 (i) and (ii).

- **Asymptotic normality**:

In both cases (i) and (ii), the regression residual has a normal distribution because of the normality of r^* in Assumption 1 and the normality of ξ in Assumption 3 (ii). Therefore,

$$\widehat{\boldsymbol{\beta}} \sim \mathcal{N}(E[\widehat{\boldsymbol{\beta}}], Var[\widehat{\boldsymbol{\beta}}]). \tag{A.5}$$

Now, I turn to the derivation of the asymptotic variance for each case.

(i) If $Var[\xi_t] = 0$, (A.1) yields

$$\begin{aligned} &Var[n(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})] = Var[n(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{r}^*] \\ &= (\frac{\mathbf{X}'\mathbf{X}}{n})^{-1} Var[\mathbf{X}'\mathbf{r}^*]((\frac{\mathbf{X}'\mathbf{X}}{n})^{-1})' \\ &= \underbrace{(\frac{\mathbf{X}'\mathbf{X}}{n})^{-1}}\underbrace{\left(\mathbf{X}' Var[\mathbf{r}^*|\mathbf{X}]\mathbf{X}\right)}\underbrace{(\frac{\mathbf{X}'\mathbf{X}}{n})^{-1}} \\ &\xrightarrow{P} \boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}^*\boldsymbol{\Omega}^{-1}, \end{aligned} \tag{A.6}$$

which follows from Assumptions A and B.

(ii) If $Var[\xi_t] \neq 0$, (A.1) gives

$$\begin{aligned} &Var[\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})] = Var[\frac{1}{\sqrt{n}} n(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{r}^* + \sqrt{n}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Delta\boldsymbol{\xi}] \\ &= \frac{1}{n} Var[n(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{r}^*] + Var[\sqrt{n}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Delta\boldsymbol{\xi}], \end{aligned} \tag{A.7}$$

using Assumption 3 (i). The first term in (A.7) vanishes as a result of (A.6). For the second term,

$$\begin{aligned} &Var[\sqrt{n}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Delta\xi] = (\frac{\mathbf{X}'\mathbf{X}}{n})^{-1}\frac{Var[\mathbf{X}'\Delta\boldsymbol{\xi}]}{n}((\frac{\mathbf{X}'\mathbf{X}}{n})^{-1})' \\ &= \underbrace{(\frac{\mathbf{X}'\mathbf{X}}{n})^{-1}}\underbrace{\frac{\mathbf{X}' Var[\Delta\boldsymbol{\xi}]\mathbf{X}}{n}}\underbrace{(\frac{\mathbf{X}'\mathbf{X}}{n})^{-1}} \\ &\xrightarrow{P} \boldsymbol{\Omega}^{-1}\mathbf{S}\boldsymbol{\Omega}^{-1}, \end{aligned} \tag{A.8}$$

following from Assumptions A and C.

Proof of Proposition 1

(i) If $Var[\xi_t] = 0$,

$$\begin{aligned}
&Cov[\widehat{r}_i^*, \widehat{r}_{i-1}^*] = Cov[r_i^* + \mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}), r_{i-1}^* + \mathbf{X}_{i-1}'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})] \\
&= \underbrace{Cov[r_i^*, \widehat{r}_{i-1}^*]}_{=0} + Cov[\mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}), r_{i-1}^*] + Cov[\mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}), \mathbf{X}_{i-1}'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})] \\
&= Cov[\mathbf{X}_i', r_{i-1}^*] \underbrace{(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})}_{o_P(1/n)} + \underbrace{(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})'}_{o_P(1/n)} Cov[\mathbf{X}_i, \mathbf{X}_{i-1}'] \underbrace{(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})}_{o_P(1/n)} \\
&\xrightarrow{P} 0,
\end{aligned} \tag{A.9}$$

as a result of the properties of the semimartingale in Assumption 1 and Theorem 1 (i).

(ii) If $Var[\xi_t] \neq 0$,

$$\begin{aligned}
&Cov[\widehat{r}_i^*, \widehat{r}_{i-1}^*] = Cov[r_i^* + \mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) + \Delta\xi_i, r_{i-1}^* + \mathbf{X}_{i-1}'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) + \Delta\xi_{i-1}] \\
&= \underbrace{Cov[r_i^*, \widehat{r}_{i-1}^*]}_{=0} + Cov[\mathbf{X}_i', r_{i-1}^*] \underbrace{(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})}_{o_P(1/\sqrt{n})} + \underbrace{(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})'}_{o_P(1/\sqrt{n})} Cov[\mathbf{X}_i, \mathbf{X}_{i-1}'] \underbrace{(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})}_{o_P(1/\sqrt{n})} \\
&+ \underbrace{Cov[\mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}), \Delta\xi_{i-1}]}_{=0} + \underbrace{Cov[\Delta\xi_i, r_{i-1}^* + \mathbf{X}_{i-1}'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})]}_{=0} + \underbrace{Cov[\Delta\xi_i, \Delta\xi_{i-1}]}_{=-Var[\xi_t]} \\
&\xrightarrow{P} -Var[\xi_t] < 0,
\end{aligned} \tag{A.10}$$

using the properties of the semimartingale in Assumption 1 along with Assumption 3 (i) and Theorem 1 (ii).

Proof of Theorem 2

Recall,

$$\begin{aligned}
\widehat{r}_i^* &= r_i - \mathbf{X}_i'\widehat{\boldsymbol{\beta}} \\
&= r_i^* + \mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) + \Delta\xi_i.
\end{aligned} \tag{A.11}$$

(i) Under H_0, (A.11) and Theorem 1 (i) yield that $\widehat{r}_i^* = r_i^* + o_P(1/n)$. Applying Example 5 from page 1062 of Kinnebrock and Podolskij (2008) for the $\widehat{r}^*$, I obtain

$$\sqrt{n}\sum_{i=1}^{n}\widehat{r}_i^*\widehat{r}_{i-1}^* \xrightarrow{st} \mathcal{MN}(0, IQ), \tag{A.12}$$

where $IQ = \int_0^1 \sigma_u^4 du$. A consistent estimator of IQ under H_0 is given by $\frac{n}{3}\sum_{i=1}^{n}\widehat{r}_i^{*4}$.

(ii) Under H_1, (A.11) and Theorem 1 (ii) imply that $\widehat{r}_i^* = \Delta\xi_i + o_P(1/\sqrt{n})$. Then, I apply Theorem 1 of Barndorff-Nielsen et al. (2008a) to obtain

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{r}_i^*\widehat{r}_{i-1}^* \xrightarrow{L} \mathcal{N}(0, 5E[\xi_t^2]^2 + Var[\xi_t^2]). \tag{A.13}$$

Therefore $\sqrt{n}RC(\widehat{p}^*) \to \infty$ and the result follows.

Proof of Proposition 2

(i) If $Var[\xi_t] = 0$,

$$\begin{aligned} Cov[r_i, \mathbf{X}_i'\widehat{\boldsymbol{\beta}}] &= Cov[r_i^* + \mathbf{X}_i'\boldsymbol{\beta}, \mathbf{X}_i'\widehat{\boldsymbol{\beta}}] \\ &= Cov[r_i^* + \mathbf{X}_i'\boldsymbol{\beta}, \mathbf{X}_i'\underbrace{(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})}_{=o_P(1/n)} + \mathbf{X}_i'\boldsymbol{\beta}] \\ &= Var[\mathbf{X}_i'\boldsymbol{\beta}] + Cov[r_i^*, \Delta\varepsilon_i] + o_P(1/n), \end{aligned} \tag{A.14}$$

using Assumptions 1 and 2 and Theorem 1 (i).

$$\begin{aligned} Cov[\widehat{r}_i^*, \mathbf{X}_i'\widehat{\boldsymbol{\beta}}] &= Cov[r_i^* + \mathbf{X}_i'\underbrace{(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})}_{=o_P(1/n)}, \mathbf{X}_i'\boldsymbol{\beta} + \mathbf{X}_i'\underbrace{(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})}_{=o_P(1/n)}] \\ &= Cov[r_i^*, \Delta\varepsilon_i] + o_P(1/n), \end{aligned} \tag{A.15}$$

using Assumptions 1 and 2, and Theorem 1 (i).

(ii) If $Var[\xi_t] \neq 0$,

$$\begin{aligned} Cov[r_i, \mathbf{X}_i^{'}\widehat{\boldsymbol{\beta}}] &= Cov[r_i^* + \mathbf{X}_i^{'}\boldsymbol{\beta} + \Delta\xi_i, \mathbf{X}_i^{'}\widehat{\boldsymbol{\beta}}] \\ &= Cov[r_i^* + \mathbf{X}_i^{'}\boldsymbol{\beta} + \Delta\xi_i, \mathbf{X}_i^{'}\underbrace{(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})}_{=o_P(1/\sqrt{n})} + \mathbf{X}_i^{'}\boldsymbol{\beta}] \\ &= Var[\mathbf{X}_i^{'}\boldsymbol{\beta}] + Cov[r_i^*, \Delta\varepsilon_i] + o_P(1/\sqrt{n}), \end{aligned} \tag{A.16}$$

using Assumptions 1, 2 and 3, and Theorem 1 (ii).

$$\begin{aligned} Cov[\widehat{r}_i^*, \mathbf{X}_i^{'}\widehat{\boldsymbol{\beta}}] &= Cov[r_i^* + \mathbf{X}_i^{'}\underbrace{(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})}_{=o_P(1/\sqrt{n})} + \Delta\xi_i, \mathbf{X}_i^{'}\boldsymbol{\beta} + \mathbf{X}_i^{'}\underbrace{(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})}_{=o_P(1/\sqrt{n})}] \\ &= Cov[r_i^*, \Delta\varepsilon_i] + o_P(1/\sqrt{n}), \end{aligned} \tag{A.17}$$

using Assumptions 1, 2 and 3, and Theorem 1 (ii).

Proof of Theorem 3

(i) Assume that $Var[\xi_t] = 0$.

$$\begin{aligned} &\sqrt{n}(RV(\widehat{p}^*) - IV) \\ &= \underbrace{\sqrt{n}\,(RV(p^*) - IV)}_{} + \underbrace{\sqrt{n}\sum_{i=1}^{n}(\mathbf{X}_i^{'}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}))^2}_{} + \underbrace{2\sqrt{n}\sum_{i=1}^{n} r_i^*\mathbf{X}_i^{'}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})}_{}. \end{aligned} \tag{A.18}$$

The first term of (A.18) is the usual term: $\sqrt{n}(RV(\widehat{p}^*) - IV) \xrightarrow{st} \mathcal{MN}(0, 2\,IQ)$ as shown in Barndorff-Nielsen and Shephard (2002). In the following, I show that the second and third terms are negligible.

For the second term of (A.18),

$$\begin{aligned} \sqrt{n}\sum_{i=1}^{n}(\mathbf{X}_i^{'}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}))^2 &= \sqrt{n}\sum_{i=1}^{n}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^{'}\mathbf{X}_i\mathbf{X}_i^{'}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) \\ &= n\sqrt{n}\underbrace{(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^{'}}_{o_P(1/n)}\underbrace{(\frac{\sum_{i=1}^{n}\mathbf{X}_i\mathbf{X}_i^{'}}{n})}_{O_P(1)}\underbrace{(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})}_{o_P(1/n)} \\ &= o_P(1), \end{aligned} \tag{A.19}$$

using Assumption A and Theorem 1 (i).

For the last term of (A.18),

$$2\sqrt{n}\sum_{i=1}^{n} r_i^* \mathbf{X}_i'(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}) = 2\underbrace{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} r_i^* \mathbf{X}_i'}_{O_P(1)}\underbrace{N(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}_{o_P(1)} = o_P(1), \tag{A.20}$$

using the Cauchy-Schwartz inequality as in (A.4) and Theorem 1 (i).

(ii) In this proof, I use the original notation for indices $Y_{i/n}$ instead of the simplified notation Y_i for a given process Y.

The two time scales estimator of Zhang et al. (2005) applied to the adjusted price $\widehat{p}^*$ is given by

$$RV^{tts}(\widehat{p}^*) = RV^{avg}(\widehat{p}^*) - \frac{\overline{n}}{n} RV(\widehat{p}^*), \tag{A.21}$$

where

$$RV^{avg}(\widehat{p}^*) = \frac{1}{K}\sum_{k=1}^{K} RV^{(k)}(\widehat{p}^*), \tag{A.22}$$

and

$$RV^{(k)}(\widehat{p}^*) = \sum_{i=1}^{n_k} (\widehat{p}^*_{k-1+K\frac{i}{n}} - \widehat{p}^*_{k-1+K\frac{i-1}{n}})^2,\ n_k = Floor\left(\frac{n-k+2}{K}\right), \tag{A.23}$$

$$\overline{n} = \frac{1}{K}\sum_{k=1}^{K} n_k. \tag{A.24}$$

The bias $RV^{tts}(\widehat{p}^*) - IV$ could be decomposed as

$$RV^{tts}(\widehat{p}^*) - IV = \underbrace{(RV^{tts}(\widehat{p}^*) - RV^{avg}(p^*))}_{error\ due\ to\ the\ noise} + \underbrace{(RV^{avg}(p^*) - IV)}_{error\ due\ to\ discretization} \tag{A.25}$$

My objective is to show

1. for the bias due to noise:

$$\sqrt{\frac{K}{\overline{n}}}\left(RV^{tts}(\widehat{p}^*) - RV^{avg}(p^*)\right) \xrightarrow{L} \mathcal{N}(0, 8(E[\xi^2])^2), \tag{A.26}$$

2. and for the bias due to discretization:

$$\sqrt{\frac{n}{K}}(RV^{avg}(p^*) - IV) \xrightarrow{st} \mathcal{MN}(0, \frac{4}{3}\ IQ). \tag{A.27}$$

Combining the two sources of errors (A.26) and (A.27) so that each will be present at the limit leads to $\frac{K}{\overline{n}}$ proportional to $\frac{n}{K}$, as in Zhang et al. (2005). Taking

$$\begin{aligned} K &= cn^{2/3}, \\ \overline{n} &= c^{-1}n^{1/3}, \end{aligned} \tag{A.28}$$

implies that neither source of error will dominate at the limit. The constant c is a tuning parameter which could be optimally determined. Let $\Gamma_\xi = \frac{8}{c^2}E[\xi^2]^2 + c\frac{4}{3}IQ$, the limiting distribution is given by $n^{1/6}\left(RV^{tts}(\widehat{p}^*) - IV\right) \xrightarrow{st} \mathcal{N}(0, \Gamma_\xi)$. For the optimal choice of c, minimizing the asymptotic variance Γ_ξ leads to $c = \left(\frac{1}{12E[\xi^2]^2}IQ\right)^{-1/3}$.

Let's start by showing (A.26). The error due to the noise of (A.25) is written as

$$\begin{aligned} \sqrt{\frac{K}{\overline{n}}}(RV^{tts}(\widehat{p}^*) - RV^{avg}(p^*)) &= \sqrt{\frac{K}{\overline{n}}}(RV^{avg}(\widehat{p}^*) - \frac{\overline{n}}{n}RV(\widehat{p}^*) - RV^{avg}(p^*)) \\ &= \sqrt{\frac{K}{\overline{n}}}(RV^{avg}(\widehat{p}^*) - RV^{avg}(p^*) - 2\overline{n}E[\xi_t^2] + 2\overline{n}E[\xi_t^2] - \frac{\overline{n}}{n}RV(\widehat{p}^*)) \\ &= \underbrace{\sqrt{\frac{K}{\overline{n}}}(RV^{avg}(\widehat{p}^*) - RV^{avg}(p^*) - 2\overline{n}E[\xi_t^2])} - \underbrace{2\sqrt{K\overline{n}}\left(\frac{RV(\widehat{p}^*)}{2n} - E[\xi_t^2]\right)}. \end{aligned} \tag{A.29}$$

The following Lemma is needed, which is in the same spirit as Lemma A.2 in Zhang et al. (2005).

Lemma 1 *Under assumptions 1, 2 and 3,*

$$\begin{aligned} &(i)\ RV(\widehat{p}^*) = RV(\xi) + O_p(1), \\ &(ii)\ RV^{avg}(\widehat{p}^*) - RV^{avg}(p^*) = RV^{avg}(\xi) + O_p\left(\frac{1}{\sqrt{K}}\right). \end{aligned}$$

Proof of Lemma 1:

(i) The realized variance of the adjusted price is given by

$$
\begin{aligned}
RV(\widehat{p}^*) &= \sum_{i=1}^{n} \widehat{r}_i^{*2} \\
&= \sum_{i=1}^{n} \left(r_i^* + \mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) + \Delta\xi_i \right)^2 \\
&= \sum_{i=1}^{n} r_i^{*2} + \sum_{i=1}^{n} \left(\mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) \right)^2 + \sum_{i=1}^{n} (\Delta\xi_i)^2 \\
&+ 2\sum_{i=1}^{n} r_i^* \Delta\xi_i + 2\sum_{i=1}^{n} r_i^* \mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) + 2\sum_{i=1}^{n} \mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})\Delta\xi_i.
\end{aligned} \tag{A.30}
$$

Since $\sum_{i=1}^{n} r_i^{*2} = O_P(1)$, $\sum_{i=1}^{n} (\Delta\xi_i)^2 = RV(\xi)$ and $2\sum_{i=1}^{n} r_i^* \Delta\xi_i = o_P(1)$ using Assumption 3 then, (A.30) becomes

$$
\begin{aligned}
RV(\widehat{p}^*) &= RV(\xi) + O_P(1) \\
&+ \sum_{i=1}^{n} \left(\mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) \right)^2 + 2\sum_{i=1}^{n} r_i^* \mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) + 2\sum_{i=1}^{n} \mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})\Delta\xi_i.
\end{aligned} \tag{A.31}
$$

In the following, I show that each of the last three terms of (A.31) is $o_P(1)$ which implies the order $O_P(1)$. First:

$$
\begin{aligned}
\sum_{i=1}^{n} \left(\mathbf{X}_i'(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) \right)^2 &= trace\left(\sum_{i=1}^{n} (\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})' \mathbf{X}_i \mathbf{X}_i' (\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) \right) \\
&= trace\left((\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})' \mathbf{X}' \mathbf{X} (\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) \right) \\
&= trace\left(\underbrace{\sqrt{n}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})'}_{o_P(1)} \underbrace{\left(\frac{\mathbf{X}'\mathbf{X}}{n} \right)}_{O_P(1)} \underbrace{\sqrt{n}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})}_{o_P(1)} \right) \\
&= o_P(1),
\end{aligned} \tag{A.32}
$$

using Assumption A and Theorem 1 (ii).

Second:

$$\sum_{i=1}^{n} r_i^* \mathbf{X}_i^{'}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) = \underbrace{\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} r_i^* \mathbf{X}_i^{'} \right)}_{O_P(1)} \underbrace{\sqrt{n}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})}_{o_P(1)} = o_P(1), \tag{A.33}$$

using (A.20) and Theorem 1 (ii).

Third:

$$\sum_{i=1}^{n} \mathbf{X}_i^{'}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) \Delta\xi_i = \underbrace{\left(\sum_{i=1}^{n} \mathbf{X}_i^{'} \Delta\xi_i \right)}_{O_P(\sqrt{n})} \underbrace{(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})}_{o_P(1/\sqrt{n})} = o_P(1). \tag{A.34}$$

using Assumption C and Theorem 1 (ii).

(ii) I need to show that $RV^{avg}(\widehat{p}^*) - RV^{avg}(p^*) = RV^{avg}(\xi) + O_p\left(\frac{1}{\sqrt{K}}\right)$.

I define the average realized covariance of two given processes U_t and V_t by $RC^{avg}(U,V)$ as an extension of the average realized variance - $RV^{avg}(.)$ - the notion introduced earlier:

$$RC^{avg}(U,V) = \frac{1}{K} \sum_{k=1}^{K} RC^{(k)}(U,V), \tag{A.35}$$

where

$$RV^{(k)}(U,V) = \sum_{i=1}^{n_k} (U_{k-1+K\frac{i}{n}} - U_{k-1+K\frac{i-1}{n}})(V_{k-1+K\frac{i}{n}} - V_{k-1+K\frac{i-1}{n}}). \tag{A.36}$$

The average realized variance of the adjusted price could be written as

$$\begin{aligned} RV^{avg}(\widehat{p}^*) &= RV^{avg}(p^* + \mathbf{F}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}) + \xi) \\ &= RV^{avg}(p^*) + \underbrace{RV^{avg}(\mathbf{F}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}))}_{(a)} + RV^{avg}(\xi) \\ &+ \underbrace{2RC^{avg}(\mathbf{F}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}), \xi)}_{(b)} + \underbrace{2RC^{avg}(p^*, \mathbf{F}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}))}_{(c)} + \underbrace{2RC^{avg}(p^*, \xi)}_{(d)}, \end{aligned} \tag{A.37}$$

as a result of the definitions in (A.22) and (A.23) as well as the average realized covariance in (A.35) and (A.36).

Using (A.37), I need to show that each of the terms (a), (b), (c) and (d) is $O_p\left(\frac{1}{\sqrt{K}}\right)$ to be able to prove Lemma 1 (ii).

$$
\begin{aligned}
(a) &= RV^{avg}(\mathbf{F}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})) = \frac{1}{K}\sum_{k=1}^{K} RV^{(k)}(\mathbf{F}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})) \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\left[(\mathbf{F}^{'}_{k-1+K\frac{i}{n}} - \mathbf{F}^{'}_{k-1+K\frac{i-1}{n}})(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})\right]^2 \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{n_k}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})^{'}(\mathbf{F}_{k-1+K\frac{i}{n}} - \mathbf{F}_{k-1+K\frac{i-1}{n}})(\mathbf{F}^{'}_{k-1+K\frac{i}{n}} - \mathbf{F}^{'}_{k-1+K\frac{i-1}{n}})(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}) \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{n_k} trace\left((\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})^{'}(\mathbf{F}_{k-1+K\frac{i}{n}} - \mathbf{F}_{k-1+K\frac{i-1}{n}})(\mathbf{F}^{'}_{k-1+K\frac{i}{n}} - \mathbf{F}^{'}_{k-1+K\frac{i-1}{n}})(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})\right) \qquad \text{(A.38)} \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{n_k} trace\left(\underbrace{(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})^{'}}_{o_P(1/n)}\underbrace{(\mathbf{F}_{k-1+K\frac{i}{n}} - \mathbf{F}_{k-1+K\frac{i-1}{n}})(\mathbf{F}^{'}_{k-1+K\frac{i}{n}} - \mathbf{F}^{'}_{k-1+K\frac{i-1}{n}})}_{O_P(1)}\right) \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{n_k} o_P(1/n) = \left(\frac{1}{K}\sum_{k=1}^{K} n_k\right) o_P(1/n) = \overline{n}\ o_P(1/n) = o_P(\frac{\sqrt{\overline{n}}}{n}),
\end{aligned}
$$

using the Normality limit distribution of $\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}$ as well as the rate of convergence of Theorem 1 (ii). I also use the assumption that the fourth moment of $\mathbf{F}_t$ is bounded and the definition of $\overline{n}$ in (A.24). Finally, as a result of condition (A.28), $(a) = o_P(\frac{\sqrt{\overline{n}}}{n}) < O_p\left(\frac{1}{\sqrt{K}}\right)$.

$$
\begin{aligned}
(b) &= RC^{avg}(\mathbf{F}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}),\xi) = \frac{1}{K}\sum_{k=1}^{K} RC^{(k)}(\mathbf{F}(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}),\xi) \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\underbrace{(\xi_{k-1+K\frac{i}{n}} - \xi_{k-1+K\frac{i-1}{n}})(\mathbf{F}^{'}_{k-1+K\frac{i}{n}} - \mathbf{F}^{'}_{k-1+K\frac{i-1}{n}})}_{O_P(1)}\underbrace{(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})}_{o_P(1/\sqrt{n})} \qquad \text{(A.39)} \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{n_k} o_P(1/\sqrt{n}) = \left(\frac{1}{K}\sum_{k=1}^{K} n_k\right) o_P(1/\sqrt{n}) = \overline{n}\ o_P(1/\sqrt{n}) = o_P\left(\sqrt{\frac{\overline{n}}{n}}\right),
\end{aligned}
$$

using Assumption 2 (ii), Theorem 1 (ii), Assumption 3 and the definition of $\overline{n}$ in (A.24).

Finally, as a result of condition (A.28), $(b) = o_P\left(\sqrt{\frac{\overline{n}}{n}}\right) = o_p\left(\frac{1}{\sqrt{K}}\right) < O_p\left(\frac{1}{\sqrt{K}}\right)$.

$$
\begin{aligned}
(c) &= RC^{avg}(p^*, \mathbf{F}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})) = \frac{1}{K}\sum_{k=1}^{K} RC^{(k)}(p^*, \mathbf{F}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})) \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{i=1}^{n_k} \underbrace{(p^*_{k-1+K\frac{i}{n}} - p^*_{k-1+K\frac{i-1}{n}})(\mathbf{F}'_{k-1+K\frac{i}{n}} - \mathbf{F}'_{k-1+K\frac{i-1}{n}})}_{<O_P(1)} \underbrace{(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}})}_{o_P(1/\sqrt{n})} \qquad \text{(A.40)} \\
&< \left(\frac{1}{K}\sum_{k=1}^{K} n_k\right) o_P(1/\sqrt{n}) = \overline{n}\; o_P(1/\sqrt{n}) = o_p\left(\frac{1}{\sqrt{K}}\right) \\
&< O_p\left(\frac{1}{\sqrt{K}}\right),
\end{aligned}
$$

using Assumptions 1, 2 and Theorem 1 (ii). The derivation of the asymptotic order in (A.40) results from the definition of $\overline{n}$ in (A.24) and the condition (A.28).
$(d) = RC^{avg}(p^*, \xi) = O_p\left(\frac{1}{\sqrt{K}}\right)$ is a direct result from Lemma A.2 part (b) relative to the multiple grid case, page 1408 from Zhang et al. (2005); which completes the proof of Lemma 1.

Now, I turn to the proof of (A.26) using Lemma 1. Being back to the two components of (A.29), I derive the joint limit distribution of these two components in order to show (A.26). In expression (A.15) of their Appendix, Zhang et al. (2005) show that for an i.i.d. and exogenous noise ξ, the joint distribution of $RV(\xi)$ and $RV^{avg}(\xi)$ is given by

$$
\frac{1}{\sqrt{n}}\begin{pmatrix} RV(\xi) - 2NE[\xi^2] \\ RV^{avg}(\xi)K - 2\overline{N}KE[\xi^2] \end{pmatrix} \xrightarrow{L} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4E[\xi^4] & 4Var[\xi^2] \\ 4Var[\xi^2] & 4E[\xi^4] \end{pmatrix}\right). \qquad \text{(A.41)}
$$

Using Lemma 1, the limiting result (A.41) becomes

$$
\begin{aligned}
&\begin{pmatrix} \sqrt{n}\left(\frac{RV(\widehat{p}^*)}{2n} - E[\xi_t^2]\right) \\ \sqrt{\frac{K}{\overline{n}}}\left(RV^{avg}(\widehat{p}^*) - RV^{avg}(p^*) - 2\overline{n}E[\xi^2]\right) \end{pmatrix} \\
&\xrightarrow{L} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} E[\xi^4] & 2Var[\xi^2] \\ 2Var[\xi^2] & 4E[\xi^4] \end{pmatrix}\right),
\end{aligned} \qquad \text{(A.42)}
$$

which yields the limit distribution of (A.26).
To complete the proof, it remains to show (A.27). This result is derived in Section 3.4 of Zhang et al. (2005).

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